2.3.3 Symplectic reductions

In this subsection, we consider a Hamiltonian action of a compact Lie group G on a symplectic manifold (M, ω) with moment map $\mu : M \to \mathfrak{g}^*$.

We say that $\alpha \in \mathfrak{g}^*$ is a regular value of μ if $\mu^{-1}(\alpha) = \emptyset$ or for any $x \in \mu^{-1}(\alpha)$, $d\mu : T_xM \to T_{\mu(x)}\mathfrak{g}^* = \mathfrak{g}^*$ is surjective. By the Sard theorem, the measure of the set of the points which are not the regular value is zero.

If $\alpha \in \mathfrak{g}^*$ is a regular value of μ, then $\mu^{-1}(\alpha)$ is a smooth manifold. Recall that its coadjoint orbit is $\mathcal{O}_\alpha = G \cdot \alpha = \mathrm{Ad}_G^* \alpha$.

Proposition 2.3.10. *For any $\alpha \in \mathfrak{g}^*$, G acts on $\mu^{-1}(\mathcal{O}_\alpha)$. If $\alpha \in \mathfrak{g}^*$ is a regular value of μ, then for any $x \in \mu^{-1}(\alpha)$, the map*

$$\mathfrak{g} \to T_x(G \cdot x), \quad \xi \mapsto \xi_x^M \tag{2.3.56}$$

is bijective and

$$T_x(G \cdot x) = T_x(\mu^{-1}(\alpha))^{\perp_\omega}. \tag{2.3.57}$$

In this case, the G-action is locally free on $\mu^{-1}(G \cdot \alpha)$, that is, for any $x \in \mu^{-1}(G \cdot \alpha)$, there exists an open neighborhood U of $e \in G$ such that U acts freely on $\mu^{-1}(G \cdot \alpha)$. Inversely, if G acts locally free on $\mu^{-1}(G \cdot \alpha)$, then α is a regular value of μ.

Proof. By (2.3.3), we know if $x \in \mu^{-1}(\mathcal{O}_\alpha)$, then for any $g \in G$, $\mu(gx) \in \mathcal{O}_\alpha$, thus G acts on $\mu^{-1}(\mathcal{O}_\alpha)$.

For any $X \in T_xM$, $\xi \in \mathfrak{g}$, by (2.3.2), we have

$$(d\mu(X), \xi) = \omega(\xi_x^M, X). \tag{2.3.58}$$

From (2.3.58), $d\mu$ is surjective at x if and only if the equation $\xi_x^M = 0$ implies $\xi = 0$, i.e., G acts locally free near x in $\mu^{-1}(G \cdot \alpha)$. Thus (2.3.56) and the last part of Proposition 2.3.10 hold.

For $X \in T_x(\mu^{-1}(\alpha))$ and $\xi \in \mathfrak{g}$, we have $d\mu(X) = 0$. By (2.3.58), we get $\xi_x^M \in T_x(\mu^{-1}(\alpha))^{\perp_\omega}$. Since $\dim T_x(\mu^{-1}(\alpha))^{\perp_\omega} = \dim M - \dim \mu^{-1}(\alpha) = \dim G$, we get (2.3.57). The proof of Lemma 2.3.10 is completed. ☐

We state now the following main result of this subsection which gives an effective way to construct the symplectic manifolds from the known symplectic manifolds. This fundamental construction has many applications in geometry, in particular to get the natural symplectic structure on the different moduli spaces.

Theorem 2.3.11. *If G acts freely on $\mu^{-1}(0)$, then the orbit space $M_G := G \backslash \mu^{-1}(0)$ is a smooth manifold, and the natural projection $\pi : \mu^{-1}(0) \to M_G$ is a principle G-bundle, and there exists a unique symplectic form ω_G on M_G satisfying $\pi^*\omega_G = \imath^*\omega$, with $\imath : \mu^{-1}(0) \hookrightarrow M$ the natural embedding.*

Definition 2.3.12. The symplectic manifold (M_G, ω_G) is called the Mardsen-Weinstein symplectic reduction of (M, ω) with respect to the Hamiltonian G-action, it is also called the reduced space or symplectic quotient.

The following technical Lemma allows us to obtain the symplectic form on the reduction space from the original manifold.

Let H be a Lie group and $\pi : P \to M$ be a principal H-bundle. We say that a form β on P is horizontal if for any $v \in \ker(d\pi)$, $i_v\beta = 0$. Set

$$\begin{aligned} \Omega^\bullet(P)^H &:= \{\beta \in \Omega^\bullet(P) : g^*\beta = \beta, \quad \text{for any } g \in H\}, \\ \Omega^\bullet(P)_{\mathrm{bas}} &:= \{\beta \in \Omega^\bullet(P)^H : \beta \text{ is horizontal}\}. \end{aligned} \tag{2.3.59}$$

Then $\Omega^\bullet(P)^H$ is the H-invariant subspace of $\Omega^\bullet(P)$ and we say that β is a basic form on P if $\beta \in \Omega^\bullet(P)_{\rm bas}$.

Lemma 2.3.13. *The exterior differential d^P preserves $\Omega^\bullet(P)_{\rm bas}$, thus $(\Omega^\bullet(P)_{\rm bas}, d^P)$ is a subcomplex of the de Rham complex $(\Omega^\bullet(P), d^P)$. The pull-back map π^* induces an isomorphism of complexes $(\Omega^\bullet(M), d^M)$ and $(\Omega^\bullet(P)_{\rm bas}, d^P)$, i.e.,*

$$\text{For any } k\in\mathbb{N},\ \pi^*:\Omega^k(M)\to\Omega^k(P)_{\rm bas} \text{ is an isomorphism and } \pi^*d^M = d^P\pi^*. \tag{2.3.60}$$

In particular, $d^M\bar\alpha = 0$ if and only if $d^P\pi^*\bar\alpha = 0$.

If $\bar\alpha \in \Omega^2(M)$, then $\bar\alpha$ is nondegenerate if and only if $\ker(\pi^*\bar\alpha) = \ker(d\pi)$.

Proof. If $\alpha \in \Omega^k(P)_{\rm bas}$, then $g^*d^P\alpha = d^Pg^*\alpha = d^P\alpha$ for any $g \in H$, thus $d^P\alpha \in \Omega^{k+1}(P)^H$. Moreover, as $\alpha \in \Omega^k(P)_{\rm bas}$, for any $\xi \in \mathfrak{g}$, by (2.2.2), $L_{\xi^P}\alpha = L_\xi\alpha = 0$ and $i_{\xi^P}\alpha = 0$. Thus by the Cartan formula (1.2.20),

$$i_{\xi^P}d^P\alpha = L_{\xi^P}\alpha - d^Pi_{\xi^P}\alpha = 0. \tag{2.3.61}$$

This means that $d^P\alpha$ is horizontal. Thus $d^P:\Omega^k(P)_{\rm bas}\to\Omega^{k+1}(P)_{\rm bas}$ and it is a subcomplex of $(\Omega^\bullet(P), d^P)$.

Note that for any $\bar\alpha \in \Omega^k(M)$, $Y_1,\dots,Y_k \in T_pP$, $p\in P$,

$$(\pi^*\bar\alpha)(Y_1,\dots,Y_k) = \bar\alpha(d\pi(Y_1),\dots,d\pi(Y_k)). \tag{2.3.62}$$

From (2.3.62), $\pi^*\bar\alpha$ is horizontal and for any $g\in G$, as $\pi\circ g = \pi$, we have

$$(g^*\pi^*\bar\alpha)(Y_1,\dots,Y_k) = \bar\alpha(d(\pi\circ g)(Y_1),\dots,d(\pi\circ g)(Y_k)) = \pi^*\bar\alpha(Y_1,\dots,Y_k). \tag{2.3.63}$$

Thus $\pi^*\bar\alpha$ is basic, and π^* maps $\Omega^k(M)$ to $\Omega^k(P)_{\rm bas}$. As $d\pi: T_pP\to T_{\pi(p)}M$ is surjective for any $p\in P$, from (2.3.62), $\pi^*:\Omega^k(M)\to\Omega^k(P)_{\rm bas}$ is injective.

Assume now $\alpha\in\Omega^k(P)_{\rm bas}$, we define $\bar\alpha\in\Omega^k(M)$ by

$$\bar\alpha_x(X_1,\dots,X_k) := \alpha_p(Y_{1,p},\dots,Y_{k,p}), \tag{2.3.64}$$

for any $x\in M$, $p\in\pi^{-1}(x)$ and $X_j\in T_xM$, $1\le j\le k$, where $Y_{j,p}\in T_pP$ such that $d\pi(Y_{j,p}) = X_j$. Then $\bar\alpha$ does not depend on the choices of p, $Y_{j,p}$. In fact, fixing $p\in\pi^{-1}(x)$ first, if $Y'_{j,p}\in T_pP$ such that $d\pi(Y'_{j,p}) = X_j$ we have $Y_{j,p}-Y'_{j,p}\in\ker(d\pi)$, thus

$$i_{Y_{j,p}}\alpha - i_{Y'_{j,p}}\alpha = i_{Y_{j,p}-Y'_{j,p}}\alpha = 0. \tag{2.3.65}$$

For $p,p'\in\pi^{-1}(x)$, then there exists $g\in G$, such that $p' = p\cdot g$. By (2.3.65), we can choose $Y_{j,pg} = dg(Y_{j,p})$, then

$$\begin{aligned}\alpha_{pg}(Y_{1,pg},\dots,Y_{k,pg}) &= \alpha_{pg}(dg(Y_{1,p}),\dots,dg(Y_{k,p})) = (g^*\alpha)_p(Y_{1,p},\dots,Y_{k,p})\\ &= \alpha_p(Y_{1,p},\dots,Y_{k,p}).\end{aligned} \tag{2.3.66}$$

From (2.3.65) and (2.3.66), $\bar\alpha$ is well-defined and $\alpha = \pi^*\bar\alpha$. Thus $\pi^*:\Omega^k(M)\to\Omega^k(P)_{\rm bas}$ is surjective.

As the pull-back map commutes with the exterior differential, we have $\pi^*d^M = d^P\pi^*$. Thus we have established (2.3.60). From (2.3.60), $d^M\bar\alpha = 0$ if and only if $d^P\pi^*\bar\alpha = 0$.

Finally, from (2.3.62),

$$\begin{aligned}\bar\alpha\in\Omega^2(M) \text{ is nondegenerate} &\Leftrightarrow \text{if } i_X\bar\alpha = 0, \text{then } X = 0\\ &\Leftrightarrow \text{if } Y\in\ker(\pi^*\bar\alpha), \text{then } d\pi(Y) = 0\\ &\Leftrightarrow \ker(\pi^*\bar\alpha) = \ker(d\pi).\end{aligned} \tag{2.3.67}$$

The proof of Lemma 2.3.13 is completed. □

Proof of Theorem 2.3.11. As ω is G-invariant and G preserves $\mu^{-1}(0)$ (cf. (2.3.3)), we know $\imath^*\omega \in \Omega^2(\mu^{-1}(0))^G$. For any $x \in \mu^{-1}(0)$, $\xi \in \mathfrak{g}$ and $X \in T_x\mu^{-1}(0)$,

$$\omega(\xi_x^M, X) = \langle d\mu(X), \xi\rangle_x = 0. \tag{2.3.68}$$

So $i_{\xi_x^M}\imath^*\omega = 0$. By Lemma 2.3.13, there exists a unique $\bar{\omega} \in \Omega^2(M_G)$, such that $\pi^*\bar{\omega} = \imath^*\omega$. As $d\imath^*\omega = \imath^* d\omega = 0$, this implies $d\bar{\omega} = 0$. By Proposition 2.3.10,

$$\ker(\imath^*\omega) = T(\mu^{-1}(0))^{\perp,\omega} = T(G\cdot x) = \ker(d\pi).$$

So $\bar{\omega}$ is nondegenerate. □

In general, for a regular value $\alpha \in \mathfrak{g}^*$ of μ, we consider the reduction on its coadjoint orbit $\mathcal{O}_\alpha$. Recall that the stabilizer of α is

$$G_\alpha = \{g \in G : \mathrm{Ad}_g^*\alpha = \alpha\}. \tag{2.3.69}$$

By (2.3.3), G acts on $\mu^{-1}(\mathcal{O}_\alpha)$. If α is a regular value of μ, by Proposition 2.3.10, the G-action is locally free on $\mu^{-1}(\mathcal{O}_\alpha)$ and the G_α-action is locally free on $\mu^{-1}(\alpha)$.

Lemma 2.3.14. *The group G acts freely on $\mu^{-1}(\mathcal{O}_\alpha)$ if and only if G_α acts freely on $\mu^{-1}(\alpha)$. Moreover, the map $\psi : G_\alpha\backslash\mu^{-1}(\alpha) \to G\backslash\mu^{-1}(\mathcal{O}_\alpha)$ induced by the injection $\mu^{-1}(\alpha) \hookrightarrow \mu^{-1}(\mathcal{O}_\alpha)$, is a diffeomorphism.*

Proof. It is obvious that if G acts freely on $\mu^{-1}(\mathcal{O}_\alpha)$, then G_α acts freely on $\mu^{-1}(\alpha)$. Now we assume G_α acts freely on $\mu^{-1}(\alpha)$. If $g \in G$ and $x \in \mu^{-1}(\mathcal{O}_\alpha)$ such that $gx = x$, then as $\mu(x) \in \mathcal{O}_\alpha$ there exists $h \in G$ such that $\mu(x) = \mathrm{Ad}_h^*\alpha$. So by (2.3.3), $y := h^{-1}x \in \mu^{-1}(\alpha)$, and

$$h^{-1}ghy = h^{-1}ghh^{-1}x = h^{-1}x = y, \quad \alpha = \mu(y) = \mu(h^{-1}ghy) = \mathrm{Ad}^*_{h^{-1}gh}\alpha. \tag{2.3.70}$$

Thus $h^{-1}gh \in G_\alpha$. As G_α acts freely on $\mu^{-1}(\alpha)$, we know $h^{-1}gh = e$, thus $g = e$. So G acts freely on $\mu^{-1}(\mathcal{O}_\alpha)$.

About the map ψ, it is obvious that it is well-defined. For any $x \in \mu^{-1}(\mathcal{O}_\alpha)$, there exists $h \in G$ such that $h^{-1}x \in \mu^{-1}(\alpha)$. So the map ψ is surjective. If $x, y \in \mu^{-1}(\alpha)$ and $h \in G$ such that $hx = y$, then

$$\alpha = \mu(y) = \mathrm{Ad}_h^*\mu(x) = \mathrm{Ad}_h^*\alpha. \tag{2.3.71}$$

So $h \in G_\alpha$. Thus the map ψ is injective.

We claim that for $x \in \mu^{-1}(\alpha)$,

$$T_x(\mu^{-1}(\alpha)) \cap T_x(G\cdot x) = T_x(G_\alpha \cdot x). \tag{2.3.72}$$

It is obvious that $T_x(G_\alpha\cdot x) \subset T_x(\mu^{-1}(\alpha)) \cap T_x(G\cdot x)$. For $\xi \in \mathfrak{g}$, if $\xi_x^M \in T_x(\mu^{-1}(\alpha)) \cap T_x(G\cdot x)$, we have $d\mu(\xi_x^M) = 0$. Thus by (2.3.3) and $\mu(x) = \alpha$, we get

$$\begin{aligned} \mathrm{ad}_\xi^*\alpha = \frac{d}{dt}\Big|_{t=0} \mathrm{Ad}^*_{\exp(t\xi)}\mu(x) = \frac{d}{dt}\Big|_{t=0} \mu(\exp(t\xi)x) \\ = d\mu(\xi_x^M) = 0. \end{aligned} \tag{2.3.73}$$

So $\xi \in \mathfrak{g}_\alpha$ and $\xi_x^M \in T_x(G_\alpha\cdot x)$. Thus (2.3.72) holds.

As ψ is induced by the injection $\mu^{-1}(\alpha) \hookrightarrow \mu^{-1}(\mathcal{O}_\alpha)$ which is $\mathscr{C}^\infty$, thus ψ is $\mathscr{C}^\infty$. Finally from (2.3.72), we know $d\psi : T(G_\alpha\backslash\mu^{-1}(\alpha)) \to T(G\backslash\mu^{-1}(\mathcal{O}_\alpha))$ is also bijective, thus ψ is a diffeomorphism. □

Theorem 2.3.15. *If G acts freely on $\mu^{-1}(\mathcal{O}_\alpha)$, then there exists a unique $\omega_\alpha \in \Omega^2(G\backslash\mu^{-1}(\mathcal{O}_\alpha))$ such that $\imath^*\pi^*\omega_\alpha = \imath^*\omega \in \Omega^2(\mu^{-1}(\alpha))$ and ω_α is a symplectic form, where $\pi : \mu^{-1}(\mathcal{O}_\alpha) \to M_\alpha := G\backslash\mu^{-1}(\mathcal{O}_\alpha)$ is the projection, and $\imath : \mu^{-1}(\alpha) \hookrightarrow M$ is the natural embedding.*

The symplectic manifold $(M_\alpha := G\backslash\mu^{-1}(\mathcal{O}_\alpha), \omega_\alpha)$ is called the Marsden-Weinstein symplectic reduction of (M, ω) at $\alpha \in \mathfrak{g}^*$ with respect to the Hamiltonian G-action, or the reduction at the coadjoint orbit $\mathcal{O}_\alpha$.

First proof. For any $x \in M$, $X \in T_x(\mu^{-1}(\alpha))$ and $\xi \in \mathfrak{g}_\alpha$, $(i_{\xi_x^M}\omega)(X) = \omega(\xi_x^M, X) = \langle i_X d\mu, \xi\rangle_x = 0$, as $X(\mu)_x = 0$. Certainly, for $g \in G_\alpha$, $g^*(\imath^*\omega) = \imath^*\omega$, thus by Lemma 2.3.13, there exists a unique

$$\bar{\omega} \in \Omega^2(G_\alpha\backslash\mu^{-1}(\alpha)) = \Omega^2(G\backslash\mu^{-1}(\mathcal{O}_\alpha)) \ \text{ such that } \ \pi_1^*\bar{\omega} = \imath^*\omega, \tag{2.3.74}$$

where $\pi_1 : \mu^{-1}(\alpha) \to G_\alpha\backslash\mu^{-1}(\alpha)$ is the natural projection. Since $d\omega = 0$, by Lemma 2.3.13, we get $d\bar{\omega} = 0$. By Proposition 2.3.10 and (2.3.72), we have

$$\begin{aligned} \ker(\imath^*\omega) &= T_x(\mu^{-1}(\alpha)) \cap T_x(\mu^{-1}(\alpha))^{\perp\omega} = T_x(\mu^{-1}(\alpha)) \cap T_x(G\cdot x) \\ &= T_x(G_\alpha \cdot x) = \ker(d\pi_1). \end{aligned} \tag{2.3.75}$$

By Lemma 2.3.13, $\bar{\omega}$ is nondegenerate. □

Second proof. We denote by $\omega^{\mathcal{O}_\alpha}$ the symplectic form on $\mathcal{O}_\alpha$ defined by (2.3.51). We consider a new symplectic manifold $(M \times \mathcal{O}_\alpha, \omega - \omega^{\mathcal{O}_\alpha})$ with a G-action induced by the G-actions on M and $\mathcal{O}_\alpha$:

$$g(x, \beta) = (g\cdot x, \mathrm{Ad}_g^*\beta) \quad \text{ for } g \in G, x \in M, \beta \in \mathcal{O}_\alpha. \tag{2.3.76}$$

By Proposition 2.3.9, $\mu_\alpha(\beta) = \beta$ is the moment map for the G-action on $(\mathcal{O}_\alpha, \omega_{\mathcal{O}_\alpha})$. Thus, the map

$$\tilde{\mu} : M \times \mathcal{O}_\alpha \to \mathfrak{g}^*, \quad \tilde{\mu}(x, \beta) = \mu(x) - \beta \tag{2.3.77}$$

is a moment map for the G-action on $M \times \mathcal{O}_\alpha$. Then the map

$$\mu^{-1}(\mathcal{O}_\alpha) \to \tilde{\mu}^{-1}(0) = \{(x, \mu(x)) : x \in \mu^{-1}(\mathcal{O}_\alpha)\}, \quad x \mapsto (x, \mu(x)) \tag{2.3.78}$$

is bijective. Since G acts freely on $\mu^{-1}(\mathcal{O}_\alpha)$ and $\tilde{\mu}^{-1}(0)$, we have $G\backslash\mu^{-1}(\mathcal{O}_\alpha) \simeq G\backslash\tilde{\mu}^{-1}(0)$. By Proposition 2.3.11, we get Theorem 2.3.15. Moreover, for $\Psi : \mu^{-1}(\alpha) \hookrightarrow \mu^{-1}(\mathcal{O}_\alpha) \to \tilde{\mu}^{-1}(0)$, we have $\Psi^*(\omega - \omega^{\mathcal{O}_\alpha}) = \imath^*\omega$, thus we get the same symplectic form on $G \setminus \mu^{-1}(\mathcal{O}_\alpha)$ as in the first proof. □

Proposition 2.3.16. *Let G, H be two Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{h}$, acting on a symplectic manifold (M, ω). We suppose that G is compact and $g(hx) = h(gx)$ for any $g \in G, h \in H, x \in M$, and that*

$$\mu = (\mu_G, \mu_H) : M \to \mathfrak{g}^* \oplus \mathfrak{h}^* \tag{2.3.79}$$

is a moment map for the $G \times H$-action on (M, ω). For $\alpha \in \mathfrak{g}^*$, if G_α acts freely on $\mu_G^{-1}(\alpha)$, then H acts naturally and Hamiltonianly on $M_\alpha := G_\alpha\backslash\mu_G^{-1}(\alpha)$, with the associated moment map $\mu_{H,\alpha}$ defined by: for $y \in M_\alpha$,

$$\mu_{H,\alpha}(y) = \mu_H(x) \quad \text{and } x \in \mu^{-1}(\alpha) \text{ such that } \quad \pi_1(x) = y, \tag{2.3.80}$$

with $\pi_1 : \mu^{-1}(\alpha) \to M_\alpha$ the natural projection.

Proof. Observe first that for $x \in M$, $g \in G, h \in H$, $\mu((g,h)x) = \mathrm{Ad}^*_{(g,h)}\mu(x)$ if and only if

$$\mu_G((g,h)x) = \mathrm{Ad}^*_g \mu_G(x), \quad \mu_H((g,h)x) = \mathrm{Ad}^*_h \mu_H(x). \tag{2.3.81}$$

And $d(\mu,(\beta,\xi)) = i_{(\beta,\xi)^M}\omega$ for any $\beta \in \mathfrak{g}, \xi \in \mathfrak{h}$ if and only if

$$d(\mu_G,\beta) = i_{\beta^M}\omega, \quad \text{ and } d(\mu_H,\xi) = i_{\xi^M}\omega. \tag{2.3.82}$$

Thus $\mu_G : M \to \mathfrak{g}^*$, $\mu_H : M \to \mathfrak{h}^*$ are moment maps for the G and H-actions on M. Inversely, if the G, H actions are Hamiltonian and (2.3.81) holds, then the $G\times H$-action on M is Hamiltonian and $\mu = (\mu_G, \mu_H)$ is the associated moment map.

By (2.3.81), H acts on $\mu_G^{-1}(\alpha)$, thus for $h \in H$, $y \in M_\alpha$, let $x \in \mu^{-1}(\alpha)$ such that $\pi_1(x) = y$, we define that

$$h \cdot y = \pi_1(hx). \tag{2.3.83}$$

Then $\pi_1(hx)$ and $\mu_H(x)$ do not depend on the choice of x, as if $x' \in \mu^{-1}(\alpha)$ such that $\pi_1(x') = y$, then there is $g \in G_\alpha$ such that $x' = gx$, and by (2.3.81), we get $\mu_H(x) = \mu_H(x')$.

Finally, for $\xi \in \mathfrak{h}$, $X \in T_yM_\alpha$, we take $\widetilde{X} \in T_x\mu^{-1}(\alpha)$ such that $d\pi_1(\widetilde{X}) = X$, then by (2.3.81), we get

$$X(\mu_{H,\alpha},\xi)_y = \widetilde{X}(\mu_H,\xi)_x = \omega(\xi^M_x, \widetilde{X}) = \omega_\alpha(\xi^{M_\alpha}_y, X), \tag{2.3.84}$$

as $\xi^M_x \in T_x\mu^{-1}(\alpha)$ and $d\pi_1(\xi^M_x) = \xi^{M_\alpha}_y$. Moreover, (2.3.80) and (2.3.81) imply $\mu_{H,\alpha}(hy) = \mathrm{Ad}^*_h \mu_{H,\alpha}(y)$. Thus $\mu_{H,\alpha}$ is a moment map for the H-action on M_α. □

Example 2.3.17. (Projective space) The group $\mathbb{S}^1$ acts on $(\mathbb{C}^n, \omega_{st})$ by

$$e^{\sqrt{-1}\theta}(z_1,\dots,z_n)^t := (e^{\sqrt{-1}\theta}z_1,\dots,e^{\sqrt{-1}\theta}z_n)^t. \tag{2.3.85}$$

By Subsection 2.3.2, the $\mathbb{S}^1 \subset U(n)$ action is a Hamiltonian action and $\mu_{\mathbb{S}^1} = i^* \circ \mu_{U(n)}$ is the moment map by the functional property, where $i : \mathrm{Lie}(\mathbb{S}^1) \to u(n)$ is the embedding, and

$$i(\xi) = \xi \cdot \mathrm{Id}_{\mathbb{C}^n} \quad \text{ for } \xi = \sqrt{-1}\theta \in \mathrm{Lie}(\mathbb{S}^1) = \sqrt{-1}\mathbb{R}. \tag{2.3.86}$$

By (2.3.35) and (2.3.39), for $z \in \mathbb{C}^n$, we have

$$(\mu_{\mathbb{S}^1}(z),\xi) = (\mu_{U(n)}(z), i(\xi)) = -\mathrm{Tr}\Big[-\frac{\sqrt{-1}}{2}zz^*\xi\cdot \mathrm{Id}_{\mathbb{C}^n}\Big] = \frac{\sqrt{-1}}{2}|z|^2\xi. \tag{2.3.87}$$

We use the bilinear form (scalar product) $-\xi\eta$ for $\xi,\eta \in \mathrm{Lie}(\mathbb{S}^1) = \sqrt{-1}\mathbb{R}$ to identify $\mathrm{Lie}(\mathbb{S}^1)^*$ and $\mathrm{Lie}(\mathbb{S}^1)$, then

$$\mu_{\mathbb{S}^1}(z) = -\frac{\sqrt{-1}}{2}|z|^2 \in \mathrm{Lie}(\mathbb{S}^1)^* = \sqrt{-1}\mathbb{R}. \tag{2.3.88}$$

Now we consider the symplectic reduction on $-\frac{\sqrt{-1}}{2}t \in \mathrm{Lie}(\mathbb{S}^1)^*, t > 0$. By (2.3.88), we get

$$\mu_{\mathbb{S}^1}^{-1}\Big(-\frac{\sqrt{-1}}{2}\Big) = \{z \in \mathbb{C}^n : \ |z|^2 = 1\} = \mathbb{S}^{2n-1}, \tag{2.3.89}$$

the $2n-1$-dimensional unit sphere. So the symplectic reduction on $-\frac{\sqrt{-1}}{2} \in \mathrm{Lie}(\mathbb{S}^1)^*$ is $\mathbb{S}^1\backslash\mathbb{S}^{2n-1}$, the projective space $\mathbb{CP}^{n-1}$. We show now that the symplectic form $\bar{\omega}$ on the symplectic reduction $\mathbb{CP}^{n-1}$ induced by ω_{st} is just π times the Fubini-Study form ω_{FS}.

For the $\mathbb{C}^\times$-principal bundle $\widetilde{\pi} : \mathbb{C}^n \setminus \{0\} \to \mathbb{CP}^{n-1}$ with the $\mathbb{C}^\times$ action on $\mathbb{C}^n \setminus \{0\}$ by $a \cdot (z_1, \dots, z_n)^t = (az_1, \dots, az_n)^t$ for $a \in \mathbb{C}^\times$. At first, $\widetilde{\omega}_{FS} \in \Omega^2(\mathbb{C}^n \setminus \{0\})^{\mathbb{C}^\times}$ (cf. (1.4.6)), as $\widetilde{\omega}_{FS}$ is a $(1,1)$-form and the $\mathbb{C}^\times$ action is holomorphic, we only need to verify it for $u, v \in T_z^{(1,0)}\mathbb{C}^n$, then by (1.4.6),

$$(a^*\widetilde{\omega}_{FS})(u, \overline{v})_z = \widetilde{\omega}_{FS}(au, \overline{av})_{az} = \widetilde{\omega}_{FS}(u, \overline{v})_z.$$

Moreover, $\ker \widetilde{\omega}_{FS} = \ker(d\widetilde{\pi})$. By using a linear transformation, we only need to verify it for $x = (1, 0, \dots, 0)^t$, but

$$\widetilde{\omega}_{FS,x} = \frac{\sqrt{-1}}{2\pi} \sum_{j=2}^n dz_j \wedge d\overline{z}_j, \quad \text{ and } \ker \widetilde{\omega}_{FS,x} = \mathbb{R}\Big\{\frac{\partial}{\partial z_1}, \frac{\partial}{\partial \overline{z}_1}\Big\} = \ker(d\widetilde{\pi})_x.$$

Thus from Lemma 2.3.13 and the construction in Example 1.4.2, $\omega_{FS} \in \Omega^2(\mathbb{CP}^{n-1})$ is the unique 2-form such that

$$\widetilde{\omega}_{FS} = \widetilde{\pi}^*\omega_{FS}. \tag{2.3.90}$$

But by (1.4.6) and $\sum_{j=1}^n z_j d\overline{z}_j = 0$ on $\mathbb{S}^{2n-1}$, we know as differential forms on $\mathbb{S}^{2n-1}$,

$$\widetilde{\omega}_{FS} = \frac{\sqrt{-1}}{2\pi} \sum_{j=1}^n dz_j \wedge d\overline{z}_j = \frac{1}{\pi}\omega_{st}|_{\mathbb{S}^{2n-1}} \in \Omega^2(\mathbb{S}^{2n-1}). \tag{2.3.91}$$

By applying Lemma 2.3.13, (2.3.90) and (2.3.91) for the $\mathbb{S}^1$-principal bundle $\pi : \mathbb{S}^{2n-1} \to \mathbb{CP}^{n-1}$, we get

$$\omega_{FS} = \frac{1}{\pi}\bar{\omega}. \tag{2.3.92}$$

Now we consider $\mathrm{U}(n) \times \mathbb{S}^1$-action on $\mathbb{C}^n$ by: for $z = (z_1, \dots, z_n)^t$,

$$(A, e^{i\theta})(z_1, \dots, z_n)^t = e^{i\theta} A(z_1, \dots, z_n)^t. \tag{2.3.93}$$

Then the $\mathrm{U}(n)$, $\mathbb{S}^1$ actions commute. By (2.3.40) and (2.3.88), the associated moment map is

$$\mu(z) := (\mu_{\mathrm{U}(n)}(z), \mu_{\mathbb{S}^1}(z)) = \Big(-\frac{\sqrt{-1}}{2}zz^*, -\frac{\sqrt{-1}}{2}|z|^2\Big) \in \mathfrak{u}(n) \oplus \mathrm{Lie}(\mathbb{S}^1). \tag{2.3.94}$$

Thus for $\alpha = -\frac{\sqrt{-1}}{2} \in \mathrm{Lie}(\mathbb{S}^1)$, the induced $\mathrm{U}(n)$ action on $\mathbb{CP}^{n-1}$ is defined by

$$B[z] := [Bz] \quad \text{ for } B \in \mathrm{U}(n), \tag{2.3.95}$$

here $[z]$ is the homogenous coordinate of $\mathbb{CP}^{n-1}$, and as $\frac{z}{|z|} \in \pi^{-1}([z])$, the moment map for the $\mathrm{U}(n)$ action on $(\mathbb{CP}^{n-1}, \pi\omega_{FS})$ is given by

$$(\mu([z]), A) = \Big(\mu_{\mathrm{U}(n)}(\frac{z}{|z|}), A\Big) = \sqrt{-1}\frac{\mathrm{Tr}[Azz^*]}{2|z|^2} = \sqrt{-1}\frac{\langle Az, z\rangle}{2|z|^2} \quad \text{ for } A \in \mathfrak{u}(n). \tag{2.3.96}$$

Exercise 2.3.1. In the context of Lemma 2.3.2, for $\eta \in \mathfrak{g}$, let $L_\eta f$ be the Lie derivative of η on $f \in \mathscr{C}^\infty(M)$, verify that

$$\{\mu(\eta), f\} = -L_{\eta^M} f = L_\eta f. \tag{2.3.97}$$

Exercise 2.3.2. Let $Q = \left\{ \begin{pmatrix} t & b \\ 0 & 1/t \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) \right\}$, $Q^+ = \left\{ \begin{pmatrix} t & b \\ 0 & 1/t \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) : t > 0 \right\}$, and we identify $U(1)$ as a subgroup of SU(2) via $t \in U(1) \to \begin{pmatrix} t & 0 \\ 0 & 1/t \end{pmatrix} \in \mathrm{SU}(2)$.

1. Verify that Q, Q^+ are closed subgroups of $\mathrm{SL}(2,\mathbb{C})$ and $Q = U(1) \cdot Q^+$.
2. For any $A \in \mathrm{SL}(2,\mathbb{C})$, there exists a unique decomposition $A = UB$ with $U \in \mathrm{SU}(2), B \in Q^+$.
3. Using Example 2.1.32 to show that $\mathrm{SL}(2,\mathbb{C})/Q \simeq \mathrm{SU}(2)/U(1) \simeq \mathbb{CP}^1$.
4. Verify that $\mathfrak{sl}(2,\mathbb{C}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in M_2(\mathbb{C}) \right\}$, $\mathfrak{su}(2) = \left\{ \begin{pmatrix} a & b \\ -\bar{b} & -a \end{pmatrix} : a \in \sqrt{-1}\mathbb{R}, b \in \mathbb{C} \right\}$, and
$$\mathfrak{sl}(2,\mathbb{C}) \ni \xi \to \{\eta \to -\mathrm{Tr}_{\mathbb{C}^2}[\xi\eta]\} \in \mathfrak{sl}(2,\mathbb{C})^* \tag{2.3.98}$$
is an isomorphism of $\mathrm{SL}(2,\mathbb{C})$-representations.
5. The left multiplication of $\mathrm{SL}(2,\mathbb{C})$ on $\mathrm{SL}(2,\mathbb{C})/Q \simeq \mathbb{CP}^1$ induces an $\mathrm{SL}(2,\mathbb{C})$-Hamiltonian action on $T^*\mathbb{CP}^1$ with moment map $\mu : T^*\mathbb{CP}^1 \to \mathfrak{sl}(2,\mathbb{C})^* \simeq \mathfrak{sl}(2,\mathbb{C})$. Verify that $\mathrm{Im}(\mu) = \mathrm{Ad}_{\mathrm{SL}(2,\mathbb{C})} \begin{pmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{pmatrix} = \mathrm{Ad}_{\mathrm{SU}(2)} \begin{pmatrix} 0 & \mathbb{C} \\ 0 & 0 \end{pmatrix}$.
6. The left multiplication of SU(2) on $\mathrm{SU}(2)/U(1) \simeq \mathbb{CP}^1$ induces an SU(2)-Hamiltonian action on $T^*\mathbb{CP}^1$ with moment map $\nu : T^*\mathbb{CP}^1 \to \mathfrak{su}(2)^* \simeq \mathfrak{su}(2)$. Verify that
$$\mathrm{Im}(\nu) = \left\{ \mathrm{Ad}_{\mathrm{SU}(2)} \begin{pmatrix} 0 & z \\ -\bar{z} & 0 \end{pmatrix} : z \in \mathbb{C} \right\}.$$

Exercise 2.3.3. Let $\mu : M \to \mathfrak{g}^*$ be a moment map.

1. For $\alpha \in \mathfrak{g}^*$, $x \in \mu^{-1}(\mathcal{O}_\alpha)$, verify that $G \cdot x \in \mu^{-1}(\mathcal{O}_\alpha)$.
2. Verify that as differential forms on $G \cdot x$,
$$\mu^*(\omega^{\mathcal{O}_\alpha}) = \omega \in \Omega^2(G \cdot x). \tag{2.3.99}$$
Assume that G acts freely on $\mu^{-1}(\mathcal{O}_\alpha)$.
3. Verify that for $\pi : \mu^{-1}(\mathcal{O}_\alpha) \to G\backslash\mu^{-1}(\mathcal{O}_\alpha)$, we have $\ker(d\pi)_x = \{\xi_x^M : \xi \in \mathfrak{g}\}$. For $x \in \mu^{-1}(\mathcal{O}_\alpha)$, $\xi, \eta \in \mathfrak{g}$, prove that
$$\omega(\xi_x^M, \eta_x^M) = \langle \mu, [\xi, \eta] \rangle_x .$$
Conclude that in general, $\ker(d\pi) \subset \ker(\jmath^*\omega)$ need not to hold for $\jmath : \mu^{-1}(\mathcal{O}_\alpha) \to M$ the natural embedding. Thus we can not apply directly Lemma 2.3.13 for this principal G-bundle to get the symplectic form on the reduction space $G \setminus \mu^{-1}(\mathcal{O}_\alpha)$.
4. Verify from Proposition 2.3.9 that for $Y \in T_\beta\mathcal{O}_\alpha$, $\xi \in \mathfrak{g}$,
$$(Y, \xi) = \omega^{\mathcal{O}_\alpha}(\xi_\beta^{\mathcal{O}_\alpha}, Y). \tag{2.3.100}$$
5. Let $\omega^\mu = \omega - \mu^*\omega^{\mathcal{O}_\alpha} \in \Omega^2(\mu^{-1}(\mathcal{O}_\alpha))$, verify that ω^μ is closed and
$$\omega^\mu(\xi^M, X) = 0 \quad \text{for } X \in T\mu^{-1}(\mathcal{O}_\alpha), \xi \in \mathfrak{g}.$$

6. Verify that
$$\ker(\omega_x^\mu) \subset (T_x\mu^{-1}(\alpha))^{\perp_\omega} = T_x(G\cdot x).$$
Conclude that there exists $\overline{\omega}^\mu \in \Omega^2(G\backslash\mu^{-1}(\mathcal{O}_\alpha))$ such that $\omega^\mu = \pi^*\overline{\omega}^\mu$. Verify that $\overline{\omega}^\mu = \overline{\omega}$ in Theorem 2.3.15.

Exercise 2.3.4. The nondegenerate bilinear form on $\mathfrak{u}(n)$ is defined by: for $\xi, \eta \in \mathfrak{u}(n)$
$$B_{\mathfrak{u}(n)}(\xi,\eta) = -\operatorname{Tr}_{\mathbb{C}^n}[\xi\,\eta]. \tag{2.3.101}$$

1. Verify that $\operatorname{Tr}_{\mathbb{C}^n}[\xi\,\eta] \in \mathbb{R}$ for $\xi,\eta \in \mathfrak{u}(n)$, and $B_{\mathfrak{u}(n)}$ is $\operatorname{Ad}_{\mathrm{U}(n)}$-invariant, i.e., $B_{\mathfrak{u}(n)}(\xi,\eta) = B_{\mathfrak{u}(n)}(\operatorname{Ad}_g\xi, \operatorname{Ad}_g\eta)$ for any $g \in \mathrm{U}(n)$.
2. By the identification (1.1.59), verify that with $B_{\mathfrak{sp}(2n)}$ in (2.3.35)
$$B_{\mathfrak{sp}(2n)}(\tau(\xi),\tau(\eta)) = B_{\mathfrak{u}(n)}(\xi,\eta). \tag{2.3.102}$$
3. Verify that (2.3.36) holds. Thus
$$\xi \in \mathfrak{u}(n) \to \beta = \{\eta \in \mathfrak{u}(n) \to \beta(\eta) = B_{\mathfrak{u}(n)}(\xi,\eta)\} \in \mathfrak{u}(n)^* \tag{2.3.103}$$
is an identification of $\mathfrak{u}(n)$ to $\mathfrak{u}(n)^*$ via the Euclidean metric $B_{\mathfrak{u}(n)}$ on $\mathfrak{u}(n)$.
4. Verify directly that (2.3.40) is a moment map for the natural $\mathrm{U}(n)$-action on $(\mathbb{C}^n, \omega_0)$.

Exercise 2.3.5. Let $M_{n,r}(\mathbb{K})$ be the vector space of $n\times r$ matrices over $\mathbb{K}$. We define the action of $\mathrm{U}(n)\times\mathrm{U}(r)$ on $M_{n,r}(\mathbb{C})$ by
$$(A,B)\cdot Z = AZB^{-1} \text{ for } A \in \mathrm{U}(n), B\in\mathrm{U}(r), Z \in M_{n,r}(\mathbb{C}). \tag{2.3.104}$$
By the identification $M_{n,r}(\mathbb{C}) \ni Z = (z_{jk}) \to (z_{jk}) \in \mathbb{C}^{nr}$, the canonical symplectic form on $M_{n,r}(\mathbb{C})$ is given by
$$\omega = \sum_{j=1}^n\sum_{k=1}^r \frac{\sqrt{-1}}{2} dz_{jk}\wedge d\overline{z}_{jk}.$$

1. Verify that under the identification (2.3.103), $\mu(Z) = -\frac{\sqrt{-1}}{2}(ZZ^*, -Z^*Z) \in \mathfrak{u}(n)\times\mathfrak{u}(r)$ is a moment map for the $\mathrm{U}(n)\times\mathrm{U}(r)$ action on $M_{n,r}(\mathbb{C})$.
2. Verify that the symplectic reduction at $\frac{\sqrt{-1}}{2}I_r$ for the $U(r)$-action is $\mathbb{G}(r,n)$ the Grassmannian of r-planes in $\mathbb{C}^n$.
3. Verify that $U(n)$ acts Hamiltonianly on $\mathbb{G}(r,n)$ with the symplectic form induced by ω on $M_{n,r}(\mathbb{C})$ and the associated moment map is given by
$$\big(\mu_{\mathrm{U}(n)}([Z]), A\big) = \frac{\sqrt{-1}}{2}\operatorname{Tr}_{\mathbb{C}^n}\big[Z(Z^*Z)^{-1}Z^*A\big]$$
for $A \in \mathfrak{u}(n)$, and $Z \in M_{n,r}(\mathbb{C})$ with $[Z]$ the homogenous coordinate of $\mathbb{G}(r,n)$.

2.4 Symplectic cuts

The Lie algebra Lie($\mathbb{S}^1$) is $\sqrt{-1}\mathbb{R}$ with the usual exponential map: for $\xi \in$ Lie($\mathbb{S}^1$), $e^\xi \in \mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$. Now we identify $\mathbb{R}$ with Lie($\mathbb{S}^1$) by the map $\theta \to \sqrt{-1}\theta$. Then under this identification, the scalar product on Lie($\mathbb{S}^1$) is the canonical scalar product on $\mathbb{R}$ defined by $\langle u, v\rangle = uv$ for $u, v \in \mathbb{R}$.

Let $\mathbb{S}^1$ act on $\mathbb{C}$ by multiplication and the canonical symplectic form on $\mathbb{C}$ is $\omega_{st}(z) = \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}$ for $z \in \mathbb{C}$. Then under our identification of $\mathbb{R}$ and Lie($\mathbb{S}^1$), the associated moment map is

$$\mu_{\mathbb{S}^1}(z) = -\frac{1}{2}|z|^2 \quad \text{for } z \in \mathbb{C}. \tag{2.4.1}$$

Let $\mathbb{S}^1$ act Hamiltonianly on a symplectic manifold (M, ω) with moment map $\mu : M \to \mathbb{R}$. Let $G = \mathbb{S}^1$ act on $M \times \mathbb{C}$ by

$$g \cdot (x, z) = (g \cdot x, gz) \quad \text{for } x \in M, z \in \mathbb{C}, g \in G = \mathbb{S}^1. \tag{2.4.2}$$

Then the moment map $\widetilde{\mu}$ for the $\mathbb{S}^1$-action on $(M \times \mathbb{C}, \widetilde{\omega} = \omega + \omega_{st})$ is

$$\widetilde{\mu}(x, z) = \mu(x) - \frac{1}{2}|z|^2 \quad \text{for } x \in M, z \in \mathbb{C}. \tag{2.4.3}$$

We assume that $\mathbb{S}^1$ acts freely on $\mu^{-1}(0)$. Then $\mathbb{S}^1$ acts freely on

$$\widetilde{\mu}^{-1}(0) = \{(x, z) \in M \times \mathbb{C} : \mu(x) = \frac{1}{2}|z|^2\}, \tag{2.4.4}$$

as we have a $\mathbb{S}^1$ -equivariant diffeomorphism

$$\psi : \mu^{-1}(\mathbb{R}^*_+) \times \mathbb{S}^1 \to \widetilde{\mu}^{-1}(0) \setminus \mu^{-1}(0) \times \{0\} =: \widetilde{M}_1, \quad (x, e^{i\theta}) \to (x, e^{i\theta}\sqrt{\mu(x)}). \tag{2.4.5}$$

Thus the reduction space $M_{\geq 0} := \mathbb{S}^1 \backslash \widetilde{\mu}^{-1}(0)$ is a symplectic manifold with symplectic form $\omega_{\geq 0}$ induced by $\widetilde{\omega}$, we call $M_{\geq 0}$ a symplectic cut of M with respect to the $\mathbb{S}^1$-action.

Theorem 2.4.1. *The manifold $(M_{\mathbb{S}^1}, \omega_{\mathbb{S}^1})$ imbeds in $M_{\geq 0}$ as a symplectic submanifold of codimension 2, and its complements is symplectic diffeomorphic to the open subset $\mu^{-1}(\mathbb{R}^\times_+)$ of M. And $M_{\geq 0}$ is the disjoint union of $M_{\mathbb{S}^1}$ and $\mu^{-1}(\mathbb{R}^\times_+)$.*

Proof. As the symplectic form $\omega_{\mathbb{S}^1}$ is the restriction of $\omega_{\geq 0}$ on $M_{\mathbb{S}^1}$, we get the first part of Theorem 2.4.1. Observe that $\psi^*\widetilde{\omega} = \omega|_{\mu^{-1}(\mathbb{R}^\times_+)}$, we get the second part of Theorem 2.4.1. □

Now let $H = \mathbb{S}^1$ acts on $M \times \mathbb{C}$ by

$$g \cdot (x, z) = (g \cdot x, z) \quad \text{for } x \in M, z \in \mathbb{C}, g \in H = \mathbb{S}^1. \tag{2.4.6}$$

Then the associated moment map is $\mu_H(x, z) = \mu(x)$.

As G, H-actions on $M \times \mathbb{C}$ commute, we get a $G \times H$ Hamiltonian action on $M \times \mathbb{C}$. By Proposition 2.3.16, H acts Hamiltonianly on $M_{\geq 0}$ with the moment map $\mu_{\geq 0}$ given by

$$\begin{aligned} \mu_{\geq 0}(y) = & 0 && \text{if } y \in M_{\mathbb{S}^1}, \\ & \mu(y) && \text{if } y \in \mu^{-1}(\mathbb{R}^\times_+). \end{aligned} \tag{2.4.7}$$

Finally the fixed point set $M^{\mathbb{S}^1}_{\geq 0}$ of H-action on $M_{\geq 0}$ is the union $(M^{\mathbb{S}^1} \cap \mu^{-1}(\mathbb{R}^\times_+)) \cup M_{\mathbb{S}^1}$, i.e;, in addition to the fixed point set that existed prior to cutting, there is one new fixed point set created by cutting: the reduced space $M_{\mathbb{S}^1}$.

In the same way, by using the symplectic form $\omega - \omega_{st}$ on $M \times \mathbb{C}$, we get the symplectic cut $M_{\leq 0} = M_{\mathbb{S}^1} \cup \mu^{-1}(]-\infty, 0[)$. The procedure of splitting M as two symplectic manifolds $M_{\geq 0}$ and $M_{\leq 0}$ is a symplectic analogue of degenerations of varieties in algebraic geometry.

2.5 Bibliographic notes

The reference [18] is again a nice reference for this chapter. Another reference is [32].

Theorem 2.1.25 is a result of E. Cartan [19] which is in fact at the origin of the notion of Lie algebra cohomology. Then Chevalley and Eilenberg [21] gave a systematic treatment of the methods by which topological questions concerning compact Lie groups may be reduced to algebraic questions concerning Lie algebras, in particular, they gave a complete proof of Theorem 2.1.25 which we followed here.

For basic material on cohomology groups with integer coefficient and homotopy groups, cf. [15], [45]. For Remark 2.1.30 on Weyl's unitary trick, cf. [33, §3.6] or [38]. Theorem 2.1.31 was established in Bott [12], and a classical reference is Milnor's book [46], we can also find a proof on $\pi_2(G) = 0$ in [16, Proposition 5.7.5]. For the Hurewicz's theorem, cf. [15, Theorem 17.21].

For compact Lie groups, we can use the book [16]. Knapp's books [38], [39] are very complete references for Lie groups.

The symplectic reduction was introduced in mathematics by Marsden and Weinstein [44], even it was used in mechanics.

Symplectic cuts were first introduced by Lerman [42] as a symplectic analogy of degeneration of varieties.