Soit (M, ω) variété compacte symplectique

(L, h^L, ∇^L) fibré en droites prequantifier

$$\begin{tikzcd} (L, h^L, \nabla^L) \arrow[d] \\ (M, \omega) \end{tikzcd}$$

$$\omega = \frac{\sqrt{-1}}{2\pi} R^L.$$

Soit G un groupe de Lie compacte connexe

$$\begin{tikzcd} & L \arrow[d] \\ G \arrow[ur, phantom, "\curvearrowright"] \arrow[r, phantom, "\curvearrowright"] & M \end{tikzcd}$$

On voudrait montrer $[Q, R] = 0$.

$$D^L : \Omega^{0, \overset{\text{pair}}{*}}(M, L) \longrightarrow \Omega^{0, \overset{\text{impair}}{*}}(M, L)$$

Prop : soit $Q \in \mathrm{End}(\Lambda(T^{*(0,1)}M))$ change la parité et auto-adjoint, G-invariant, alors

$$(D^L+Q)_+ : \Omega^{0,\mathrm{pair}}(M,L) \longrightarrow \Omega^{0,\mathrm{impair}}(M,L).$$

est un operateur de Fredholm, G-inv, et

$$\mathrm{Ind}(D^L_+) = \mathrm{Ind}(D^L+Q)_+ \in R(G)$$

Rp : $\mathrm{Ind}\, D^L_+ = \ker D^L_+ - \ker D^L_- \in R(G)$.

Dém : $t \in \mathbb{R} \longrightarrow D^L + tQ$ est une famille de l'op. elliptique d'ordr 1, G-inv. bien que $\ker(D^L+tQ)$ peut varier, mais

$$\ker(D^L+tQ)_+ - \underbrace{\ker(D^L+tQ)_-}_{=\operatorname{coker}(D^L+tQ)_+} \in R(G)$$

ne dep pas en t. //

μ l'application moment. $\mu : M \longrightarrow \underline{\mathfrak{g}}^*$

On fixe $\langle\ \rangle$ un produit scalaire G-inv sur $\mathfrak{g}$

$\rightsquigarrow \mu : M \longrightarrow \mathfrak{g}$. $\qquad \underset{\mathfrak{g}}{\langle K, v\rangle} = \underset{\mathfrak{g}^*}{(K^*, v)}$

$\mu(x) \in \mathfrak{g}$.

On def.

$$\mu^M(x) = \left(\mu(x)\right)^M_x \in \mathcal{C}^\infty(M, TM).$$

Soit V_i base ON de $(\mathfrak{g}, \langle\ \rangle)$, $\rightsquigarrow V^i$ base dual

$$\mu = \sum_i \mu_i(x) V^i \rightsquigarrow \mu = \sum_i \mu_i(x) V_i$$

$$\mu^M(x) = \sum_j \mu_j(x) V_j^M(x).$$

$$i_{V_j^M}\omega = d\langle \mu, V_i\rangle = d\mu_i.$$

$$i_{\mu^M}\omega = \sum_j \mu_j(x)\, i_{V_j^M}\omega = \sum_j \mu_j d\mu_j = \frac{1}{2} d|\mu|^2_{\mathfrak{g}^*}.$$

$2\mu^M$ est le champ de v. ham. de $\mathcal{H} = |\mu|^2$.

$$X_{\mathcal{H}} = 2\mu^M$$

Lemme : $X_{\mathcal{H}}$ est G-inv, $\mathcal{H}$ est une foncti

Lemme : $X_{\mathcal{H}} = 2\mu^M$ est G-inv, $\mathcal{H} = |\mu|^2$ est une fonction G-inv.

$$\{\mu = 0\} = \{x \in M,\ d\mathcal{H} = 0\} = \{X_{\mathcal{H}} = 0\}$$

Dem : $\forall g \in G$, $g^* \mathcal{H}(x) = \mathcal{H}(gx) = |\mu(gx)|^2 = |Ad_g^* \mu(x)|^2_{\mathfrak{g}^*}$

$= |\mu(x)|^2 = \mathcal{H}(x)$.

$\langle\ \rangle$ est Ad_g^*-inv $\quad i_{\mu^M}\omega = \frac{1}{2}\underbrace{d\mathcal{H}}_{G\text{-inv}} \Rightarrow \mu^M$ est G-inv!

Def : Deformation de Tian-Zhang (Inv Math 1998).

Pour $T \in \mathbb{R}$, $D_T^L = D^L + \sqrt{-1}\, T\, \underbrace{c(\mu^M)}_{\Omega^{0,\text{pair}} \to \Omega^{0,\text{impair}}} : \Omega^{0,*}(M, L)$ ↺.

$X = u + \bar{u} \in TM$, $u \in T^{1,0}M$,

$$c(X) = \sqrt{2}\,(u^* \wedge - i_{\bar{u}}).$$

$$\langle c(X) s_1, s_2 \rangle = \sqrt{2}\,\langle (u^* - i_{\bar{u}}) s_1, s_2 \rangle$$

$$= \sqrt{2}\,\langle s_1, (-u^* + i_{\bar{u}}) s_2 \rangle = \langle s_1, -c(X) s_2 \rangle.$$

$\rightsquigarrow c(X)^* = -c(X) \rightsquigarrow \sqrt{-1}\, T\, c(\mu^M)$ est auto-adj.

$\forall T \in \mathbb{R}$, $\mathrm{Ind}(D_+^L) = \mathrm{Ind}(D_{T,+}^L) \in R(G)$.

On va prendre $T \to \infty$, et montrer

$$\mathrm{Ind}(D_{T,+}^L)^G \longrightarrow \mathrm{Ind}(D^L)^G.$$

On localise la contribution de $\mathrm{Ind}(D_{T,+}^L)^G$ au vois de $\mu^{-1}(0)$.

<u>Problème</u> : $[D^L, c(\mu^M)]$ op diff d'ordre 1

Pour $\xi \in T^*M \simeq TM$

$$\sigma([D^L, c(\mu^M)])(\xi) = [\sigma(D^L)(\xi), c(\mu^M)]$$

$$= \underbrace{\sqrt{-1}\,[c(\xi), c(\mu^M)]}_{} = -2\sqrt{-1}\langle \xi, \mu^M \rangle \neq 0.$$

Rq : Deformation de Witten : Soit f une fonction de Morse sur X compacte. Pour $T>0$. (1982).

$$d_T = e^{-Tf} d e^{Tf}, \quad d_T^* = e^{Tf} d^* e^{-Tf}.$$

$$D_T = d_T + d_T^* = d + d^* + T \underbrace{(df\wedge + i_{df})}_{V}$$

$$D_T^2 = D^2 + T \underbrace{[D,V]}_{\text{ordre } 0} + T^2 |df|^2.$$

$\xrightarrow{T\to\infty}$ localise la cohomologie aux voisinages de $\{df=0\} = \mathrm{Crit}(f)$.

$\rightsquigarrow$ Inégalité de Morse.

$B_r = \{x \in \mathrm{Crit}(f),\ \mathrm{ind}(f)_x = r\}$, $\longrightarrow$ $\exists$ coord local. $f(y) = f(x) - \sum_{j=1}^{r} y_j^2 + \sum_{j=r+1}^{\dim X} y_j^2$.

Ineq de Morse

$\forall q$ (forte) $\displaystyle\sum_{j=0}^{q} (-1)^{q-j} \dim H^j(X,\mathbb{R}) \le \sum_{j=0}^{q} (-1)^{q-j} \# B_j$.

faible : $\dim H^q(X,\mathbb{R}) \le \# B_q$.

Th $\quad D_T^{L} = D^{L} + \sqrt{-1}\, T\, c(\mu^M)$ ✓

$$D_T^{L,2} = D^{L,2} + \frac{\sqrt{-1}}{2} T \sum_j c(e_j)\, c(\nabla^{TM}_{e_j}\mu^M) - \sqrt{-1}\, T \operatorname{Tr}\left[\nabla^{TM}\mu^M|_{T^{1,0}M}\right]$$
$$+ \frac{T}{2}\sum_{i=1}^{\dim G}\left(\sqrt{-1}\, c(JV_i^M)\, c(V_i^M) + |V_i^M|^2\right) + 4\pi T \mathcal{H}.$$
$$-2\sqrt{-1}\, T \sum_{i=1}^{\dim G} \mu_i L_{V_i} + T^2|\mu^M|^2.$$

Dém : $D_T^{L,2} = D^{L,2} + \sqrt{-1}\, T \sum_j \left(c(e_j)\, \nabla^{cl}_{e_j} c(\mu^M) + c(\mu^M)\, c(e_j)\nabla^{cl}_{e_j}\right) + T^2|\mu^M|^2$

$c(\nabla^{TM}_{e_j}\mu^M) + c(\mu^M)\nabla^{cl}_{e_j}$ $\qquad \langle \mu^M, e_j\rangle$

$D^L = \sum c(e_j)\nabla^{cl\otimes L}_{e_j}$.

$$= D^{L,2} + \sqrt{-1}\, T \sum_j c(e_j)\, c(\nabla^{TM}_{e_j}\mu^M) \quad (1)$$
$$-2\sqrt{-1}\, T\, \nabla^{cl}_{\mu^M} + T^2|\mu^M|^2.$$

$\mu^M = \mu_i V_i^M$.

Lemme si $V \in \mathfrak{g}$

$$\nabla^{cl}_{V^M} = L_V + 2\pi\sqrt{-1}\langle \mu, V\rangle + \frac{1}{4}\sum c(e_j)\, c(\nabla^{TM}_{e_j}\mu^M)$$
$$+ \frac{1}{2}\operatorname{Tr}\Big[\underbrace{\nabla^{TM}\mu^M|_{T^{1,0}M}}_{P^{T^{1,0}M}\nabla^{TM}\mu^M}\Big].$$

$$\nabla^{cl}_{\mu^M} = \sum_i \mu_i \nabla^{cl}_{V_i^M} = \sum_i \Big\{ \mu_i L_{V_i} + \mu_i 2\pi\sqrt{-1}\langle \mu, V_i\rangle \mu_i$$

$$+ \frac{1}{4}\sum_{j=1}^{2n} c(e_j)\, c(\nabla^{TM}_{e_j} V_i^M)\cdot \mu_i + \mu_i \frac{1}{2}\mathrm{Tr}\left[\nabla^{TM} V_i^M \big|_{T^{(1,0)}M}\right]$$

$$= \sum_i \mu_i L_{V_i} + \underbrace{2\pi\sqrt{-1}\,\mathcal{H}}_{\otimes_5} + \frac{1}{4}\sum_j c(e_j)\Big(\underbrace{c(\nabla^{TM}_{e_j}(\mu_i V_i^M))}_{\otimes_3} - c(V_i^M)\, e_j(\mu_i)\Big)$$

$$+ \frac{1}{2}\underbrace{\mathrm{Tr}\left[\nabla^{TM}(\mu_i V_i^M)\big|_{T^{(1,0)}M}\right]}_{\otimes_4} - \frac{1}{2}\mathrm{Tr}\left[V_i^M \overset{I_1}{\otimes} (d\mu_i)^{1,0}\right].$$

μ^M

$$d\mu_i = d\langle \mu, V_i\rangle = i_{V_i^M}\omega = \langle J V_i^M, \cdot\rangle \overset{I_2}{\rightsquigarrow} e_j(\mu_i) = \langle J V_i^M, e_j\rangle$$

$$I_1 = -\frac{1}{4} c(JV_i^M)\, c(V_i^M)$$

ON $T^{1,0}M$.

$$I_2 = -\frac{1}{2}\sum_j (d\mu_i, w_j)\langle V_i^M, \overline{w}_j\rangle = -\frac{1}{2}\langle JV_i^M, w_j\rangle\langle V_i^M, \overline{w}_j\rangle$$

$$= \frac{\sqrt{-1}}{2}\langle V_i^M, w_j\rangle\langle V_i^M, \overline{w}_j\rangle = \frac{\sqrt{-1}}{2}\frac{1}{2}|V_i^M|^2 = \frac{\sqrt{-1}}{4}|V_i^M|^2.$$

$$\rightsquigarrow D^{L,2}_T = D^{L,2} + \sqrt{-1}\,T\Big(\underbrace{1}_{\otimes}-\underbrace{\tfrac{2}{4}}_{\otimes_3}\Big)\sum_j c(e_j)\, c(\nabla^{TM}_{e_j}\mu^M) - 2\sqrt{-1}\,T(I_1+I_2)$$

$$-\sqrt{-1}\,T\,\underbrace{\mathrm{Tr}\left[\nabla^{TM}\mu^M \big|_{T^{1,0}M}\right]}_{\otimes_4} - \underbrace{2\sqrt{-1}\,T(2\pi\sqrt{-1})\mathcal{H}}_{\otimes_5}$$

$$-2\sqrt{-1}\,T\sum \mu_i L_{V_i} + T^2|\mu^M|^2. \qquad //$$

Lemme $L_V = \nabla^{d}_{V^M} - \frac{1}{4}\sum_j c(e_j) c(\nabla^{TM}_{e_j} V^M) - \frac{1}{2}\mathrm{Tr}[P^{T^{1,0}M}\nabla^{TM} V^M]$.

$V \in \mathfrak{g}$

Dem : w_j ON de $T^{1,0}M$, $\rightarrow e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j)$

$e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j)$.

Γ^{TX} form de con de ∇^{TM} p.r. à $\{e_j\}$.

on note $\overline{w}^I = \overline{w}^{i_1} \wedge \dots \wedge \overline{w}^{i_k}$, $I = \{i_1 < \dots < i_k\}$.

sous le repère $\{\overline{w}^I\}$ de $\Lambda(T^{*(0,1)}M)$ sur U carte locale.

$\nabla^{d}_{V^M}(f\overline{w}^I) = V^M(f)\overline{w}^I + \frac{1}{4}\langle \Gamma^{TM}(V^M) e_i, e_j\rangle c(e_i)c(e_j) + \frac{1}{2}\mathrm{Tr}[\Gamma^{TM}(V^M)|_{T^{1,0}M}]$

L_V preserve $T^{1,0}M$ et $T^{0,1}M$. $([L_V, J] = 0)$.

$L_V \overline{w}^i = ?$ $\quad (L_V\overline{w}^i, \overline{w}_j) = -(\overline{w}^i, L_V\overline{w}_j) = -\langle w_i, L_V \overline{w}_j\rangle_{g^{TM}\,\mathbb{C}\text{-bil}}$

$\underset{L_V g^{TM} = 0}{=} \langle L_V w_i, \overline{w}_j\rangle$. $\rightsquigarrow L_V\overline{w}^i = \sum_i \langle L_V w_i, \overline{w}_j\rangle \overline{w}^j$.

$\rightarrow L_V(f\overline{w}^I) = V^M(f)\overline{w}^I + \langle L_V w_i, \overline{w}_j\rangle \overline{w}^j i_{\overline{w}_i} f\overline{w}^I$.

$\rightarrow L_V$ est une section réel B de $\mathrm{End}(TM)$, anti-sym et commute à J. t.q

$$Bw_i = L_V w_i = \langle L_V w_i, \overline{w}_j\rangle w_j.$$

B-anti-sym $\langle L_V w_i, \overline{w}_j\rangle \overset{L_V g^{TM}=0}{=} L_V \underbrace{\langle w_i, \overline{w}_j\rangle}_{=0} - \langle w_i, L_V \overline{w}_j\rangle$

Pour $X \in \mathscr{C}^\infty(M, TM)$, on a

$$L_V X = L_{V^M} X = \nabla^{TM}_{V^M} X - \nabla^{TM}_{X} V^M.$$

$$\rightsquigarrow \nabla^{TM}_{V^M} = \nabla^{TM}_{\cdot} V^M + L_V$$

$$\rightsquigarrow \Gamma^{TM} = \nabla^{TM}_{\cdot} V^M + B \leftarrow \text{anti-sym, comm à } J.$$

Lemme si $A \in \mathrm{End}(TM)$ anti-sym, commute à J, alors

$$\frac14 \langle A e_i, e_j \rangle c(e_i) c(e_j) = \langle A w_i, \bar{w}_j \rangle \bar{w}^j \wedge i_{\bar{w}_i} - \frac12 \langle A w_j, \bar{w}_j \rangle$$

$$\rightarrow \frac14 \langle \Gamma^{TM}(V^M) e_i, e_j \rangle c(e_i) c(e_j) + \frac12 \mathrm{Tr}\left[\Gamma^{TM} |_{T^{1,0}M} \right]$$

$$= \frac14 \langle \nabla^{TM}_{e_i} V^M, e_j \rangle c(e_i) c(e_j) + \langle B w_i, \bar{w}_j \rangle \bar{w}^j \wedge i_{\bar{w}_i}$$

$$- \frac12 \langle B w_j, \bar{w}_j \rangle + \frac12 \langle \Gamma^{TM}(V^M) w_j, \bar{w}_j \rangle$$

$$= \langle L_V w_i, \bar{w}_j \rangle \bar{w}^j \wedge i_{w_i} + \frac14 \langle \nabla^{TM}_{e_i} V^M, e_j \rangle c(e_i) c(e_j)$$

$$+ \frac12 \langle (\nabla^{TM} V^M) w_j, \bar{w}_j \rangle.$$

Dem de Lemme $\quad h(e_k) c(e_k) = h(w_k) c(\bar{w}_k) + h(\bar{w}_k) c(w_k)$

$$\frac14 \langle A e_i, e_j \rangle c(e_i) c(e_j) = \frac14 \langle A \underset{\bar{w}_i}{\cancel{e_i}}, w_j \rangle c(\underset{w_i}{\cancel{e_i}}) c(\bar{w}_j) + \frac14 \langle A \underset{w_i}{\cancel{e_i}}, \bar{w}_j \rangle c(\underset{\bar{w}_i}{\cancel{e_i}}) c(w_j)$$

$$= \frac{1}{4}\langle A\bar{w}_j, w_i\rangle \underbrace{c(w_j)}_{\sqrt{2}\,\bar{w}^j}\,\underbrace{c(\bar{w}_i)}_{\sqrt{2}\, i_{\bar{w}_i}} + \frac{1}{4}\langle A w_i, \bar{w}_j\rangle c(\bar{w}_i)\, c(w_j).$$

$$= \frac{1}{2}\underbrace{\langle A\bar{w}_j, w_i\rangle}_{-\langle \bar{w}_j, A w_i\rangle} \bar{w}^j\, i_{\bar{w}_i} + \frac{1}{2}\langle A w_i, \bar{w}_j\rangle \underbrace{i_{\bar{w}_i}\bar{w}^j}_{\delta_{ij} - \bar{w}^j \wedge i_{\bar{w}_i}}.$$

$$= \langle A w_i, \bar{w}_j\rangle \bar{w}^j \wedge i_{\bar{w}_i} - \frac{1}{2}\langle A w_j, \bar{w}_j\rangle \quad //.$$

Lemme $\nabla^{TM} V^M \in \mathrm{End}(TM)$ est anti-sym!

Dem: $\forall X, Y, Z \in \mathcal{C}^\infty(M, TM)$

$$(L_X g^{TM})(Y,Z) = X\langle Y, Z\rangle - \langle [X,Y], Z\rangle - \langle Y, [X,Z]\rangle$$

$$= \cancel{\langle \nabla^{TM}_X Y, Z\rangle} + \cancel{\langle Y, \nabla^{TM}_X Z\rangle} - \langle \cancel{\nabla^{TM}_X Y} - \nabla^{TM}_Y X, Z\rangle$$
$$- \langle Y, \cancel{\nabla^{TM}_X Z} - \nabla^{TM}_Z X\rangle = \underline{\langle \nabla^{TM}_Y X, Z\rangle + \langle Y, \nabla^{TM}_Z X\rangle}$$

Mais $\underline{L_{V^M} g^{TM} = 0} \rightsquigarrow$. //.

Cor : Pour $p \in \mathbb{N}^*$, sur $\Omega^{0,*}(M, L^p)^G$

$$D_T^{L^p,2} = \overbrace{D^{L^p,2}}^{\geq c_0\|s\|_1^2 - C_1\|s\|_0^2} + T\Big(\underbrace{\tfrac{\sqrt{-1}}{2}\sum_j c(e_j)c(\nabla^{TM}_{e_j}\mu^M) - \sqrt{-1}\,\mathrm{Tr}[\nabla^{TM}\mu^M|_{T^{1,0}M}]}_{A_1}\Big)$$

$$+\frac{T}{2}\Big(\underbrace{\sum_{i=1}^{\dim G}\big(\sqrt{-1}\,c(JV_i^M)c(V_i^M) + |V_i^M|^2\big)}_{A_2}\Big) + 4\pi p T\mathcal{H} + T^2\underbrace{|\mu^M|^2}_{\geq 0 \text{ sur } M\setminus U \; > 0}$$

Dem : $\forall s \in \Omega^{0,*}(M, L^p)^G$, $\forall V \in \mathfrak{g}$, $L_V s = 0$ //

Def : $F_T^{L^p} = D_T^{L^p,2} + 2\sqrt{-1}\,T\sum_i \mu_i L_{V_i} : \Omega^{0,*}(M, L^p)^G$.

Rq : le coef de T dans $D_T^{L^p,2}$ est un op diff d'ord 1 on ne peut pas le contrôle, mais le coef de T dans $F_T^{L^p}$ est un tenseur, (d'ord 0 !), on va appli... localisation analytique de Bismut-Lebeau.

Th Soit U un vois ouvert de $\mu^{-1}(0)$, alors $\exists C>0$, $b>0$ t.q $\forall T \geq 1$, $\forall s \in \Omega^{0,*}(M, L^p)$, supp $s \subset M\setminus U$, on a

$$\mathrm{Re}\langle F_T^{L^p}s, s\rangle \geq C\big(\|s\|_1^2 + (T-b)\|s\|_0^2\big). \quad (*)$$

Dem : a) $\exists p_0 \in \mathbb{N}$, t.q $\forall p > p_0$, (*) vrai. C'est trivial

$$\langle D^{L^p,2} s, s\rangle = \|D^{L^p} s\|_{L^2}^2 \geq C_0 \|s\|_1^2 - C_p \|s\|_0^2 .$$

b) Si $\operatorname{supp} s \subset \{x, \mu^M(x) \neq 0\}$

$$\rightsquigarrow T^2 \langle |\mu^M|^2 s, s\rangle \geq C\, T^2 \|s\|_0^2 \xrightarrow[T \to \infty]{\text{domine}} \text{vrai}$$

c) $x_0 \in \underbrace{M \setminus U}_{\mu(x_0) \neq 0}$ et $x_0 \in \{x \in M, \mu^M(x_0) = 0\} = \operatorname{Crit}(\mathcal{H})$

$\in T^{1,0}M \quad \in T^{0,1}M$,

Lemme pour $W = w^{1,0} + w^{0,1} \in TM$,

$$\sqrt{-1}\, c(JW) c(W) = 4 w^{1,0*} \wedge i_{w^{0,1}} - \underbrace{|W|^2}$$

$\widehat{\Lambda}^{*(0,1)}M \quad (w^{1,0*}, u) = \langle w^{1,0}, u\rangle$

Dem : $\sqrt{-1}\, c(JW)c(W) = \sqrt{-1}\left(\sqrt{-1}\, c(w^{1,0}) - \sqrt{-1}\, c(w^{0,1})\right)\left(c(w^{1,0}) + c(w^{0,1})\right)$

$= -\sqrt{2}\left(w^{1,0*} + i_{w^{0,1}}\right)\sqrt{2}\left(w^{1,0*} - i_{w^{0,1}}\right) = -2\left(w^{1,0*}(-i_{w^{0,1}}) + i_{w^{0,1}} w^{1,0*}\right)$

$= 4 w^{1,0*} i_{w^{0,1}} - 2|w^{1,0}|^2 \quad |w^{0,1}|^2 - w^{1,0*} i_{w^{0,1}}$

$\rightsquigarrow A_2 = \mathbb{E}\left(\sqrt{-1}\, c(JV_i^M) c(V_i^M) + |V_i^M|^2\right) \geq 0$

Pour A_1 ? $\mathcal{H} = |\mu|^2$. $x_0 \in \mathrm{Crit}\,\mathcal{H}$.

Près de x_0, choisit carte locale $(y_1, \ldots, y_{2n})$ t.q $0 \rightsquigarrow x_0$,

et $f_j = \left\{ \frac{\partial}{\partial y_j} \right\}_{x_0}$ est une base ON de $(T_{x_0}M, g^{TM})$, et

$$\mathcal{H}(y) = \mathcal{H}(x_0) + \sum_{j=1}^{2n} a_j y_j^2 + O(|y|^3).$$

a_j peut 0.

$$D_T^L = D^L + \sqrt{-1}\, T\, c(\mu^M) \circlearrowleft \Omega^{0,*}(M,L)^G$$

$$\mathrm{Ind}(D_T^L)^G = \mathrm{Ind}(D^L)^G \in \underset{\mathbb{Z}}{\cancel{R(G)}} \quad \forall\, T \cancel{\in \mathbb{R}} \geq 0$$

$$\Big\downarrow T \to \infty$$

$$\mathrm{Ind}(D^{L_G})$$

<u>Théorème</u> soit U voisinage ouvert G-inv de $\mu^{-1}(0)$ alors $\exists\, C>0,\ b>0$ t.q $\forall\, T \geq 1$, $\forall\, s \in \Omega^{0,*}(M,L)$ supp $s \subset M \setminus U$, on a

$$\langle F_T^L s, s\rangle \geq C\left(\|s\|_1^2 + (T-b)\|s\|_0^2\right).$$

$$F_T^L = \underbrace{D_T^{L,2}} + 2\sqrt{-1}\, T \sum_i \mu_i L_{V_i}.$$

$\rightsquigarrow F_T^L$ le coeff de T dans F_T^L est d'ordre 0.

$$F_T^L = D_T^{L,2} \circlearrowleft \Omega^{0,*}(M,L)^G$$

$T \to \infty$ localisation analytique de Bismut-Lebeau

$$F_T^L = D^{L,2} + T\underbrace{\left(\frac{\sqrt{-1}}{2}\sum_j c(e_j)\, c\left(\nabla^{TM}_{e_j}\mu^M\right) - \sqrt{-1}\,\mathrm{Tr}\left[\nabla^{TM}\mu^M|_{T^{1,0}M}\right]\right)}_{A_1}$$

$$+\frac{T}{2}\sum_i\left(\underbrace{\sqrt{-1}\,c(JV_i^M)\,c(V_i^M) + |V_i^M|^2}_{A_2\geq 0}\right) + 4\pi T\mathcal{H} + \underline{T^2|\mu^M|^2}.$$

T^2

- Si $\mu^M(x_0)\neq 0$, pas problème
- $x_0\in M\backslash U$, $\mu^M(x_0)=0$, $\rightsquigarrow \mu(x_0)\neq 0$. $\rightsquigarrow \underline{\mathcal{H}(x_0)>0}$

$x_0\in \mathrm{Crit}(\mathcal{H})$ $\quad |\mu(x_0)|^2 = \mathcal{H}(x_0)$

Près de x_0, choisit une carte locale $(y_1,\cdots,y_{2n})$ t.q. $0\rightsquigarrow x_0$, $\{\underline{f_j = \left(\frac{\partial}{\partial y_j}\right)_{x_0}}\}$ est une base ON de $(T_{x_0}M, g^{TM})$

et
$$\mathcal{H}(y) = \mathcal{H}(x_0) + \sum_{j=1}^{2n} a_j y_j^2 + O(|y|^3).$$

"$a_j=0$ possible"

$A(y) = A(x_0) + O(|y|)$

Lemme : en x_0, $A_1(x_0)\geq -\sum_{j=1}^{2n}|a_j|$

equalité est strict si $\exists j$, t.q $a_j<0$.

Dem : $2\mu^M = X_{\mathcal{H}} = -J(d\mathcal{H})^* = -2a_j y_j \underbrace{J(dy_j)^*}_{Jf_j} + O(|y|^2)$

$$\mu^M = \frac{1}{2} X_{g\ell} = -\sum_j h_j(y)\, J f_j, \quad h_j(y) = a_j y_j + O(|y|^2).$$

en x_0, on a.

$$A_1(x_0) = -\frac{\sqrt{-1}}{2} \sum_j a_j\, c(f_j)\, c(J f_j) \qquad \sqrt{-1}\sum_j \langle p^{T^{1,0}M} \nabla^{TM}_{f_j} \mu^M, f_j \rangle$$

$$= -\sqrt{-1} \sum_j \langle \tfrac{1}{2}(1-\sqrt{-1}J)(-a_j J f_j), f_j \rangle$$

$$= \frac{1}{2} \sum_j a_j \left(\sqrt{-1}\, c(f_j)\, c(J f_j) - |f_j|^2 \right) \qquad \langle J f_j, f_j \rangle = 0$$

$$= \sum_j \left(2 a_j f_j^{1,0*} \wedge i_{f_j^{0,1}} - a_j \right) = -\sum_j 2 a_j\, i_{f_j^{0,1}} \wedge f_j^{1,0*} \geq -\sum_j |a_j|$$

$$D^{L,2} = \Delta^{d} + \frac{r^X}{4} + \frac{1}{2}\left(R^L + \frac{1}{2} R^{\det}\right)(e_i, e_j)\, c(e_i)\, c(e_j)$$

$$= \Delta^{d} + \cancel{O(1)}$$

$$\Delta^{d} = -\sum_j \left((\nabla^{d}_{e_j})^2 - \nabla^{d}_{\nabla^{TM}_{e_j} e_j} \right) \qquad |\mu^M|^2$$

Près de x_0.

$$F_T^L \geq \Delta^{d} - T \sum_{j=1}^{2n} |a_j| + 4\pi T\, \mathcal{H}(x_0) + T^2 h_j^2 + \cancel{O(|HT y|)}$$

$$A(y) = \cancel{\sum_{j=1}} + O(|y|)$$

Soit $(\nabla^{cl}_{f_j})^*$ l'adj de $\nabla^{cl}_{e_j}$ sur $U_{x_0} = B(x_0, \alpha)$.

$\forall s_1, s_2 \in \Omega^{0,*}_0(B(x_0,\alpha), L)$.

$$\langle \nabla^{cl}_{f_j} s_1, s_2 \rangle = \int_{U_{x_0}} \left(f_j \langle s_1, s_2 \rangle - \langle s_1, \nabla^{cl}_{f_j} s_2 \rangle \right) dv_X$$

$$= \int_{U_{x_0}} \left(\langle s_1, -\nabla^{cl}_{f_j} s_2 \rangle - \langle s_1, s_2 \rangle \frac{L_{f_j} dv_X}{dv_X} \right) dv_X + \int_{U_{x_0}} L_{f_j} (\langle s_1, s_2 \rangle dv) \quad \underset{0}{\parallel}$$

$$\rightsquigarrow (\nabla^{cl}_{f_j})^* = -\nabla^{cl}_{f_j} - \frac{L_{f_j} dv_X}{dv_X}.$$

On pose $\Delta_T = \sum_j \left((\nabla^{cl}_{f_j})^* + T(\mathrm{sgn}\, a_j) h_j \right) \left(\nabla^{cl}_{f_j} + T(\mathrm{sgn}\, a_j) h_j \right)$

$$\rightsquigarrow \langle \Delta_T s_1, s_1 \rangle = \sum_j \left\| \left(\nabla^{cl}_{e_j} + T(\mathrm{sgn}\, a_j) h_j \right) s_1 \right\|^2_{L^2} \geq 0.$$

$$\Delta_T = \Delta^{cl} - T \sum_j |a_j| + T \sum_j h_j^2 + O(\partial + 1 + T|y|)$$

∂ : op diff d'ordre 1

$= \sum b_j(y) \frac{\partial}{\partial y_j} + d(y)$.

$b_j(y) = O(1)$,

$d(y) = O(1 + T|y|)$.

$$\langle F^L_T s, s \rangle \geq \langle (\Delta_T + 4\pi T \delta(x_0) + O(\partial + 1 + T|y|)) s, s \rangle$$

$$\geq \frac{\Delta_T}{k} \geq \frac{\Delta^{cl}}{k} - \frac{T}{k} \sum_j |a_j| + \cancel{\frac{T}{k} \sum_j h_j^2} + \frac{O}{k}(\partial + 1 + T|y|)$$

$$\langle F_T^{\frac{1}{T}} s, s\rangle \geq \left\langle \left(\frac{1}{k}\Delta^{d} - \frac{T}{k}\sum_j |a_j| + 4\pi T\, \mathscr{H}(x_0) + O(\partial + 1 + T|y|)\right)s, s\right\rangle$$

$$\langle \Delta^{d} s, s\rangle = \|\nabla^{d} s\|_{L^2}^2 \geq \|s\|_1^2 - \|s\|_0^2.$$

Cauchy

$$|\langle O(\partial)s, s\rangle| \leq c_3 \alpha \|s\|_1^2 + \frac{c_4}{\alpha}\|s\|_0^2.$$

$$|\langle O(1+T|y|)s, s\rangle| \leq c_5(1+T\alpha)\|s\|_0^2. \qquad \|s\|_0^2$$

$$\rightsquigarrow \langle F_T^{\frac{1}{T}} s, s\rangle \geq \left(\frac{1}{k} - c_3\alpha\right)\|s\|_1^2 + T\left(4\pi \mathscr{H}(x_0) - c_5\alpha - \frac{1}{k}\sum_j |a_j|\right)\|s\|_0^2$$

$$- \left(\frac{c_4}{\alpha} + c_5\right)\|s\|_0^2.$$

Comme $\mu(x_0) \neq 0$. $\rightsquigarrow \mathscr{H}(x_0) > 0$. prend $k \gg 1$ t-q

$$2\pi \mathscr{H}(x_0) - \frac{1}{k}\sum_j |a_j| > 0$$

prend α assez petit

$$\pi \mathscr{H}(x_0) - c_5\alpha > 0.$$

$$\frac{1}{2k} - c_3\alpha > 0.$$

Donc $\exists \alpha > 0$ t-q $\forall s \in \Omega_0^{0,*}(B(x_0,\alpha), L)$, $\forall T \geq 1$, on a

$$\langle F_T^{\frac{1}{T}} s, s\rangle \geq c_{x_0}\|s\|_1^2 + (T\pi \mathscr{H}(x_0) - c'_{x_0})\|s\|_0^2.$$

Résumé : $\forall x_0 \in M \setminus U$, $\exists \alpha_{x_0} > 0$, $c_{x_0} > 0$, $b_{x_0} > 0$ t.q

$\forall s \in \Omega_0^{0,*}(B(x_0, \alpha_{x_0}), L)$, $\forall T \geq 1$, on a

$$\left[\langle F_T^L s, s \rangle \geq c_{x_0} \left(\|s\|_1^2 + (T - b_{x_0}) \|s\|_0^2 \right) \right.$$

Comme $M \setminus U$ compact. $\rightsquigarrow \exists \{x_j\}_{j \in I \text{ fini}}$ t.q $M \setminus U \subset \bigcup_j B(x_j, \alpha_{x_j})$

Soit $\varphi_j \in \mathcal{C}_0^\infty(B(x_j, \alpha_{x_j}))$ t.q $\sum_j \varphi_j^2 \equiv 1$ sur $M \setminus U$.

ajoute U, prend $\psi_j \in \mathcal{C}_0^\infty(B(x_j, \alpha_{x_j}))$, $\psi_0 \in C_0^\infty(U)$ t.q

$\psi_0 + \sum_j \psi_j \equiv 1$ $\rightsquigarrow$ $\varphi_j = \dfrac{\psi_j}{\psi_0^2 + \sum \psi_j^2}$

$\forall s \in \Omega_0^{0,*}(M \setminus U, L)$

$$\langle F_T^L s, s \rangle = \sum_j \langle F_T^L (\varphi_j^2 s), s \rangle$$

$$= \sum_j \langle (\varphi_j F_T^L + [F_T^L, \varphi_j])(\varphi_j s), s \rangle$$

$$= \sum_j \langle F_T^L(\varphi_j s), \varphi_j s \rangle + \langle \underbrace{[D^{L,2}, \varphi_j]}_{\text{op diff d'ord 1}} (\varphi_j s), s \rangle$$

pas

$$\geq C \sum_j \|\varphi_j s\|_1^2 + (T - b') \underbrace{\sum_j \|\varphi_j s\|_0^2}_{\|s\|_0^2}) - C' \|s\|_1 \|\varphi_j s\|_{L^2}$$

$$\sum_j \|\varphi_j s\|_1^2 = \sum_j \|\nabla^{L}(\varphi_j s)\|_{L^2}^2 + \underbrace{\|\varphi_j s\|_{L^2}^2}_{\|s\|_{L^2}^2} \geq \|\nabla^L s\|^2 - C\|s\|_0^2$$

$$\rightsquigarrow \langle F_T^{\frac{1}{}} s, s\rangle \geq C\left(\|s\|_1^2 + (T-b)\|s\|_0^2\right). \quad //.$$

Localisation? $Jg^M = N_{Y/M}$ $\quad Y = \mu^{-1}(0) \hookrightarrow M$.

sur $Y=\mu^{-1}(0)$, decomp ON $\quad g^M \xrightarrow{J} Jg^M$. $\Omega^{0,*}(M, L)$

$$TM|_Y = T^H Y \oplus g^M \oplus J g^M$$

N_J $\quad M_G = G\backslash Y$. $\quad \Omega^{0,*}(M_G, L_G)$

$$g^M_x = \{K^M_x \in T_x M,\ K \in g\}$$

$T^H Y$ J-inv. induit J_G str. presque complex sur M_G

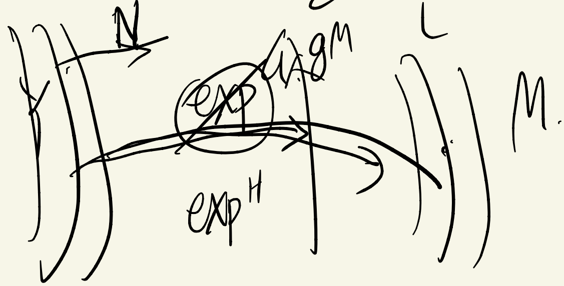
$$\rightsquigarrow \left(T^{H(1,0)} Y\right)_G \longrightarrow T^{1,0} M_G$$

$$\mathscr{C}^\infty_G(Y, \Lambda(T^{H*(0,1)} Y) \otimes L)^G \simeq \mathscr{C}^\infty(M_G, \Lambda(T^{*(0,1)} M_G) \otimes L)$$

D_G? $\quad D^{L_G}$ $\quad N_J \otimes \mathbb{C} = N_J^{1,0} \oplus N_J^{0,1}$.

$$\Lambda(T^{*(0,1)} M)|_Y = \Lambda(T^{H*(0,1)} Y) \otimes \Lambda(N_J^{*(0,1)})$$

Pour $\varepsilon > 0$; $B_\varepsilon = \{ Z \in N_{Y/M}, |Z| < \varepsilon \}$



Sur $U \supset \mu^{-1}(0)$, $T^H U = (\mathfrak{g}^M)^\perp$; $X = U/G$.

Soit $\nabla^{T^H U} = \pi^* \nabla^{TX}$ con sur $T^H U$. induit par ∇^{TX}. con L.C.

$\underline{\exp^H}$: $t \longrightarrow x_t = \exp^H_y(tW) \subset U$.

t.q. $x_t|_{t=0} = y$, $\frac{dx}{dt}|_{t=0} = W$, $\underline{\frac{dx}{dt} \in T^H U}$, et

$$\nabla^{T^H U}_{\frac{dx}{dx}} \frac{dx}{dx} = 0$$

le relèvement de géodésique $\overline{x}_t$ sur (X, g^{TX})

$$N_{Y/M} \supset B_\varepsilon \ni (y, Z) \longrightarrow \exp^H_y(Z) \in M.$$

identification G-equiv. de vois B_ε de Y dans N à $\mathcal{U}_\varepsilon$ vois de Y dans M.

Pour $x=(y,Z)\in \mathcal{U}_\varepsilon$, on identifie $(\Lambda(T^{*(0,1)}M)\otimes L)_x$ à $(\Lambda T^{*(0,1)}M \otimes L)_y$ par transp. parall. p.r. à $\nabla^{\mathcal{L}^*}$ le long $t\longrightarrow (y,tZ)$. $\pi_N : N\to Y$

$$(\Lambda(T^{*(0,1)}M)\otimes L)|_{\mathcal{U}_\varepsilon} = \pi_N^*\left(\left(\Lambda(T^{*(0,1)}M)\otimes L\right)|_Y\right).$$

<u>Def</u> : $\overset{\theta}{E}$ (resp $\mathbb{E}^\theta$, resp F^θ) l'esp de section G-inv de $\Lambda(T^{*(0,1)}M)\otimes L$ sur M, (resp π_N^* sur N ; resp $\Lambda(T^{*(0,1)}Y)\otimes L|_Y$ sur Y).

$$\mathbb{E}^0 : \langle s_1, s_2\rangle = \int_Y\left(\int_N \langle s_1, s_2\rangle(y,Z)\,dv_N(Z)\right)dv_Y(y).$$

$$F^0 \longleftrightarrow \Omega^{0,*}(M_G, L_G).$$

<u>Def</u> : $\forall T>0$, soit I_T :

$$\sigma\in F^0 \longrightarrow (I_T\sigma)(y,Z) = \alpha_T^{1/2}\,g(Z)\exp\left(-\frac{T}{2}|d\mu_y(Z)|^2\right)\sigma(y)$$

$\in \mathbb{E}^0$

$$|d\mu_y(Z)|^2 \geq C|Z|^2$$