Chapter 4

Localizations and Duistermaat-Heckman measures

Let (M, ω) be a compact symplectic manifold with a Hamiltonian torus action. In 1982, Duistermaat and Heckman discovered that the push-forward of the symplectic volume form on M under the associated moment map, so-called now the Duistermaat-Heckman (DH) measure, is locally polynomial with respect to the Lebesgue measure on the dual of its Lie algebra. If the fixed point sets of the torus action are isolated, they can evaluate explicitly the Fourier transformation of the DH measure by using the geometric data on the fixed point sets. Note that the classical stationary phase formula only gives an asymptotic formula. Thus all these interesting results were very surprising for many people and motivated many mathematicians to understand this formula deeply. Soon after, Berline and Vergne established a general localization formula which evaluates the integral of a differential form by using the geometric data on the fixed point sets for any smooth manifold, inspired by Bott's work on the localization of characteristic classes in 1967, and Atiyah and Bott developed a more topological approach. Witten applied formally the DH localization formula on loop spaces and deduced the famous Atiyah-Singer index theorem for pure Dirac operators. Atiyah explained these ideas to mathematicians in a conference in honor of Laurent Schwartz in 1983 at École Polytechnique, and it is exactly this point motivated Bismut to move from probability to geometry. All these evens are exciting even we look back 30 years later.

In this chapter, after introduced the theory on connections and curvatures for vector bundles in Section 4.1, we explain in detail the construction of equivariant Euler forms of real vector bundles. Then we present Bismut's proof of the Berline-Vergne (BV) localization formula, an advantage of this approach is that one sees how the equivariant Euler form appears geometrically in the localization formula. We explain also the theory behind the localization formula: equivariant cohomology. Then in Section 4.3, as an application, we deduce the DH localization formula from the BV localization formula, and establish that the DH measure is locally polynomial with respect the Lebesgue measure.

4.1 Connections and curvatures

In this section we introduce the theory on connections which is simply a way to do the differential calculus on manifolds. In different geometric situations, connections need to be compatible with extra structures, this motivates us to introduce Hermitian connections which are compatible with metrics, and Chern connections which are compatible with complex structures on manifolds and metrics on vector bundles.

4.1.1 Connection

Let E be a complex vector bundle over a smooth manifold M. Let $\mathscr{C}^\infty(M, E)$ be the space of smooth sections of E on M. Let $\Omega^q(M, E) := \mathscr{C}^\infty(M, \Lambda^q(T^*M) \otimes E)$ be the spaces of smooth q-forms on M with values in E.

A linear map $\nabla^E : \mathscr{C}^\infty(M, E) \to \mathscr{C}^\infty(M, T^*M \otimes E)$ is called a *connection* on E if for any $\varphi \in \mathscr{C}^\infty(M, \mathbb{C})$, $s \in \mathscr{C}^\infty(M, E)$ and $U \in TM$, we have

$$\nabla^E_U(\varphi s) := (\nabla^E(\varphi s))_U = U(\varphi)\, s + \varphi \nabla^E_U\, s\,. \tag{4.1.1}$$

Connections on E always exist. Indeed, let $\{V_k\}_k$ an open covering of M such that $E|_{V_k}$ is trivial. If $\{\eta_{kl}\}_l$ is a local frame of $E|_{V_k}$, any section $s \in \mathscr{C}^\infty(V_k, E)$ has the form $s = \sum_l s_l \eta_{kl}$ with uniquely determined $s_l \in \mathscr{C}^\infty(V_k, \mathbb{C})$. We define a connection on $E|_{V_k}$ by $\nabla^E_k s := \sum_l ds_l \otimes \eta_{kl}$. Consider now a partition of unity $\{\psi_k\}_k$ subordinated to $\{V_k\}_k$. Then $\nabla^E s := \sum_k \nabla^E_k(\psi_k s)$, for $s \in \mathscr{C}^\infty(M, E)$, defines a connection on E.

If ∇^E_1 is another connection on E, then by (4.1.1), for any $\varphi \in \mathscr{C}^\infty(M)$, $(\nabla^E_1 - \nabla^E)(\varphi s) = \varphi(\nabla^E_1 - \nabla^E)s$, thus

$$\nabla^E_1 - \nabla^E \in \Omega^1(M, \mathrm{End}(E)). \tag{4.1.2}$$

Inversely, if $A \in \Omega^1(M, \mathrm{End}(E))$, then $\nabla^E + A$ satisfies also (4.1.1), thus $\nabla^E + A$ defines a connection on E. Thus the space of connections on E is an infinite dimensional affine space $\nabla^E + \Omega^1(M, \mathrm{End}(E))$.

A connection ∇^E on E induces naturally a connection ∇^{E^*} on its dual vector bundle E^* by: for $s \in \mathscr{C}^\infty(M, E)$, $s' \in \mathscr{C}^\infty(M, E^*)$,

$$d(s', s) = (\nabla^{E^*} s', s) + (s', \nabla^E s)\,. \tag{4.1.3}$$

We verify that (4.1.1) holds for ∇^{E^*}, thus it is a connection on E^*.

If ∇^F is a connection on another vector bundle F on M, then ∇^E, ∇^F induce a natural connection $\nabla^{E\otimes F}$ on $E \otimes F$: for $s \in \mathscr{C}^\infty(M, E)$, $\sigma \in \mathscr{C}^\infty(M, F)$, $U \in TM$,

$$\nabla^{E\otimes F}_U(s \otimes \sigma) = \nabla^E_U s \otimes \sigma + s \otimes \nabla^F_U \sigma. \tag{4.1.4}$$

We denote it formally as

$$\nabla^{E\otimes F} = \nabla^E \otimes \mathrm{Id}_F + \mathrm{Id}_E \otimes \nabla^F. \tag{4.1.5}$$

In this way, a connection ∇^E on E induces naturally a connection on $\mathrm{End}(E)$, etc: for example, for $A \in \mathscr{C}^\infty(M, \mathrm{End}(E))$, $s \in \mathscr{C}^\infty(M, E)$, $U \in TM$, we have

$$(\nabla^{\mathrm{End}(E)}_U A)s = \nabla^E_U(As) - A(\nabla^E_U s). \tag{4.1.6}$$

For a smooth curve $\gamma : [0, 1] \to M$, the parallel transport $\tau_s(v) \in E_{\gamma(s)}$ of $v \in E_{\gamma(0)}$ along the curve γ with respect to a connection ∇^E is the solution of the equation

$$\nabla^E_{\dot\gamma} \tau_\cdot(v) = 0 \quad \text{and} \quad \tau_0(v) = v, \tag{4.1.7}$$

with $\dot\gamma(s) = \frac{\partial}{\partial s}\gamma(s) \in T_{\gamma(s)}M$. Then by the existence and uniqueness of the solutions of ordinary differential equations, we know $\tau_s : E_{\gamma(0)} \to E_{\gamma(s)}$ is a linear isomorphism and is $\mathscr{C}^\infty$ on s.

If ∇^E is a connection on E, then there exists a unique extension

$$\nabla^E : \Omega^\bullet(M,E) \to \Omega^{\bullet+1}(M,E)$$

verifying the Leibniz rule: for $\alpha \in \Omega^q(M)$, $s \in \Omega^p(M,E)$, we have

$$\nabla^E(\alpha \wedge s) = d\alpha \wedge s + (-1)^q \alpha \wedge \nabla^E s\,. \tag{4.1.8}$$

Lemma 4.1.1. *For $s \in \mathscr{C}^\infty(M,E)$ and vector fields U, V on M, $\varphi \in \mathscr{C}^\infty(M)$, we have*

$$(\nabla^E)^2(U,V)s = \nabla^E_U\nabla^E_V s - \nabla^E_V\nabla^E_U s - \nabla^E_{[U,V]} s\,, \tag{4.1.9a}$$

$$(\nabla^E)^2(U,V)(\varphi s) = (\nabla^E)^2(U,\varphi V)s = (\nabla^E)^2(\varphi U,V)s = \varphi(\nabla^E)^2(U,V)s. \tag{4.1.9b}$$

Proof. Let $\{\xi_l\}_l$ be a local frame of E on an open set W, then on W, we have

$$\nabla^E s = \sum_l \alpha_l \xi_l \quad \text{with } \alpha_l \in \Omega^1(W). \tag{4.1.10}$$

From (1.2.12), (4.1.8) and (4.1.10), we get $(\nabla^E)^2 s = \sum_l (d\alpha_l)\xi_l - \alpha_l \wedge \nabla^E \xi_l$, thus

$$\begin{aligned}(\nabla^E)^2(U,V)s &= (d\alpha_l)(U,V)\xi_l - \Big(\alpha_l(U)\nabla^E_V\xi_l - \alpha_l(V)\nabla^E_U\xi_l\Big)\\ &= U(\alpha_l(V))\xi_l - V(\alpha_l(U))\xi_l - \alpha_l([U,V])\xi_l - \alpha_l(U)\nabla^E_V\xi_l + \alpha_l(V)\nabla^E_U\xi_l\\ &= \nabla^E_U(\alpha_l(V)\xi_l) - \nabla^E_V(\alpha_l(U)\xi_l) - \nabla^E_{[U,V]}s.\end{aligned} \tag{4.1.11}$$

By (4.1.10) and (4.1.11), we get (4.1.9a).

By (1.2.12), (4.1.1) and (4.1.9a), we get directly (4.1.9b). The proof of Lemma 4.1.1 is completed. □

By (4.1.9b), for $x \in M$, $((\nabla^E)^2(U,V)s)_x \in E_x$ depends only on U_x, V_x, s_x, thus $(\nabla^E)^2_x \in \Lambda^2(T^*_xM) \otimes \mathrm{End}(E_x)$. By (1.2.15) and (4.1.8), for any $\alpha \in \Omega^q(M)$, $s \in \mathscr{C}^\infty(M,E)$, we get

$$(\nabla^E)^2(\alpha \wedge s) = \alpha \wedge (\nabla^E)^2 s. \tag{4.1.12}$$

Definition 4.1.2. The *curvature* of ∇^E is the tensor $R^E \in \Omega^2(M, \mathrm{End}(E))$ such that $(\nabla^E)^2$ is given by multiplication with R^E, i.e., $(\nabla^E)^2 s = R^E s \in \Omega^2(M,E)$ for $s \in \mathscr{C}^\infty(M,E)$.

From Definition 4.1.2 and (4.1.8), the Bianchi identity holds: as a 3-form with values in $\mathrm{End}(E)$, we have

$$[\nabla^E, R^E] = 0. \tag{4.1.13}$$

Let $\{\xi_l\}_{l=1}^r$ be a local frame of E. The *connection form* $\vartheta = (\vartheta^l_k)$ of ∇^E with respect to $\{\xi_l\}_{l=1}^r$ is defined by, with local 1-forms ϑ^l_k,

$$\nabla^E \xi_k = \vartheta^l_k \xi_l\,. \tag{4.1.14}$$

Thus under the trivialization of E by using the frame $\{\xi_l\}_{l=1}^r$, we have

$$\nabla^E = d + \vartheta, \quad \text{and } R^E = d\vartheta + \vartheta \wedge \vartheta, \tag{4.1.15}$$

here d is the usual differential acting on $\mathbb{C}^r$-valued functions, as $\nabla^E(f_k\xi_k) = df_k \otimes \xi_k + f_k\vartheta^l_k\xi_l$, by (4.1.14). Especially, if $\mathrm{rk}(E) = 1$, then R^E is a closed 2-form, as $\vartheta \wedge \vartheta = 0$ and $\mathrm{End}(E) = \mathbb{C}$.

Let h^E be a *Hermitian metric* on E, i.e., a smooth family $\{h^E_x\}_{x\in M}$ of sesquilinear maps $h^E_x : E_x \times E_x \to \mathbb{C}$ such that $h^E_x(\xi,\xi) > 0$ for any $\xi \in E_x \smallsetminus \{0\}$. We call (E, h^E) a Hermitian vector bundle on M, and we denote also $h^E(u,v)$ by $\langle u,v\rangle_{h^E}$ or $\langle u,v\rangle$. There always exist Hermitian metrics on E by using the partition of unity argument as above.

Definition 4.1.3. A connection ∇^E is said to be a *Hermitian connection* on (E, h^E) if ∇^E preserves h^E, i.e., for any $s_1, s_2 \in \mathscr{C}^\infty(M, E)$, $X \in \mathscr{C}^\infty(M, TM)$, we have

$$X\langle s_1, s_2\rangle_{h^E} = \langle \nabla^E_X s_1, s_2\rangle_{h^E} + \langle s_1, \nabla^E_X s_2\rangle_{h^E}. \tag{4.1.16}$$

There always exist Hermitian connections. In fact, let ∇^E_0 be a connection on E, then

$$\langle \nabla^E_1 s_1, s_2\rangle_{h^E} = d\langle s_1, s_2\rangle_{h^E} - \langle s_1, \nabla^E_0 s_2\rangle_{h^E} \tag{4.1.17}$$

defines the adjoint connection ∇^E_1 of ∇^E_0 on E. Now $\nabla^E = \frac{1}{2}(\nabla^E_0 + \nabla^E_1)$ is a Hermitian connection on (E, h^E).

Let ∇^E be a Hermitian connection on (E, h^E), and R^E be the associated curvature. Then by (4.1.9a) and (4.1.16), for any $U, V \in \mathscr{C}^\infty(M, TM)$, we have

$$\langle R^E(U, V)s_1, s_2\rangle_{h^E} + \langle s_1, R^E(U, V)s_2\rangle_{h^E} = 0. \tag{4.1.18}$$

Thus R^E is a 2-form with values in the skew-symmetric endomorphisms of E.

Remark 4.1.4. If E is a real vector bundle on M, certainly, all above discussion still holds, especially, h^E is a Euclidean metric on E if it is a smooth family $\{h^E_x\}_{x\in M}$ of bilinear forms $h^E_x : E_x \times E_x \to \mathbb{R}$ such that $h^E_x(\xi, \xi) > 0$ for any $\xi \in E_x \smallsetminus \{0\}$, and a connection ∇^E is said to be a Euclidean connection on the Euclidean vector bundle (E, h^E) if it preserves the Euclidean metric h^E. As in (4.1.18), the curvature R^E of a Euclidean connection ∇^E on (E, h^E) is a 2-form with values in the antisymmetric endomorphisms of E.

Let $\pi : N \to M$ be a smooth map between two manifolds, let E be a vector bundle on M with connection ∇^E, then the pull back vector bundle π^*E on N is defined by $\pi^*E = \cup_{y\in N} E_{\pi(y)}$ with the smooth structure induced by π and E. The pull back connection ∇^{π^*E} on π^*E over N is defined by: for $\varphi \in \mathscr{C}^\infty(N)$, $s \in \mathscr{C}^\infty(M, E)$, $U \in T_yN$, $y \in N$,

$$\nabla^{\pi^*E}_U(\varphi\pi^*s) = U(\varphi)\pi^*s(y) + \varphi(y)\pi^*(\nabla^E_{d\pi(U)}s), \tag{4.1.19}$$

and $(\pi^*s)(y) := s(\pi(y))$ defines a smooth section π^*s on π^*E over N. If h^E is a metric on E, then the pull-back metric π^*h^E on π^*E is defined by $\pi^*h^E_y(u, v) = h^E_{\pi(y)}(u, v)$.

Let $\pi : E \to M$ be a vector bundle equipped with a connection ∇^E. We apply (4.1.19) to the vector bundle π^*E over E, then we get a connection ∇^{π^*E} on π^*E over E.

For $x \in M$, the tangent bundle along the fiber (or vertical tangent bundle) of $\pi : E \to M$ is canonically identified with E_x, thus we will identify π^*E as the vertical tangent bundle of $\pi : E \to M$, which is a subbundle of TE. Let $v \in \mathscr{C}^\infty(E, \pi^*E)$ be the tautological section, i.e., $v(x, w) = (w, w) \in \pi^*E$. Then

$$\nabla^{\pi^*E} v \in \mathscr{C}^\infty(E, T^*E \otimes \pi^*E) = \mathscr{C}^\infty(E, \mathrm{Hom}(TE, \pi^*E)) . \tag{4.1.20}$$

For any $(x, w) \in E$, $u \in E_x$ as a vertical vector at (x, w) along the fiber E_x, by (4.1.19), we have

$$\nabla^{\pi^*E}_u v = u \in \pi^*E . \tag{4.1.21}$$

Thus for any $(x, w) \in E$, $\nabla^{\pi^*E}_{\cdot} v : T_{(x,w)}E \to \pi^*E_x$ is surjective.

Set

$$T^H E = \ker(\nabla^{\pi^*E} v) \subset TE . \tag{4.1.22}$$

As $\nabla^{\pi^*E}_{\cdot} v : T_{(x,w)}E \to \pi^*E_x$ is surjective, the rank of $T^H E$ is locally constant on E, thus $T^H E$ is a subbundle of TE and we call $T^H E$ the horizontal subbundle of TE induced by ∇^E. As vector bundles over E, we have

$$TE = T^H E \oplus \pi^*E . \tag{4.1.23}$$

As $d\pi : TE \to TM$ is surjective and $d\pi(\pi^* E) = 0$, we know $d\pi : T^H E \to TM$ is surjective. Since they have the same dimension, $d\pi : T^H_{(x,w)} E \to T_x M$ is an isomorphism. Thus for $Y \in T_x M$, there exists a unique $Y^H \in T^H_{(x,w)} E$ such that

$$d\pi(Y^H) = Y \ . \tag{4.1.24}$$

We call $Y^H \in T^H E$ the lifting of Y. Now it is clear on E,

$$TE = T^H E \oplus \pi^* E \simeq \pi^* TM \oplus \pi^* E \ . \tag{4.1.25}$$

Proposition 4.1.5. *For any $\sigma \in \mathscr{C}^\infty(E, \pi^* E)$, by considering σ as a vecotr field on E, we have*

$$\nabla^{\pi^* E}_{X^H} \sigma = [X^H, \sigma] \quad \text{for any } X \in \mathscr{C}^\infty(M, TM). \tag{4.1.26}$$

Proof. Let U be a local chart with trivialization $\psi : E|_U \simeq U \times E_0$. Then by (4.1.14), there exists $\vartheta \in \Omega^1(U, \mathrm{End}(E_0))$ such that

$$\nabla^E = d + \vartheta \ .$$

By (4.1.19), for $(x, u) \in U \times E_0$,

$$v_{(x,u)} = u \ , \quad \text{ and } (\nabla^{\pi^* E} v)_{(x,u)} = du + \vartheta u. \tag{4.1.27}$$

From (4.1.22) and (4.1.27), for $X \in T_x M$, we get

$$T^H_{(x,u)} E = \ker(\nabla^{\pi^* E} v) = \{(Y, -\vartheta(Y)u) : Y \in T_x M\} \ , \quad X^H_{(x,u)} = (X, -\vartheta(X)u). \tag{4.1.28}$$

By (4.1.28), for $s \in \mathscr{C}^\infty(M, E)$, $X \in \mathscr{C}^\infty(M, TM)$, by considering $\pi^* s$ as a vector field on E, we can compute the connection ∇^E by

$$\pi^*(\nabla^E_X s) = [X^H, \pi^* s]. \tag{4.1.29}$$

From (4.1.19) and (4.1.29), we get (4.1.26). □

If ∇^E is a Euclidean connection on (E, h^E), then by (4.1.19), $\nabla^{\pi^* E}$ is a Euclidean connection on $(\pi^* E, \pi^* h^E)$. By (4.1.22), for $Y \in TM$, we get

$$Y^H |v|^2_{\pi^* h^E} = \left\langle \nabla^{\pi^* E}_{Y^H} v, v \right\rangle + \left\langle v, \nabla^{\pi^* E}_{Y^H} v \right\rangle = 0. \tag{4.1.30}$$

As ∇^E is Euclidean, by (4.1.29), for $s_1, s_2 \in \mathscr{C}^\infty(M, E)$, $X \in \mathscr{C}^\infty(M, TM)$,

$$\begin{aligned}(L_{X^H} \pi^* h^E)(\pi^* s_1, \pi^* s_2) &= X^H \langle \pi^* s_1, \pi^* s_2 \rangle - \langle L_{X^H} \pi^* s_1, \pi^* s_2 \rangle - \langle \pi^* s_1, L_{X^H} \pi^* s_2 \rangle \\ &= \pi^* \left(X \langle s_1, s_2 \rangle - \left\langle \nabla^E_X s_1, s_2 \right\rangle - \left\langle s_1, \nabla^E_X s_2 \right\rangle \right) = 0. \end{aligned} \tag{4.1.31}$$

Thus

$$L_{X^H} \pi^* h^E = 0. \tag{4.1.32}$$

4.1.2 Chern connection

A complex vector bundle E on a complex manifold X is a holomorphic vector bundle if the transition functions $\psi_{ji} \in \mathscr{C}^\infty(U_i \cap U_j, M_r(\mathbb{C}))$ in Definition 1.2.5 are holomorphic.

Let E be a holomorphic vector bundle over a complex manifold X. Let h^E be a Hermitian metric on E. We call (E, h^E) a holomorphic Hermitian vector bundle.

The complex structure J induces a splitting $TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X$, where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$, respectively. Let $T^{*(1,0)}X$ and $T^{*(0,1)}X$ be the corresponding dual bundles. Let

$$\Omega^{p,q}(X,E) := \mathscr{C}^\infty(X, \Lambda^p(T^{*(1,0)}X) \otimes \Lambda^q(T^{*(0,1)}X) \otimes E)$$

be the spaces of smooth (p,q)-forms on X with values in E.

The operator $\bar\partial^E : \mathscr{C}^\infty(X,E) \to \Omega^{0,1}(X,E)$ is well-defined. Any section $s \in \mathscr{C}^\infty(X,E)$ has the local form $s = \sum_l \varphi_l \xi_l$ where $\{\xi_l\}_{l=1}^r$ is a local holomorphic frame of E and φ_l are smooth functions. In holomorphic coordinates $(z_1, \dots, z_n)$, we set

$$\bar\partial^E s = \sum_l (\bar\partial \varphi_l)\, \xi_l \quad \text{with } \bar\partial \varphi_l = \sum_j d\bar z_j \frac{\partial}{\partial \bar z_j} \varphi_l. \tag{4.1.33}$$

Definition 4.1.6. A connection ∇^E on E is said to be a *holomorphic connection* if $\nabla^E_U s = i_U(\bar\partial^E s)$ for any $U \in T^{(0,1)}X$ and $s \in \mathscr{C}^\infty(X,E)$.

Theorem 4.1.7. *There exists a unique* holomorphic Hermitian connection ∇^E *on a holomorphic Hermitian vector bundle* (E, h^E)*, called the* Chern connection *. With respect to a local holomorphic frame* $\{\xi_l\}$*, denote by* $h = (h_{lk} = \langle \xi_k, \xi_l \rangle_{h^E})$ *the matrix of* h^E *with respect to* $\{\xi_l\}_{l=1}^r$*, then the connection matrix is given by* $\vartheta = h^{-1} \cdot \partial h$.

Proof. By Definition 4.1.6, we have to define ∇^E_U just for $U \in T^{(1,0)}X$. Relation (4.1.16) implies for $U \in T^{(1,0)}X$, $s_1, s_2 \in \mathscr{C}^\infty(X,E)$,

$$U\langle s_1, s_2 \rangle_{h^E} = \langle \nabla^E_U s_1, s_2 \rangle_{h^E} + \langle s_1, \nabla^E_{\overline{U}} s_2 \rangle_{h^E} . \tag{4.1.34}$$

Since $\nabla^E_{\overline{U}} s_2 = i_{\overline{U}}(\bar\partial^E s_2)$, the above equation defines ∇^E_U uniquely. Moreover, if $\{\xi_l\}_{l=1}^r$ is a local holomorphic frame, from (4.1.16) we deduce that $\vartheta = h^{-1} \cdot \partial h$. □

Since E is holomorphic, similar to (4.1.8), the operator $\bar\partial^E$ extends naturally to $\bar\partial^E : \Omega^{\bullet,\bullet}(X,E) \to \Omega^{\bullet,\bullet+1}(X,E)$ verifying: for $\alpha \in \Omega^q(X)$, $s \in \Omega^{\bullet,\bullet}(X,E)$, we have

$$\bar\partial^E(\alpha \wedge s) = \bar\partial \alpha \wedge s + (-1)^q \alpha \wedge \bar\partial^E s. \tag{4.1.35}$$

Then from $\bar\partial^2 = 0$, we verify that $(\bar\partial^E)^2 = 0$.

Let ∇^E be the holomorphic Hermitian connection on (E, h^E). Then we have a decomposition of ∇^E according to bidegree

$$\begin{aligned} &\nabla^E = (\nabla^E)^{1,0} + (\nabla^E)^{0,1}, \\ &(\nabla^E)^{0,1} = \bar\partial^E : \Omega^{\bullet,\bullet}(X,E) \to \Omega^{\bullet,\bullet+1}(X,E), \\ &(\nabla^E)^{1,0} : \Omega^{\bullet,\bullet}(X,E) \longrightarrow \Omega^{\bullet+1,\bullet}(X,E). \end{aligned} \tag{4.1.36}$$

By (4.1.34), (4.1.36) and $(\bar\partial^E)^2 = 0$ we have

$$(\bar\partial^E)^2 = \big((\nabla^E)^{1,0}\big)^2 = 0, \quad (\nabla^E)^2 = \bar\partial^E (\nabla^E)^{1,0} + (\nabla^E)^{1,0} \bar\partial^E. \tag{4.1.37}$$

Combining with (4.1.18), the curvature $R^E \in \Omega^{1,1}(X, \mathrm{End}(E))$ is a $(1,1)$-form with values in the skew-symmetric endomorphisms of E. If $\mathrm{rk}(E) = 1$, $\mathrm{End}(E)$ is trivial and R^E is canonically identified to a $(1,1)$-form on X, such that $\sqrt{-1}R^E$ is real.

The complex $(\Omega^{\bullet,\bullet}(X,E), \overline{\partial}^E)$ is called the Dolbeault complex and its cohomology, called Dolbeault cohomology of X with values in E, is denoted by $H^{\bullet,\bullet}(X,E)$, i.e., for $p, q \in \mathbb{N}$,

$$H^{p,q}(X,E) := \frac{\ker(\overline{\partial}^E|_{\Omega^{p,q}(X,E)})}{\mathrm{Im}(\overline{\partial}^E|_{\Omega^{p,q-1}(X,E)})}. \tag{4.1.38}$$

Exercise 4.1.1. Let E, F be two $\mathbb{K}$-vector bundles on M and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Let $A : \mathscr{C}^\infty(M,E) \to \mathscr{C}^\infty(M,F)$ be a $\mathbb{K}$-linear map. Assume for any $\varphi \in \mathscr{C}^\infty(M)$, $s \in \mathscr{C}^\infty(M,E)$, $x \in M$, we have $(A(\varphi s))_x = \varphi(x)(As)_x$. Verify that

1. If $s|_U = 0$ on a neighborhood U of x_0, then $(As)_{x_0} = 0$.
2. If $s_{x_0} = 0$, then $(As)_{x_0} = 0$. (Hint: $s = \sum_l a_l \xi_l$ and $a(x_0) = 0$ for $\{\xi_l\}$ a local frame of E, we write $s = \sum_l (\psi a_l)(\psi \xi_l)$ as in (1.3.11) near x_0.)
3. For $u \in E_x$, taking $s \in \mathscr{C}^\infty(M,E)$ such that $s_x = u$ and we define $A_x u = (As)_x$. Then $A \in \mathscr{C}^\infty(M, \mathrm{Hom}(E,F))$ and for any $s \in \mathscr{C}^\infty(M,E)$, $x \in M$, $(As)_x = A_x s_x$.

Exercise 4.1.2. For ∇^E a connection on a Hermitian vector bundle (E, h^E) on M, we define for $s_1, s_2 \in \mathscr{C}^\infty(M,E)$,

$$(\nabla^E h^E)(s_1, s_2) := d\langle s_1, s_2\rangle - \langle \nabla^E s_1, s_2\rangle - \langle s_1, \nabla^E s_2\rangle.$$

1. Verify that for $f \in \mathscr{C}^\infty(M)$, $X \in \mathscr{C}^\infty(M, TM)$, we have
$$(\nabla^E h^E)_{fX}(s_1, s_2) = (\nabla^E h^E)_X(f\, s_1, s_2) = (\nabla^E h^E)_X(s_1, f\, s_2) = f(\nabla^E h^E)_X(s_1, s_2).$$
2. Conclude that $\nabla^E h^E \in \Omega^1(M, E^* \otimes \overline{E}^*)$, here $\overline{E}^*$ means the space of antilinear maps from E to $\mathbb{C}$. In particular, we define $(\nabla^E h^E)(s_1, s_2) = \langle A s_1, s_2\rangle$, then $A \in \Omega^1(M, \mathrm{End}(E))$. Verify that for $X \in T_x M$, $A(X) \in \mathrm{End}(E_x)$ is self-adjoint. We denote A by $(h^E)^{-1}\nabla^E h^E$.
3. Let ∇^{E*} be the adjoint connection of ∇^E with respect to h^E. Verify that
$$\nabla^{E*} = \nabla^E + (h^E)^{-1}\nabla^E h^E.$$

Exercise 4.1.3. In the context of (4.1.23), let P^V be the projection from TE onto $\pi^* E$ via the decomposition (4.1.23). For $X, Y \in \mathscr{C}^\infty(M, TM)$, set

$$T(X^H, Y^H) = -P^V[X^H, Y^H].$$

Verify that $T(X^H, Y^H)_{(x,u)}$ depends only on $X_x, Y_x \in T_x M$ and $u \in E_x$. Thus $T \in \mathscr{C}^\infty(E, \pi^*(\Lambda^2(T^*M) \otimes E))$. Prove that

$$T(X^H, Y^H)_{(x,u)} = R^E_x(X_x, Y_x)u \quad \text{i.e., } T_{(x,u)} = (\pi^* R^E_x)u.$$

Exercise 4.1.4. Let U be an open ball in $\mathbb{R}^m$ with center 0 and let E be a trivial vector bundle over U with connection ∇^E and curvature R^E. Let x_i be the coordinates on U and let ∂_i be the corresponding partial derivatives. Then the radial vector field on U is defined by

$$\mathcal{R} = \sum_i x_i \partial_i.$$

1. Verify that the parallel transport along the curve $t \to tZ$, $Z \in U$ with respect to ∇^E, gives a smooth trivialization of E on U. We denote this new trivialization by $\psi : E \to U \times E_0$ and by $\vartheta \in \Omega^1(U, \mathrm{End}(E_0))$ the associated connection form of (E, ∇^E).
2. Verify that $i_{\mathcal{R}}\vartheta = 0$. Conclude that $L_{\mathcal{R}}\vartheta = i_{\mathcal{R}}R^E$.
3. Verify that $L_{\mathcal{R}}x_i = x_i$ and $L_{\mathcal{R}}dx_i = dx_i$. Conclude that

$$\vartheta_x = \frac{1}{2}R_0^E(\mathcal{R}, \cdot) + \mathscr{O}(|x|^2).$$

Exercise 4.1.5. Let (M, ω) be a symplectic manifold. A connection ∇^M on TM is called a symplectic connection if it is torsion free and preserves the symplectic form ω. We will show that the space of symplectic connections is an infinite dimensional affine space.

1. Let $\varepsilon : T^*M \otimes \Lambda^\bullet(T^*M) \to \Lambda^{\bullet+1}(T^*M)$ be the exterior product map. Let ∇^0 be a torsion free connection on TM, we denote still by ∇^0 the induced connection on $\Lambda^\bullet(T^*M)$ via

$$\nabla^0_X(\alpha \wedge \beta) = (\nabla^0_X\alpha) \wedge \beta) + \alpha \wedge \nabla^0_X\beta \text{ for any } X \in \mathscr{C}^\infty(M, TM), \alpha, \beta \in \Omega(M).$$

   Prove that $d = \varepsilon \circ \nabla^0$.
2. For $X, Y, Z \in \mathscr{C}^\infty(M, TM)$, we define

$$(\nabla^0_X\omega)(Y, Z) = \omega(N(X, Y), Z).$$

   Prove that

$$\omega(N(X, Y), Z) + \omega(N(Y, Z), X) + \omega(N(Z, X), Y) = 0.$$

3. Prove that

$$\nabla_X Y = \nabla^0_X Y + \frac{1}{3}N(X, Y) + \frac{1}{3}N(Y, X)$$

   is a symplectic connection on TM.
4. Let ∇_1 be another connection on TM. Set $A = \nabla_1 - \nabla \in \Omega^1(M, \mathrm{End}(TM))$. Prove that ∇_1 is a symplectic connection if and only if the 3-tensor $\omega(A(\cdot)\cdot, \cdot)$ is symmetric.

4.2 Localizations and equivariant cohomology

In this section, we explain first the zero sets of a Killing vector field are totally geodesic submanifolds, then we introduce equivariant Euler classes which will appear in the main result of this section: the localization formula of Berline-Vergne, Theorem 4.2.18. Finally, we explain briefly the context of the localization formula: the equivariant cohomology.

4.2.1 Killing vector fields and Levi-Civita connections

Let M be a smooth manifold and g^{TM} be a Riemannian metric on M, as usual, we denote also g^{TM} by $\langle\,,\,\rangle$. Let ∇^{TM} be the Levi-Civita connection on (M, g^{TM}). Then the Levi-Civita connection ∇^{TM} is characterized by: for $X, Y, Z, W \in \mathscr{C}^\infty(M, TM)$,

$$\begin{aligned} \text{torsion free} : \nabla^{TM}_X Y - \nabla^{TM}_Y X &= [X, Y] \,, \\ \text{preserving metric} : \langle \nabla^{TM}_X Y, Z\rangle + \langle Y, \nabla^{TM}_X Z\rangle &= X\langle Y, Z\rangle \,. \end{aligned} \tag{4.2.1}$$