Chapter 5

Symplectic reductions and prequantizations

In classical Hamiltonian mechanic, the classical phase space is a symplectic manifold, and in quantum mechanic, the quantum phase space is a Hilbert space. In the mathematical theory of quantization, one attempts to associate to a symplectic manifold (X, ω) a Hilbert space H and a mapping from the space of functions on X into the space of operators on H, and this in a canonical way. The mapping should give some reasonable relationship between the Poisson bracket on the function side and the commutator on the operator side. It is generally acknowledged that there is no canonical way to construct a quantization of X without making use of certain additional structures. In the theory of the geometric quantization of Kostant and Souriau, (X, ω) is assumed to be prequantizable, that is, there exists a prequantum line bundle.

This chapter is organized as follows. In Section 5.1, we discuss first the problem on the existence of a prequantum line bundle on a symplectic manifold, and for a symplectic group action, when we can lift the action on the prequantum line bundle. An important fact is the Kostant formula which says that if the group action lifts on the prequantum line bundle, then automatically, it is a Hamiltonian action. In Section 5.2, we review some facts on maximal torus and roots of a compact Lie group. Then in Section 5.3, we discuss how the geometric objects can be inherited from the original manifold after the symplectic reduction, and we specify the Kähler situation in Section 5.4, in particular, the reduction space is still Kähler and the induced line bundle is holomorphic.

5.1 Prequantizations

5.1.1 Prequantized symplectic manifolds

Let M be a manifold. Let (L, h^L, ∇^L) be a Hermitian line bundle on M with Hermitian connection ∇^L and curvature $R^L = (\nabla^L)^2$. The first Chern form of (L, h^L, ∇^L) is defined by

$$c_1(L, \nabla^L) = \frac{\sqrt{-1}}{2\pi} R^L. \tag{5.1.1}$$

Proposition 5.1.1. *The first Chern form $c_1(L, \nabla^L)$ is a real closed 2-form. Its cohomology class $[c_1(L, \nabla^L)] \in H^2(M, \mathbb{R})$ does not depend on ∇^L, h^L, we denote it by $c_1(L)$ and call it as the first Chern class of L.*

Proof. As ∇^L is Hermitian, then by (4.1.18), $R^L \in \Omega^2(M, \mathrm{End}^{\mathrm{anti}}(L))$ is a 2-form with coefficient in $\mathrm{End}^{\mathrm{anti}}(L)$, the skew-symmetric endomorphism of L. Since $\mathrm{rk}(L) = 1$, we have an isomorphism

$$\mathrm{End}(L) = \mathbb{C}, \quad \mathrm{End}^{\mathrm{anti}}(L) = \sqrt{-1}\mathbb{R}. \tag{5.1.2}$$

Thus $c_1(L, \nabla^L) \in \Omega^2(M, \mathbb{R})$.

Let $\phi : L|_U \to U \times \mathbb{C}$ be a trivialization of L on an open set $U \subset M$. Then there exists $\alpha \in \Omega^1(U, \mathbb{C})$ such that

$$\nabla^L|_U = d + \alpha. \tag{5.1.3}$$

By (5.1.3), on U we have

$$R^L = d\alpha. \tag{5.1.4}$$

Thus $c_1(L, \nabla^L)$ is closed.

If ∇_1^L is another Hermitian connection (Hermitian with respect to another Hermitian metric on L), then there exists $\beta \in \Omega^1(M, \mathbb{C})$ such that

$$\nabla_1^L = \nabla^L + \beta. \tag{5.1.5}$$

By (5.1.5), we get

$$R_1^L = \left(\nabla_1^L\right)^2 = R^L + d\beta. \tag{5.1.6}$$

As $R_1^L, R^L \in \Omega^2(M, \sqrt{-1}\mathbb{R})$, by (5.1.6), we get $R_1^L = R^L + \sqrt{-1} d\,\mathrm{Im}(\beta)$. This means that the class $[c_1(L, \nabla^L)] \in H^2(M, \mathbb{R})$ does not depend on ∇^L, h^L. □

Remark 5.1.2. In fact, we can lift $c_1(L)$ as an element of $H^2(M, \mathbb{Z})$. By the sheaf theory, we can prove that the map c_1 from $\mathrm{Pic}(M)$, the group of isomorphism classes of complex line bundles on M, to $H^2(M, \mathbb{Z})$, is an isomorphism of groups. In particular, the integration of $c_1(L, \nabla^L)$ on any closed surface is an integer.

Definition 5.1.3 (Kostant, Souriau)**.** Let (M, ω) be a symplectic manifold, if there is a Hermitian line bundle (L, h^L, ∇^L) on M with Hermitian connection ∇^L such that

$$\omega = c_1(L, \nabla^L), \tag{5.1.7}$$

then we say that (M, ω) is prequantized by (L, h^L, ∇^L), and (L, h^L, ∇^L) is called a prequantum line bundle on (M, ω).

Let G be a compact Lie group acting on M and let E be a complex vector bundle on M.

We say that the G-action on M lifts on E if G acts smoothly on E, and for $x \in M$, $g \in G$, we have $g(E_x) = E_{gx}$ and the map $g : E_x \ni v \to g \cdot v \in E_{gx}$ is $\mathbb{C}$-linear.

Assume now the G-action on M lifts on E. We define the G-action on $\mathscr{C}^\infty(M, E)$ by: for $s \in \mathscr{C}^\infty(M, E)$, $g \in G$, $x \in M$,

$$(g \cdot s)(x) := g\left(s(g^{-1}x)\right). \tag{5.1.8}$$

We verify easily from (5.1.8) that $g_1 \cdot (g_2 \cdot s) = (g_1 g_2) \cdot s$ for any $g_1, g_2 \in G$, thus this defines a G-action on $\mathscr{C}^\infty(M, E)$. We say $s \in \mathscr{C}^\infty(M, E)$ is G-invariant if $g \cdot s = s$ for any $g \in G$.

For $g \in G$, a Hermitian metric h^E on E, we define another Hermitian metric $g \cdot h^E$ on E by: for $x \in M$, $u, v \in E_x$,

$$\left(g \cdot h^E\right)(u,v)_x := \left((g^{-1})^* h^E\right)(u,v)_x := h^E(g^{-1}u, g^{-1}v)_{g^{-1}x}. \tag{5.1.9}$$

Then $(g, h^E) \to g \cdot h^E$ defines a G-action on the space of Hermitian metrics on E, i.e., $g_1 \cdot (g_2 \cdot h^E) = (g_1 g_2) \cdot h^E$. We say h^E is G-invariant if $g \cdot h^E = h^E$ for any $g \in G$.

The G-action on the space of connections on E is defined as follows: For $g \in G$, a connection ∇^E on E, the connection $\nabla^{E,g}_X s := g \cdot \nabla^E$ is defined by: for $X \in \mathscr{C}^\infty(M, TM)$, $s \in \mathscr{C}^\infty(M, E)$,

$$\nabla^{E,g}_X s := (g \cdot \nabla^E)_X s := g \cdot \left(\nabla^E_{g_*^{-1}X}(g^{-1}s)\right). \tag{5.1.10}$$

We claim that $\nabla^{E,g}$ is a connection on E. In fact, for any $f \in \mathscr{C}^\infty(M)$, we have

$$\begin{aligned} \nabla^{E,g}_X(fs) &= g \cdot \left(\nabla^E_{g_*^{-1}X}(g^{-1}(fs))\right) = g \cdot \left(\nabla^E_{g_*^{-1}X}(g^{-1} \cdot f)(g^{-1}s)\right) = (Xf)s + f\nabla^{E,g}_X s, \\ \nabla^{E,g}_{fX}(s) &= g \cdot \left(\nabla^E_{g_*^{-1}(fX)}(g^{-1}(s))\right) = g \cdot \left((g^{-1} \cdot f)\nabla^E_{g_*^{-1}X}(g^{-1}s)\right) = f\nabla^{E,g}_X s. \end{aligned} \tag{5.1.11}$$

We call a connection ∇^E is G-invariant if $\nabla^{E,g} = \nabla^E$ for any $g \in G$.

Theorem 5.1.4. *Let (M, ω) be a symplectic manifold with a complex line bundle L such that*

$$c_1(L) = [\omega] \in H^2(M, \mathbb{R}). \tag{5.1.12}$$

Let G be a compact Lie group. We assume that G acts symplectically on M and that this action lifts on L. Then there is a G-invariant Hermitian metric h^L on L and a G-invariant Hermitian connection ∇^L on (L, h^L) such that

$$\omega = c_1(L, \nabla^L). \tag{5.1.13}$$

In particular, the existence of h^L, ∇^L on L such that (5.1.7) holds is equivalent to the topological condition (5.1.12).

A naturel question for a symplectic manifold (M, ω) is whether there exists a prequantum line bundle. By Remark 5.1.2 and Theorem 5.1.4, the obstruction is that the class $[\omega] \in H^2(M, \mathbb{R})$ can be lifted to $H^2(M, \mathbb{Z})$. The lifting from $\mathbb{R}$ to $\mathbb{Z}$ explains more or less the terminology of quantization.

Proof of Theorem 5.1.4. Let $d\mu$ be the Haar measure on G. Let h^L_0 be a Hermitian metric on L and ∇^L_0 be a Hermitian connection on (L, h^L_0). We define

$$h^L = \int_{g \in G} g \cdot h^L_0 \, d\mu(g), \quad \nabla^L_1 = \int_{g \in G} (g \cdot \nabla^L_0) \, d\mu(g). \tag{5.1.14}$$

Then h^L is a G-invariant Hermitian metric on L, as for $g_1 \in G$,

$$g_1 \cdot h^L = \int_{g \in G} L^*_{g_1}(g \cdot h^L_0 \; L^*_{g_1^{-1}} d\mu(g)) = \int_{g \in G} g \cdot h^L_0 \; L^*_{g_1^{-1}} d\mu(g) = h^L, \tag{5.1.15}$$

here as in (2.1.67), we understand $g \to g \cdot h^L_0$ as a function on G. As the total mass of $d\mu$ is 1, ∇^L_1 is a connection on L, and it is G-invariant as in (5.1.15). But in general, ∇^L_1 is not Hermitian with respect to h^L.

Let ∇_1^{L*} be the adjoint connection on L of ∇_1^L with respect to h^L. Then

$$\nabla_2^L = \frac{1}{2}(\nabla_1^L + \nabla_1^{L*}) \tag{5.1.16}$$

is a Hermitian connection with respect to h^L. Since h^L, ∇_1^L are G-invariant, by (4.1.17), ∇_1^{L*} and ∇_2^L are G-invariant.

By Proposition 5.1.1, there exists $\alpha \in \Omega^1(M, \mathbb{R})$ such that

$$\sqrt{-1}\left(\nabla_2^L\right)^2 + d\alpha = 2\pi\omega. \tag{5.1.17}$$

Set

$$\begin{aligned} \alpha_2 &= \int_{g\in G} (g^{-1})^* \alpha \, d\mu(g), \\ \nabla^L &= \nabla_2^L - \sqrt{-1}\alpha_2. \end{aligned} \tag{5.1.18}$$

Then as α_2 is a real 1-form, ∇^L is a G-invariant Hermitian connection on (L, h^L). As ∇_2^L and ω are G-invariant, by (5.1.17) and (5.1.18), we get

$$\sqrt{-1}(\nabla^L)^2 = \sqrt{-1}\left(\nabla_2^L\right)^2 + d\alpha_2 = 2\pi\omega. \tag{5.1.19}$$

Thus (5.1.13) holds. Now we apply (5.1.13) for $G = \{e\}$ the trivial group, we get the last part of Theorem 5.1.4. □

Example 5.1.5. Let X be a manifold. Let λ be the Liouville form defined by (1.2.42), and $\omega^{T^*X} = -d\lambda$. Then (T^*X, ω^{T^*X}) is a symplectic manifold. Let $L = \mathbb{C}$ be the trivial line bundle on T^*X with the trivial Hermitian metric h^L, i.e., $|\mathbf{1}|_{h^L}(x) = 1$ for the canonical section $\mathbf{1} \in \mathbb{C}$. Let ∇^L be the Hermitian connection on (L, h^L) defined by

$$\nabla^L = d + 2i\pi\lambda. \tag{5.1.20}$$

By (5.1.20), we get

$$R^L = 2i\pi d\lambda = -2i\pi\omega^{T^*X}. \tag{5.1.21}$$

Hence (T^*X, ω^{T^*X}) is prequantized by (L, h^L, ∇^L).

For $\varphi \in \mathrm{Diff}(X)$, we define $\varphi \cdot \mathbf{1} = \mathbf{1}$, then it defines the lifting of $\mathrm{Diff}(X)$ on L. As λ is $\mathrm{Diff}(X)$-invariant, we know the $\mathrm{Diff}(X)$–action on L preserves h^L, ∇^L.

In the sequel, we suppose that the symplectic manifold (M, ω) is prequantized by (L, h^L, ∇^L), and the compact Lie group G with Lie algebra $\mathfrak{g}$, acts on M symplectically, which lifts on L such that ω, h^L, ∇^L are G-invariant.

We recall that for $K \in \mathfrak{g}$, the vector field $K^M \in \mathscr{C}^\infty(M, TM)$ induced by K is given by, for $x \in M$

$$K_x^M = \frac{d}{dt}\Big|_{t=0} e^{tK} x. \tag{5.1.22}$$

The Lie derivation $L_K : \mathscr{C}^\infty(M, L) \to \mathscr{C}^\infty(M, L)$ is given by: for $s \in \mathscr{C}^\infty(M, L)$,

$$\left(L_K s\right)(x) = \frac{d}{dt}\Big|_{t=0} \left(e^{-tK} s\right)(x) = \frac{d}{dt}\Big|_{t=0} e^{-tK} \cdot s(e^{tK} x). \tag{5.1.23}$$

Theorem 5.1.6 (Kostant). *For $K \in \mathfrak{g}$, set*

$$2\sqrt{-1}\pi(\mu, K) = \nabla^L_{K^M} - L_K. \tag{5.1.24}$$

Then (μ, K) is a smooth function on M which is linear on K, thus it defines a smooth map $\mu : M \to \mathfrak{g}^*$. Finally μ is a moment map for the G-action on (M, ω).

Proof. By (4.1.1) and (5.1.23), for any $f \in \mathscr{C}^\infty(M)$, $s \in \mathscr{C}^\infty(M, L)$, we have

$$(\nabla^L_{K^M} - L_K)(fs) = f(\nabla^L_{K^M} - L_K)s. \tag{5.1.25}$$

Moreover, as $L_K, \nabla^L_{K^M}$ preserve h^L, we have

$$\nabla^L_{K^M} - L_K \in \mathscr{C}^\infty(M, \sqrt{-1}\mathbb{R}). \tag{5.1.26}$$

Finally both $\nabla^L_{K^M}$, L_K are linear on K, thus we get the first part of Theorem 5.1.6.

We verify now $\mu : M \to \mathfrak{g}^*$ is a moment map. By (5.1.10), ∇^L is G-invariant is equivalent to

$$g \cdot (\nabla^L_Y s) = \nabla^L_{g_* Y}(g \cdot s) \quad \text{for any } Y \in \mathscr{C}^\infty(M, TM), s \in \mathscr{C}^\infty(M, L), g \in G. \tag{5.1.27}$$

By taking $g = e^{-tK}$ for $K \in \mathfrak{g}$ and differential at $t = 0$ in (5.1.27), we get

$$L_K \nabla^L_Y s = \nabla^L_{L_{K^M} Y} s + \nabla^L_Y L_K s, \quad \text{i.e., } [L_K, \nabla^L] = 0. \tag{5.1.28}$$

By (5.1.24), (5.1.28) is equivalent to

$$\Big(- 2\sqrt{-1}\pi(\mu, K) + \nabla^L_{K^M}\Big)\nabla^L_Y s = \nabla^L_{L_{K^M} Y} s + \nabla^L_Y\Big(- 2\sqrt{-1}\pi(\mu, K) + \nabla^L_{K^M}\Big)s. \tag{5.1.29}$$

By (5.1.7), (5.1.29) is equivalent to

$$Y(\mu, K) = \frac{\sqrt{-1}}{2\pi} R^L(K^M, Y) = \omega(K^M, Y), \tag{5.1.30}$$

which is equivalent to $d(\mu, K) = i_{K^M}\omega$.

Note that by (2.2.2), for $x \in M$, $(\mathrm{Ad}_g K)^M_x = \big(g_* K^M\big)_x$. Thus from (5.1.23) and (5.1.27), we get $g \cdot (\nabla^L_{K^M} s) = \nabla^L_{(\mathrm{Ad}_g K)^M}(g \cdot s)$, and

$$g \cdot (L_K s)_x = \frac{d}{dt}\Big|_{t=0} g e^{-tK}(s(e^{tK} g^{-1} x)) = L_{\mathrm{Ad}_g K}(g \cdot s)_x. \tag{5.1.31}$$

Thus $2\sqrt{-1}\pi g \cdot ((\mu, K)s) = (\nabla^L_{(\mathrm{Ad}_g K)^M} - L_{\mathrm{Ad}_g K})(g \cdot s)$, combining with (5.1.24), we get

$$\Big(\mu(g^{-1}x), K\Big) = \Big(\mu(x), \mathrm{Ad}_g K\Big) = \Big(\mathrm{Ad}^*_{g^{-1}}\mu(x), K\Big). \tag{5.1.32}$$

By (5.1.30) and (5.1.32), μ is a moment map. The proof of Theorem 5.1.6 is completed. □

Now we turn to the problem when a symplectic action on (M, ω) can be lifted on L. By Theorem 5.1.6, such an action is always Hamiltonian. In fact, we shall prove that this is a sufficient condition provided G is compact, connected and simply connected.

Theorem 5.1.7. *Let (M,ω) be a connected compact symplectic manifold prequantized by (L, h^L, ∇^L). Let G be a compact, connected and simply connected Lie group. We suppose that there is a Hamiltonian G-action on M with moment map $\mu : M \to \mathfrak{g}^*$. Then the G-action can be lifted on L such that it preserves (h^L, ∇^L), and for $K \in \mathfrak{g}$, the Lie derivation L_K on $\mathscr{C}^\infty(M,L)$ is given by*

$$L_K = \nabla^L_{K^M} - 2\sqrt{-1}\pi(\mu, K). \tag{5.1.33}$$

Proof. Let J be a G-invariant, compatible almost complex structure on (M,ω). Then $g^{TM} = \omega(\cdot, J\cdot)$ is a G-invariant Riemannian metric on M.

Let H be a subgroup of the bundle maps of L consisting the elements which preserves (h^L, ∇^L) and which induces an isometry on (M, g^{TM}). We denote this natural map by $\phi : H \to \mathrm{Isom}(M)$, the isometry group of (M, g^{TM}). Then for $\sigma \in \ker\phi$, $\sigma : L_x \to L_x$ is an isometry for any $x \in M$. Hence, there is $a \in \mathscr{C}^\infty(M, \mathbb{S}^1)$ such that

$$\sigma : u \in L_x \to a(x)u \in L_x. \tag{5.1.34}$$

As σ preserves ∇^L, we have $da = [\nabla^L, \sigma] = 0$. As M is connected, a is a constant map. Hence we have an exact sequence

$$1 \longrightarrow \mathbb{S}^1 \longrightarrow H \xrightarrow{\ \phi\ } \mathrm{Isom}(M). \tag{5.1.35}$$

Since M is compact, $\mathrm{Isom}(M)$ is a compact Lie group. By (5.1.35), H is also a compact Lie group. We denote by H^0 the connected component of the identity in H, and $\mathfrak{h}$ its Lie algebra.

Let T^HL be the horizontal subbundle of TL induced by ∇^L (cf. (4.1.22)), and we identify M as a submanifold of L by sending $M \ni x \to (x, 0) \in L_x$. Let $\pi : L \to M$ be the natural projection.

We claim that $\mathfrak{h}$ is the space of vector fields on L which can be written as

$$X^L = X^H + 2\pi a(X) t^L_{\mathbb{R}}, \tag{5.1.36}$$

here $X^H \in T^HL$ is the lift of a Killing vector field X on M, and $t^L_{\mathbb{R}}$ is the real vector field on L induced by the rotation along the fiber $u \to e^{i\theta}u$. Note that

$$t^L_{\mathbb{R}}(x,u) = iu\frac{\partial}{\partial u} - i\overline{u}\frac{\partial}{\partial \overline{u}}. \tag{5.1.37}$$

By (4.1.28), for $s \in \mathscr{C}^\infty(M, L)$, we have

$$\pi^* L_X s = L_{X^L}\pi^* s = [X^L, \pi^* s], \quad \pi^*(\nabla^L_X s) = [X^H, \pi^* s]. \tag{5.1.38}$$

Thus the condition (5.1.36) is equivalent to

$$L_X = \nabla^L_X - 2\pi\sqrt{-1}a(X). \tag{5.1.39}$$

From (5.1.28)–(5.1.30) and (5.1.39), the condition (5.1.36) is equivalent that :

$$X \text{ is Killing, } a(X) \text{ is real and } da(X) = i_X\omega. \tag{5.1.40}$$

We verify now (5.1.36). At first, if $X^L \in \mathfrak{h}$, then $X = d\pi(X^L)$ defines a Killing vector field on M by the definition of H. As $H(M) \subset M$, we get $X = X^L|_M$. Thus there exists $b(x,u)$ such that

$$X^L - X^H = b(x,u)\frac{\partial}{\partial u} + \overline{b}(x,u)\frac{\partial}{\partial \overline{u}} \in \pi^* L, \quad \text{and } b(x,0) = 0. \tag{5.1.41}$$

As $L_X - \nabla^L_X$ is a skew-symmetric zero order differential operator acting on L, there is $f \in \mathscr{C}^\infty(M)$ such that for $s \in \mathscr{C}^\infty(M, L)$, we have

$$(L_X - \nabla^L_X)s = -\sqrt{-1} f s. \tag{5.1.42}$$

Note that as a vector field on L, $(\pi^* s)(x, u) = s(x)\frac{\partial}{\partial u}$. By (5.1.38) and (5.1.42), we have

$$-\sqrt{-1}\pi^*(fs) = [X^L - X^H, \pi^* s] = -s(x)\frac{\partial b}{\partial u}\frac{\partial}{\partial u} - s(x)\frac{\partial \overline{b}}{\partial u}\frac{\partial}{\partial \overline{u}}. \tag{5.1.43}$$

Thus $\frac{\partial b}{\partial u} = \sqrt{-1} f(x)$ and $\frac{\partial \overline{b}}{\partial u} = 0$, (5.1.41) and (5.1.43) imply that $b(x, u) = \sqrt{-1} f(x) u$.

Inversely, if (5.1.36) holds, then the flow ψ_t associated with X^L is a diffeomorphism of L verifying our condition, thus $\psi_t \in H$ for any $t \in \mathbb{R}$.

For any $K \in \mathfrak{g}$, as ∇^L is Hermitian, and (μ, K) is real, from (5.1.33), L_K preserves h^L. By (5.1.28)–(5.1.30), L_K preserves ∇^L. Moreover, K^M is a Killing vector field on (M, g^{TM}). Thus $L_K \in \mathfrak{h}$.

Now we claim that $K \in \mathfrak{g} \to -L_K \in \mathfrak{h}$ is a morphism of Lie algebras. In fact, by Proposition 2.2.3, (2.3.7), (2.3.8) and (5.1.33), we have

$$\begin{aligned} L_{[K_1,K_2]} = \nabla^L_{[K_1,K_2]^M} - 2i\pi(\mu, [K_1, K_2]) &= -\nabla^L_{[K_1^M, K_2^M]} - 2i\pi(\mu, [K_1, K_2]) \\ &= -[\nabla^L_{K_1^M}, \nabla^L_{K_2^M}] + R^L(K_1^M, K_2^M) - 2i\pi(\mu, [K_1, K_2]) \\ &= -[\nabla^L_{K_1^M}, \nabla^L_{K_2^M}] - 4i\pi(\mu, [K_1, K_2]) = -[L_{K_1^M}, L_{K_2^M}]. \end{aligned} \tag{5.1.44}$$

As G is connected, and simply connected, by the second theorem of Lie, Theorem 2.1.15, there exists a morphism of Lie groups $G \to H$ which is equivalent to the fact that the G-action on M can be lifted on L. The proof of Theorem 5.1.7 is completed. □

5.1.2 Prequantized Kähler manifolds

In the remainder of this section, we will explain the holomorphic version of Theorem 5.1.4. Let's start to explain the $\partial\overline{\partial}$-Lemma for compact Kähler mnifolds.

Lemma 5.1.8 ($\partial\overline{\partial}$-Lemma for $(1,1)$-forms)**.** *Let α be a smooth, d-exact, $(1,1)$-form on a compact Kähler manifold M, then there exists a smooth function ρ on M such that*

$$\alpha = \sqrt{-1}\partial\overline{\partial}\rho. \tag{5.1.45}$$

Moreover, if φ is real, then ρ is real.

Proof. Let $\overline{\partial}^*, \partial^*, d^*$ be the adjoint of $\overline{\partial}, \partial, d$ associated to the Kähler metric g^{TM}.

As α is d-exact, there is a 1-form β on M such that

$$\alpha = d\beta. \tag{5.1.46}$$

Let $\beta^{0,1}$ (resp. $\beta^{1,0}$) be the $(0,1)$ (resp. $(1,0)$)-component of β. As α is a $(1,1)$-form, we get

$$\varphi = \partial\psi^{0,1} + \overline{\partial}\psi^{1,0}, \quad \overline{\partial}\psi^{0,1} = 0, \quad \partial\psi^{1,0} = 0. \tag{5.1.47}$$

We claim that if θ is a $(0,1)$-form and $\overline{\partial}\theta = 0$, then there exists a function η such that

$$\partial\theta = \partial\overline{\partial}\eta. \tag{5.1.48}$$

By Hodge Theory, there exists a smooth function η such that

$$\overline{\partial}^*\theta = (\overline{\partial}^*\,\overline{\partial} + \overline{\partial}\,\overline{\partial}^*)\eta = \overline{\partial}^*\,\overline{\partial}\eta. \tag{5.1.49}$$

But from $\overline{\partial}(\theta - \overline{\partial}\eta) = \overline{\partial}\theta = 0$, we know (NEED to explain more)

$$\theta - \overline{\partial}\eta \in \ker(\overline{\partial}) \cap \ker(\overline{\partial}^*) = \ker(\Box) = \ker(\partial) \cap \ker(\partial^*). \tag{5.1.50}$$

Thus we get (5.1.48) for θ and η.

For $\beta^{1,0}$, we will apply (5.1.48) for $\overline{\beta^{1,0}}$. Thus there exists ρ such that (5.1.45) holds. If α is real, we can take ρ is real. $\Box$

Theorem 5.1.9. *Let F be a holomorphic line bundle on a connected compact Kähler manifold M. If Ω is a real, closed $(1,1)$-form on M with*

$$[\Omega] = c_1(F) \in H^2(M,\mathbb{R}), \tag{5.1.51}$$

then, up to multiplication by positive constants, there exists a unique Hermitian metric h^F on F such that $\Omega = \frac{\sqrt{-1}}{2\pi}R^F$, where R^F is the curvature associated to h^F.

If a Lie group G acts holomorphically on M such that Ω is G-invariant and that the G-action can be lifted on a holomorphic action on F, then the above h^F is G-invariant.

Proof. For any holomorphic local frame s of F on an open set U, we have

$$R^F(x) = \overline{\partial}\partial \log |s(x)|^2_{h^F} \quad \text{on } U. \tag{5.1.52}$$

Let h^F_0 be a Hermitian metric on F and let R^F_0 be the curvature associated to h^F_0. Then by (5.1.51), $\Omega - \frac{\sqrt{-1}}{2\pi}R^F_0$ is a real, d-exact, $(1,1)$-form on M. By Lemma 5.1.8, there exists a real function ρ on M such that

$$\Omega = \frac{\sqrt{-1}}{2\pi}R^F_0 + \frac{\sqrt{-1}}{2\pi}\overline{\partial}\partial\rho. \tag{5.1.53}$$

From (5.1.52) and (5.1.53), we know $-2\pi\sqrt{-1}\Omega$ is the curvature associated to the metric $e^\rho h^F_0$ on F.

Let h^F_1 be another metric on F such that $\Omega = \frac{\sqrt{-1}}{2\pi}R^F_1$. Then there is a real function ρ_1 such that $h^F_1 = e^{\rho_1}h^F$. By (5.1.52), we have

$$\overline{\partial}\partial\rho_1 = 0. \tag{5.1.54}$$

Taking the trace of both sides in (5.1.54) we get $\Delta\rho_1 = 0$ (NEED to explain more). Thus ρ_1 is a constant function on X.

We establish now the last part of Theorem. For any $g \in G$, $g \cdot \nabla^F$ is a Hermitian connection on (F, g^*h^F), we know that the g-action commutes with $\overline{\partial}^F$, thus $g \cdot \nabla^F$ is the Chern connection on (F, g^*h^F) and

$$\frac{\sqrt{-1}}{2\pi}(g \cdot \nabla^F)^2 = g^*\Omega = \Omega. \tag{5.1.55}$$

By the first part of Theorem and (5.1.55),

$$g^*h^F = \gamma(g)h^F \quad \text{with } \gamma(g) \in \mathbb{R}_{>0}, \tag{5.1.56}$$

and $\gamma(g)$ is smooth on g. Thus the image $\gamma(G)$ is a compact subgroup of $\mathbb{R}_{>0}$. But the only compact subgroup of $\mathbb{R}_{>0}$ is $\{1\}$. Thus $\gamma(G) = \{1\}$, i.e., h^F is G-invariant. $\Box$

As a corollary of Theorem 5.1.9, we get

Theorem 5.1.10. *(M, J, ω) be a connected compact Kähler manifold and let L be a holomorphic line bundle over M such that $c_1(L) = [\omega] \in H^2(M, \mathbb{R})$. If a Lie group G acts holomorphically and symplectically on M, and the G-action can be lifted on a holomorphic action on L, then up to multiplication by positive constants, there exists a unique G-invariant Hermitian metric h^L on L such that $\omega = \frac{\sqrt{-1}}{2\pi} R^L$, here R^L is the curvautre of the Chern connection on (L, h^L).*

Exercise 5.1.1. Let (L, h^L, ∇^L) be a prequantum line bundle on a symplectic manifold (M, ω). We define

$$Q : \mathscr{C}^\infty(M) \ni f \longrightarrow Q(f) = -\nabla^L_{X_f} + 2\pi i f \in \mathrm{End}(\mathscr{C}^\infty(M, L))$$

here X_f is the Hamiltonian vector field associated with f. Prove that the map Q is a morphism from the Poisson algebra $(\mathscr{C}^\infty(M), \{\quad\})$ of (M, ω) into the algebra of derivatives on $\mathrm{End}(\mathscr{C}^\infty(M, L))$, i.e.,

$$Q(\{f, g\}) = [Q(f), Q(g)].$$

5.2 Some facts on compact Lie groups

In this section, we review the structure theory of compact Lie groups, in particular, some facts on maximal torus and root system of a compact connected Lie group which will be used in Section 5.3.2 and Chapter 6.

5.2.1 Maximal torus

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. Let $Z(G)$ be the center of G. Let $\mathbb{T}$ be a maximal torus of G (i.e., if $\mathbb{T}' \subset G$ is a torus and $\mathbb{T} \subset \mathbb{T}'$, then $\mathbb{T} = \mathbb{T}'$), with Lie algebra $\mathfrak{t}$. Certainly, for any $g \in G$, $g\mathbb{T}g^{-1}$ is a maximal torus of G.

We explain here some facts on maximal torus and the structure theory of compact Lie groups without proofs.

Theorem 5.2.1. *The $\mathbb{T}$ is unique up to conjugation, i.e., if $\mathbb{T}'$ is another maximal torus then there exists $g \in G$ such that $\mathbb{T}' = g\mathbb{T}g^{-1}$. We have*

$$G = \bigcup_{g \in G} g\mathbb{T}g^{-1}, \quad Z(G) = \bigcap_{g \in G} g\mathbb{T}g^{-1}. \tag{5.2.1}$$

Let $N(\mathbb{T})$ be the normalizer of $\mathbb{T}$ in G, i.e.,

$$N(\mathbb{T}) = \{g \in G : g\mathbb{T}g^{-1} = \mathbb{T}\}. \tag{5.2.2}$$

Let $W = N(\mathbb{T})/\mathbb{T}$ be the Weyl group. By the maximality of $\mathbb{T}$, the Lie algebra of $N(\mathbb{T})$ coincides with $\mathfrak{t}$. Thus W is a finite group.

An interesting corollary of the conjugation theorem 5.2.1 is

Corollary 5.2.2. *[16, Lemma 4.2.5] Two elements of the maximal torus are conjugate in G if and only if they lie in the same orbit under the action of the Weyl group.*

Theorem 5.2.3. *If $f : G \to H$ is a surjective homomorphism of compact connected Lie groups. If $\mathbb{T} \subset G$ is a maximal torus, so is $f(\mathbb{T}) \subset H$. Furthermore* $\ker(f) \subset \mathbb{T}$ *if and only if* $\ker(f) \subset Z(G)$.

Theorem 5.2.4. *If $f : G \to H$ is a surjective homomorphism of compact connected Lie groups and $\dim G = \dim H$, then $\ker(f) \subset Z(G)$ and if $S \subset H$ is a maximal torus, then $f^{-1}(S) \subset G$ is a maximal torus. In particular, $f^{-1}(S)$ is connected and $G/f^{-1}(S) \simeq H/S$.*

Proof. For $k \in \ker(f)$, $g \in G$, we have

$$f(kgk^{-1}) = f(k)f(g)f(k^{-1}) = f(g). \tag{5.2.3}$$

But $f : G \to H$ is local diffeomorphism, thus on a neighborhood U of $e \in G$, we have $kgk^{-1} = g$ for any $g \in U$. Thus for any $g \in G$, we get

$$kgk^{-1} = g, \quad \textit{i.e.}, \ k \in Z(G). \tag{5.2.4}$$

Now assume that $S \subset H$ is a maximal torus with generating element x. Let $y \in f^{-1}(x)$. Then y is contained in some maximal torus T in G, and since $f(T)$ is compact abelian and connected, $f(T)$ is a torus in H. But $S \subset f(T)$ because $x \in f(T)$. Thus $S = f(T)$.

Since $\ker(f) \subset Z(G)$, by (5.2.1), $Z(G)$ is the intersection of all maximal tori in G, so $\ker(f) \subset T$. Thus $f^{-1}(S) \subset T$ (In fact, for any $z \in S$, there is $y_1 \in T$, such that $f(y_1) = z$, thus $f^{-1}(z) \subset T$ as $\ker(f) \subset T$). Thus $f^{-1}(S) = T$. The projection $f|_T : T \to S$ of maximal tori is also a covering map with the same group $K = \ker(f)$ of covering transformation.

Certainly, $f : G/f^{-1}(S) \to H/S$ is surjective, but if $f(y_1 f^{-1}(S)) = f(y_2 f^{-1}(S))$, then there exists $z \in S$ such that $f(y_1) = f(y_2)z$, i.e., $y_2^{-1} y_1 \in f^{-1}(z) \subset f^{-1}(S) = T$. Thus $f : G/f^{-1}(S) \to H/S$ is bijective. □

Theorem 5.2.5. *A compact connected Lie group G possesses a finite cover which is isomorphic to the direct product of a simply connected Lie group $\widetilde{G}$ and a torus S. In particular $\widetilde{G}$ is compact.*

Let $\mathbb{T}$ be a maximal torus of G. For the map $f : \widetilde{G} \times S \to G$, the maximal torus $f^{-1}(\mathbb{T})$ has the form $f^{-1}(\mathbb{T}) = \widetilde{\mathbb{T}} \times S$, and $\widetilde{\mathbb{T}}$ is a maximal torus of $\widetilde{G}$, and by Theorem 5.2.4, $\widetilde{G}/\widetilde{\mathbb{T}} \simeq G/\mathbb{T}$.

For same Lie algebra $\mathfrak{g}$, we can associate many connected Lie groups. However the following result says that for a compact connected semi-simple Lie group, its universal covering is compact and its fundamental group is finite.

Theorem 5.2.6. *If $\mathfrak{g}$ is semi-simple which is the Lie algebra of certain compact Lie group, then we can associate only one simply connected Lie group G which is compact and $\{G/Z : Z \subset Z(G)$ a subgroup$\}$ are all compact connected Lie groups associated with $\mathfrak{g}$.*

5.2.2 Roots and Weyl chambers

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. Let T be a maximum torus of G with Lie algebra $\mathfrak{t}$. Let $\mathfrak{g}^*, \mathfrak{t}^*$ be the dual of $\mathfrak{g}, \mathfrak{t}$.

Let $\Lambda \subset \mathfrak{t}$ be the integral lattice, and $\Lambda^* \subset \mathfrak{t}^*$ be the lattice of integral forms (or real weight lattice) defined by

$$\Lambda = \{a \in \mathfrak{t} : \exp(a) = 1\}, \qquad \Lambda^* = \{\beta \in \mathfrak{t}^* : \beta|_\Lambda \in 2\pi\mathbb{Z}\}. \tag{5.2.5}$$

Then $\mathbb{T} = \mathfrak{t}/\Lambda$.

For $\alpha \in \Lambda^*$, let $(\mathfrak{g}_\mathbb{C})_\alpha$ be the subspace of $\mathfrak{g}_\mathbb{C} := \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$, the complexification of $\mathfrak{g}$, defined by

$$(\mathfrak{g}_\mathbb{C})_\alpha = \{X \in \mathfrak{g} \otimes_\mathbb{R} \mathbb{C} : [H, X] = i\alpha(H)X \quad \text{for all } H \in \mathfrak{t}\}. \tag{5.2.6}$$

We call α a root, if $(\mathfrak{g}_\mathbb{C})_\alpha \neq 0$ and $\alpha \neq 0$. Let R be the set of roots, i.e.,

$$R = \{\alpha \in \Lambda^* \backslash \{0\} : (\mathfrak{g}_\mathbb{C})_\alpha \neq 0\}. \tag{5.2.7}$$

As for any $\alpha \in R$, $(\mathfrak{g}_{\mathbb{C}})_\alpha = \overline{(\mathfrak{g}_{\mathbb{C}})_\alpha}$, we get

$$\alpha \in R \text{ if and only if } -\alpha \in R. \tag{5.2.8}$$

We denote also $(\mathfrak{g}_{\mathbb{C}})_0 = \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$. Clearly, we have (cf. the argument in Corollary 3.3.1)

$$\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C} \oplus \bigoplus_{\alpha \in R} (\mathfrak{g}_{\mathbb{C}})_\alpha, \quad [(\mathfrak{g}_{\mathbb{C}})_\alpha, (\mathfrak{g}_{\mathbb{C}})_\beta] \subset (\mathfrak{g}_{\mathbb{C}})_{\alpha+\beta}. \tag{5.2.9}$$

We can reformulate (5.2.9) as

$$\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{r}, \quad \text{ with } \mathfrak{r} = [\mathfrak{t}, \mathfrak{g}] \text{ so that } \mathfrak{g}^* = \mathfrak{t}^* \oplus \mathfrak{r}^*. \tag{5.2.10}$$

Let B be a negative Ad_G-invariant bilinear form on $\mathfrak{g}$ which we extend $\mathbb{C}$-bilinearly on $\mathfrak{g}_{\mathbb{C}}$. If G is semi-simple, we can take B as the Killing form on $\mathfrak{g}$. We denote $\langle\, ,\, \rangle$ the scalar product on $\mathfrak{g}$ and $\mathfrak{g}^*$ induced by $-B$. Let $\kappa : \mathfrak{g} \to \mathfrak{g}^*$ be the isomorphism defined by

$$\kappa : \mathfrak{g} \to \mathfrak{g}^*, \quad x \to -B(x, \cdot) = \langle x, \cdot \rangle. \tag{5.2.11}$$

As $\mathfrak{t}$, $\mathfrak{r}$ are orthogonal with respect to $\langle\, ,\, \rangle$, we know

$$\kappa(\mathfrak{t}) = \mathfrak{t}^*, \quad \kappa(\mathfrak{r}) = \mathfrak{r}^*. \tag{5.2.12}$$

For $\alpha \in R$, set

$$\alpha^* = 2\kappa^{-1}\alpha / \langle \alpha, \alpha \rangle \in \mathfrak{t}. \tag{5.2.13}$$

Then $\{\alpha^*\}_{\alpha \in R}$ are called inverse roots.

Theorem 5.2.7. *For any $\alpha \in R$, $(\mathfrak{g}_{\mathbb{C}})_\alpha$ is an one dimensional complex vector space, and $[(\mathfrak{g}_{\mathbb{C}})_\alpha, (\mathfrak{g}_{\mathbb{C}})_{-\alpha}] \subset \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$ is one dimensional, and there exist unique $H_\alpha \in [(\mathfrak{g}_{\mathbb{C}})_\alpha, (\mathfrak{g}_{\mathbb{C}})_{-\alpha}] \cap i\mathfrak{t}$ and $X_\alpha \in (\mathfrak{g}_{\mathbb{C}})_\alpha$ such that*

$$(i\alpha, H_\alpha) = 2, \quad -2B(\overline{X}_\alpha, X_\alpha) = B(H_\alpha, H_\alpha) = \frac{4}{\langle \alpha, \alpha \rangle} > 0. \tag{5.2.14}$$

Finally

$$H_\alpha = [\overline{X}_\alpha, X_\alpha] = -i\,\alpha^*. \tag{5.2.15}$$

Proof. It is standard that $(\mathfrak{g}_{\mathbb{C}})_\alpha$ is an one dimensional complex vector space.

Take $0 \neq X_\alpha \in (\mathfrak{g}_{\mathbb{C}})_\alpha$, then $\overline{X}_\alpha \in (\mathfrak{g}_{\mathbb{C}})_{-\alpha}$ and $[\overline{X}_\alpha, X_\alpha] \in i\mathfrak{t} \subset \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$. For $H \in \mathfrak{t}$, by (5.2.6), we have

$$B([\overline{X}_\alpha, X_\alpha], H) = B(\overline{X}_\alpha, [X_\alpha, H]) = -B(\overline{X}_\alpha, X_\alpha)\,(i\alpha, H)\,. \tag{5.2.16}$$

From $-B(\overline{X}_\alpha, X_\alpha) > 0$ and (5.2.16), we know that $[\overline{X}_\alpha, X_\alpha]$ is non-zero, thus $[(\mathfrak{g}_{\mathbb{C}})_\alpha, (\mathfrak{g}_{\mathbb{C}})_{-\alpha}] \subset \mathfrak{t} \otimes_{\mathbb{R}} \mathbb{C}$ is one dimensional.

From (5.2.16), $(\alpha, [\overline{X}_\alpha, X_\alpha]) \neq 0$. Let $H_\alpha \in [(\mathfrak{g}_{\mathbb{C}})_\alpha, (\mathfrak{g}_{\mathbb{C}})_{-\alpha}]$ such that $(i\alpha, H_\alpha) = 2$. Then $H_\alpha \in i\mathfrak{t}$. We normalize X_α such that $-2B(\overline{X}_\alpha, X_\alpha) = B(H_\alpha, H_\alpha) > 0$.

By taking $H = H_\alpha$ in (5.2.16), we know that the first equality of (5.2.15) holds.

From (5.2.14), the first equality of (5.2.15), and (5.2.16), we get

$$B(H_\alpha, H) = \frac{1}{2}B(H_\alpha, H_\alpha)\,(i\alpha, H) = -\frac{1}{2}B(H_\alpha, H_\alpha)B(\kappa^{-1}(i\alpha), H). \tag{5.2.17}$$

Then from (5.2.17), we know

$$H_\alpha = -\frac{i}{2}B(H_\alpha, H_\alpha)\kappa^{-1}\alpha. \tag{5.2.18}$$

Finally, by the first equation of (5.2.14) and (5.2.18), we get

$$\langle \alpha, \alpha \rangle = -B(\kappa^{-1}\alpha, \kappa^{-1}\alpha) = (\alpha, \kappa^{-1}\alpha) = \frac{2(i\alpha, H_\alpha)}{B(H_\alpha, H_\alpha)} = \frac{4}{B(H_\alpha, H_\alpha)}. \tag{5.2.19}$$

From (5.2.13), (5.2.18) and (5.2.19), we get the second equality of (5.2.15). □

For each $\alpha \in R$, let $\mathcal{H}_\alpha$ be the hyperplane orthogonal to α. The hyperplanes $\{\mathcal{H}_\alpha\}_{\alpha \in R}$ devide $\mathfrak{t}^*$ into finitely many convex regions, these are called the Weyl chambers of the root system. A subset $S \subset R$ is called a system of simple roots if S is linearly independent and every root $\beta \in R$ may be written as

$$\beta = \sum_{\alpha \in S} m_\alpha \alpha, \tag{5.2.20}$$

with integers m_α such that either all $m_\alpha \geq 0$ or all $m_\alpha \leq 0$. The elements of S are called simple roots with respect to S. The associated set R_+ of positive roots consists of those roots β whose coefficients in (5.2.20) are all nonnegative.

In general, an element in a subset $R' \subset R$ is called decomposable in R' if it can be expressed as the sum of at least two elements of R'. Otherwise, it is called indecomposable.

Theorem 5.2.8. *[16, Theorem 5.4.5] For every Weyl chamber A, we assign the set S as the indecomposable elements in*

$$R_+(A) = \{\alpha \in R : \langle \alpha, u \rangle > 0 \quad \textit{for all } u \in A\}. \tag{5.2.21}$$

This map defines a bijection between the set of Weyl chambers and the set of system of simple roots.

By choosing a system of simple roots $S \subset R$, let R_+ be the set of positive roots (then $R = R_+ \cup (-R_+)$ and $R_+ \cap (-R_+) = \emptyset$). The associated positive Weyl chamber $\mathfrak{t}^*_+ \subset \mathfrak{t}^*$ is defined as

$$\mathfrak{t}^*_+ := \{\beta \in \mathfrak{t}^* : (\beta, iH_\alpha) = (\beta, \alpha^*) > 0 \text{ for any } \alpha \in R_+\}. \tag{5.2.22}$$

Then for $\gamma \in \mathfrak{t}^*_+$, we have $G_\gamma = \mathbb{T}$. Note that if $R = \emptyset$, then $\mathfrak{t}^*_+ = \mathfrak{t}^*$. By (5.2.9), as real vector spaces, we have

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha \in R_+} (\mathfrak{g}_\mathbb{C})_\alpha. \tag{5.2.23}$$

Let $\Gamma = \sum_{\alpha \in R} 2\pi\mathbb{Z}\alpha^*$ be the free abelian group generated by the inverse roots. Set

$$\varrho = \frac{1}{2} \sum_{\alpha \in R_+} \alpha. \tag{5.2.24}$$

Theorem 5.2.9. *The Weyl group W acts on $\mathfrak{t}$ by (5.2.2), and*

$$w \cdot \mathfrak{t}^*_+ \cap w' \cdot \mathfrak{t}^*_+ = \emptyset \text{ if } w, w' \in W, w \neq w', \text{ and } \mathfrak{t}^* = \cup_{w \in W} w \cdot \overline{\mathfrak{t}^*_+}. \tag{5.2.25}$$

For every $\gamma \in \overline{\mathfrak{t}^*_+}$, $W \cdot \gamma \cap \overline{\mathfrak{t}^*_+} = \{\gamma\}$. We have

a) $\varrho \in \overline{\mathfrak{t}^*_+}$, $(\varrho, \alpha^*) \in \mathbb{Z}$ for any $\alpha \in R$.

b) $R \subset \Lambda^*$, and $2\pi\alpha^* \in \Lambda$ for any $\alpha \in R$, thus $\Gamma \subset \Lambda$.

c) The fundamental group $\pi_1(G)$ of G is always abelian, and $\pi_1(G) \simeq \Lambda/\Gamma$.

Note that Γ depends only on the structure of the Lie algebra $\mathfrak{g}$, not the group G. But Λ depends really on the group G.

Set

$$\Lambda^*_+ = \Lambda^* \cap \overline{\mathfrak{t}^*_+}. \tag{5.2.26}$$

Then the set of the finite dimensional G-irreducible representations is parametrized by the set of dominant weights Λ^*_+. For $\gamma \in \Lambda^*_+$, we denote by V^G_γ the irreducible (finite dimensional complex) G-representation with highest weight γ. Then V^G_γ, $\gamma \in \Lambda^*_+$, form a $\mathbb{Z}$-basis of the representation ring $R(G)$.

5.3 Reductions and prequantizations

For a group action on a prequantum line bundle, we like to know whether the reduced space is still prequantizable. Due to the topological constraint Theorem 5.1.4, it is not possible in general. In this section, we explain these data induce naturally a prequantum line bundle on reduced space at 0 or more generally at a real weight.

5.3.1 Reduction at 0

Let (M, ω) be a symplectic manifold which is prequantized by (L, h^L, ∇^L). Let G be a compact Lie group with Lie algebra $\mathfrak{g}$, which acts symplectically on M and can be lifted on L such that the G-action preserves (h^L, ∇^L). Let $\mu : M \to \mathfrak{g}^*$ be the moment map defined by (5.1.24). Moreover, we suppose that G acts freely on $\mu^{-1}(0)$.

We recall that by Lemma 2.3.13, 0 is a regular value of μ. Thus $\mu^{-1}(0)$ is a submanifold of M. By Theorem 2.3.14, there is a natural symplectic form ω_G on the reduced space $M_G = G\backslash\mu^{-1}(0)$ induced by ω. Moreover $\pi : \mu^{-1}(0) \to M_G$ is a principal G-bundle.

The aim of this section is to construct a Hermitian line bundle (L_G, h_G, ∇^{L_G}) over M_G such that (M_G, ω_G) is prequantized by (L_G, h_G, ∇^{L_G}).

For $y \in M_G$, let $L_{G,y} = \mathscr{C}^\infty(\pi^{-1}(y), L)^G$ be the space of G-invariant sections of L over $\pi^{-1}(y)$. Since G acts freely on $\mu^{-1}(0)$, for $x \in \pi^{-1}(y)$, we have $\pi^{-1}(y) = G \cdot x$, and thus we have an isomorphism

$$L_x \ni v \longrightarrow \{s(gx) := gv\} \in L_{G,y} = \mathscr{C}^\infty(\pi^{-1}(y), L)^G. \tag{5.3.1}$$

Thus $\dim L_{G,y} = 1$. Moreover, as $\pi : \mu^{-1}(0) \to M_G$ is locally trivial, L_G is a complex line bundle over M_G. By (5.3.1), we have

$$\mathscr{C}^\infty(M_G, L_G) = \mathscr{C}^\infty(\mu^{-1}(0), L)^G. \tag{5.3.2}$$

For $y \in M_G, s \in \mathscr{C}^\infty(M_G, L_G)$, $x \in \pi^{-1}(y), U \in T_y M_G$, we define

$$\begin{aligned} &h^{L_G}(s, s)_y := h^L_x(s(x), s(x)), \\ &\nabla^{L_G}_U s := \nabla^L_V s, \quad \text{with } V \in \mathscr{C}^\infty(\pi^{-1}(y), T\mu^{-1}(0)) \text{ such that } d\pi(V) = U. \end{aligned} \tag{5.3.3}$$

Theorem 5.3.1. *The metric h^{L_G} on L_G is well-defined and ∇^{L_G} is a well-defined Hermitian connection on (L_G, h^{L_G}). The symplectic manifold (M_G, ω_G) is prequantized by $(L_G, h^{L_G}, \nabla^{L_G})$.*

Proof. For $s \in L_{G,y}$, if $x' \in \pi^{-1}(y)$, then there exists $g \in G$ such that $x' = gx$. By (5.3.1), $s(x') = g \cdot s(x)$ and as h^L is G-invariant, we get

$$h^L_{x'}(s(x'), s(x')) = h^L_{gx}(g \cdot s(x), g \cdot s(x)) = h^L_x(s(x), s(x)). \tag{5.3.4}$$

Thus the first equation of (5.3.3) does not depend on the choice of $x \in \pi^{-1}(y)$, and as $\pi : \mu^{-1}(0) \to M_G$ is locally trivial, we know h^{L_G} is a well-defined $\mathscr{C}^\infty$ metric on L_G.

We verify now ∇^{L_G} is well-defined. In fact, if $V' \in \mathscr{C}^\infty(\pi^{-1}(y), T\mu^{-1}(0))$ such that $d\pi(V) = d\pi(V')$, then for $x \in \pi^{-1}(y)$, there is $K \in \mathfrak{g}$ such that $K^M_x = V_x - V'_x$. Hence at $x \in \mu^{-1}(0)$,

$$\nabla^L_V s - \nabla^L_{V'} s = L_K s + 2i\pi(\mu, K)s = 0. \tag{5.3.5}$$

By (5.1.10), (5.3.5) and ∇^L, s are G-invariant and $d\pi(dgV) = d\pi(V)$, we get at $x \in \mu^{-1}(0)$,

$$(g \cdot (\nabla^L_V s))_x = (\nabla^L_{dgV} s)_x = (\nabla^L_V s)_x. \tag{5.3.6}$$

Thus $\nabla^L_V s \in \mathscr{C}^\infty(\pi^{-1}(y), L)^G$ and does not depend on the choice of V.

As ∇^L is Hermitian, by (5.3.3) and $\pi : \mu^{-1}(0) \to M_G$ is locally trivial, ∇^{L_G} is also Hermitian with respect to h^{L_G}.

Let U, V be the vector fields defined on some local chart $\mathcal{U}$ of M_G. Let $U', V' \in \mathscr{C}^\infty(\pi^{-1}(\mathcal{U}), T\mu^{-1}(0))$ be their lift. Then we have

$$[U', V'] \in \mathscr{C}^\infty(\pi^{-1}(\mathcal{U}), T\mu^{-1}(0)) \quad \text{and} \quad d\pi[U', V'] = [d\pi(U'), d\pi(V')] = [U, V]. \tag{5.3.7}$$

By Lemma 2.3.16, (5.3.3) and (5.3.7), we have

$$\begin{aligned} R^{L_G}(U,V)s = \nabla^{L_G}_U \nabla^{L_G}_V s - \nabla^{L_G}_V \nabla^{L_G}_U s - \nabla^{L_G}_{[U,V]} s = \nabla^L_{U'} \nabla^L_{V'} s - \nabla^L_{V'} \nabla^L_{U'} s - \nabla^L_{[U',V']} s \\ = R^L(U', V')s = -2i\pi\omega(U', V')s = -2i\pi\omega_G(U, V)s. \end{aligned} \tag{5.3.8}$$

The proof of Theorem 5.3.1 is completed. □

5.3.2 Reduction at $\gamma \in \mathfrak{g}^*$

We recall that for $\gamma \in \mathfrak{g}^*$, $\mathcal{O}_\gamma := G \cdot \gamma \subset \mathfrak{g}^*$ is the coadjoint orbit, and G_γ is the stabilizer of γ for Ad^*_G-action on $\mathfrak{g}^*$.

Assume that G_γ acts freely on $\mu^{-1}(\gamma)$. By Lemmas 2.3.13 and 2.3.17, $\mu^{-1}(\gamma)$ is a manifold. The reduced space M_γ at γ is defined by

$$M_\gamma := G \backslash \mu^{-1}(\mathcal{O}_\gamma) = G_\gamma \backslash \mu^{-1}(\gamma). \tag{5.3.9}$$

By Theorem 2.3.18, there is a canonical symplectic form ω_γ on M_γ, and $\pi : \mu^{-1}(\gamma) \to M_\gamma$ is a principal G_γ-bundle.

The aim of this section is, for some particular $\gamma \in \mathfrak{g}^*$, to construct a Hermitian line bundle $(L_\gamma, h_{L_\gamma}, \nabla^{L_\gamma})$ over $(M_\gamma, \omega_\gamma)$ such that $(M_\gamma, \omega_\gamma)$ is prequantized by $(L_\gamma, h^{L_\gamma}, \nabla^{L_\gamma})$.

Now we return to our situation where (M, ω) is a symplectic manifold prequantized by (L, h^L, ∇^L), and G is a compact connected Lie group which acts on M symplectically and the G-action can be lifted on L such that it preserves h^L, ∇^L. Let $\mu : M \to \mathfrak{g}^*$ be the moment map defined by the Kostant formula (5.1.24).

Let $2\pi\gamma \in \Lambda^* \cap \mathfrak{t}^*_+$, then $G_\gamma = \mathbb{T}$. We suppose that $\mathbb{T}$ acts freely on $\mu^{-1}(\gamma)$.
As $2\pi\gamma \in \Lambda^*$,

$$\mathbb{T} \ni g = \exp(\tau) \to \rho_\gamma(g) = \exp(2i\pi\langle\gamma, \tau\rangle) \in \mathbb{S}^1 \tag{5.3.10}$$

is a well-defined morphism from $\mathbb{T}$ to $\mathbb{S}^1$. For $y \in M_\gamma$, set

$$L_{\gamma,y} = \{s \in \mathscr{C}^\infty(\pi^{-1}(y), L) : g \cdot s = \rho_\gamma(g)s, \text{ for all } g \in \mathbb{T}\}. \tag{5.3.11}$$

Then as in (5.3.1), for $x \in \pi^{-1}(y)$, we have the isomorphism

$$L_x \ni v \to \{s_{\gamma,v}(gx) = \rho_\gamma(g^{-1})g \cdot v\} \in L_{\gamma,y}\,, \tag{5.3.12}$$

thus L_γ is a complex line bundle over M_γ, and

$$\mathscr{C}^\infty(M_\gamma, L_\gamma) = \{s \in \mathscr{C}^\infty(\mu^{-1}(\gamma), L) : g \cdot s = \rho_\gamma(g)s, \text{ for all } g \in \mathbb{T}\}. \tag{5.3.13}$$

As (5.3.3), we define the metric and connection on L_γ by: for $s \in \mathscr{C}^\infty(M_\gamma, L_\gamma)$, $x \in \pi^{-1}(y)$, $U \in T_yM_\gamma$,

$$\begin{aligned} &h^{L_\gamma}(s,s)_y := h^L_x(s(x), s(x)), \\ &\nabla^{L_\gamma}_U s := \nabla^L_V s, \quad \text{with } V \in \mathscr{C}^\infty(\pi^{-1}(y), T\mu^{-1}(\gamma)) \text{ such that } d\pi(V) = U. \end{aligned} \tag{5.3.14}$$

Theorem 5.3.2. *The metric h^{L_γ} on L_γ is well-defined and ∇^{L_γ} is a well-defined Hermitian connection on (L_γ, h^{L_γ}). The symplectic manifold $(M_\gamma, \omega_\gamma)$ is prequantized by $(L_\gamma, h^{L_\gamma}, \nabla^{L_\gamma})$.*

Proof. By (5.3.11), $h^L_x(s(x), s(x))$ does not depend on the choice of $x \in \pi^{-1}(y)$.

We need to verify ∇^{L_γ} is a well-defined connection. In fact, by (5.1.24), (5.3.13) and (5.3.14), for $s \in \mathscr{C}^\infty(M_\gamma, L_\gamma), K \in \mathfrak{t}$, we have

$$\nabla^L_{K^M} s = L_K s + 2i\pi(\mu, K)s = 0. \tag{5.3.15}$$

Thus by the same argument as in (5.3.5), ∇^{L_γ} is well-defined. Moreover ∇^{L_γ} is Hermitian with respect to h^{L_γ}. By (5.3.14), the analogue of (5.3.8) holds for ∇^{L_γ}.

The proof of Theorem 5.3.2 is completed. □

Remark 5.3.3. If $2\pi\gamma \in \Lambda^*_+ \setminus \Lambda^* \cap \mathfrak{t}^*_+$, then $\mathbb{T}$ is a strictly subgroup of G_γ, but γ defines a unique morphism $\rho_\gamma : G_\gamma \to \mathbb{S}^1$ such that its restriction to $\mathbb{T}$ is given by (5.3.10) (cf. Chapter 6). Thus the construction here also works in this situation.

As the second proof of Theorem 2.3.18, we will give another construction based on the coadjoint orbit. We give first some examples.

Example 5.3.4. Let G be a compact connected Lie group. The left action L_h for $h \in G$ induces a left action L_h on T^*G. For $h, g \in G$, $\beta \in \mathfrak{g}^*, Y \in T_gG$, set

$$L_h \cdot (g, \beta) = (hg, \beta), \quad (L^*_{g^{-1}}\beta, Y) := (\beta, dL_{g^{-1}}Y). \tag{5.3.16}$$

Then L_h defines the left action of G on $G \times \mathfrak{g}^*$ and the map

$$\Phi_L : G \times \mathfrak{g}^* \ni (g, \beta) \to (g, L^*_{g^{-1}}\beta) \in T^*G, \tag{5.3.17}$$

is a G-equivariant diffeomorphism for the left action. We apply now the construction of Example 5.1.5 to $G \times \mathfrak{g}^*$.