Let $\theta \in \Omega^1(G, \mathfrak{g})$ be the canonical 1-form with values in $\mathfrak{g}$, that is for $g \in G, X \in T_gG$, we have

$$\theta_g(X) = dL_{g^{-1}}X \in T_eG = \mathfrak{g}. \tag{5.3.18}$$

We denote also $\theta = g^{-1}dg$. We identify $X, Y \in \mathfrak{g}$ to the left invariant vector fields X, Y on G, then $\theta(X), \theta(Y)$ are constant on G, and

$$(d\theta)(X,Y) = X\theta(Y) - Y\theta(X) - \theta([X,Y]) = -[X,Y]. \tag{5.3.19}$$

For $\vartheta_1, \vartheta_2 \in \Omega^\bullet(G)$, $u, v \in \mathfrak{g}$, we define

$$[\vartheta_1 \otimes u, \vartheta_2 \otimes v] = \vartheta_1 \wedge \vartheta_2 \otimes [u,v]. \tag{5.3.20}$$

Then we can reformulate (5.3.19) as

$$d\theta = -\frac{1}{2}[\theta, \theta]. \tag{5.3.21}$$

Let $\pi_L : G \times \mathfrak{g}^* \to G$ and $\pi : T^*G \to G$ be the natural projections, then $\pi \circ \Phi_L = \pi_L$ and $L_g \circ \pi_L = \pi_L \circ L_g$. We denote by p the tautological section of $\mathfrak{g}^*$. By (1.2.42), for $(g,p) \in G \times \mathfrak{g}^*$, $X \in T_{(g,p)}(G \times \mathfrak{g}^*)$, we have

$$(\Phi_L^*\lambda)_{(g,p)}(X) = \langle L_{g^{-1}}^* p, d\pi d\Phi_L(X)\rangle = \langle p, dL_{g^{-1}} d\pi_L(X)\rangle = \langle p, (\pi_L^*\theta)(X)\rangle. \tag{5.3.22}$$

We note that λ is invariant under diffeomorphisms of G. In particular λ is left and right invariant. By (5.3.21) and (5.3.22), the symplectic form ω on $G \times \mathfrak{g}^*$ ($\simeq T^*G$) is given by

$$\omega = -d(\Phi_L^*\lambda) = -\langle dp, \pi_L^*\theta\rangle - \langle p, \pi_L^* d\theta\rangle = -\langle dp, \pi_L^*\theta\rangle + \frac{1}{2}\langle p, \pi_L^*[\theta,\theta]\rangle. \tag{5.3.23}$$

We will call the G-action on $M = G$ defined by $G \times M \ni (h,g) \to gh^{-1} \in M$ as R^{-1}-action of G on G. We define the $G \times G$-action on G by

$$I_{g_1,g_2}g = g_1 g g_2^{-1} \quad \text{for } g_1, g_2, g \in G. \tag{5.3.24}$$

It induces a $G \times G$-action on T^*G, and under the identification (5.3.17), we know by (2.3.42), for $(g, \beta) \in G \times \mathfrak{g}^*$,

$$I_{g_1,g_2}(g,\beta) = (I_{g_1,g_2}g, L^*_{g_1 g g_2^{-1}}(d(I_{g_1,g_2})^{-1})^* L^*_{g^{-1}}\beta) = (I_{g_1,g_2}g, \mathrm{Ad}^*_{g_2}\beta), \tag{5.3.25}$$

as for $X \in \mathfrak{g}$, by (5.3.16) and (5.3.24), we have

$$\begin{aligned}(L^*_{g_1 g g_2^{-1}}(d(I_{g_1,g_2})^{-1})^* L^*_{g^{-1}}\beta, X) &= (\beta, dL_{g^{-1}} dL_{g_1^{-1}} dR_{g_2} dL_{g_1 g g_2^{-1}} X)\\ &= (\beta, \mathrm{Ad}_{g_2^{-1}}X) = (\mathrm{Ad}^*_{g_2}\beta, X).\end{aligned} \tag{5.3.26}$$

For $X \in \mathfrak{g}$, as in (2.2.1), let X^L, X^R be the vector fields on $G \times \mathfrak{g}^*$ induced by X for the left action (5.3.16) and by X for the right action on G. By (5.3.25), for $g \in G, \beta \in \mathfrak{g}^*$,

$$X^L_{(g,\beta)} = (dR_g(X), 0), \quad X^R_{(g,\beta)} = (dL_g(X), -\mathrm{ad}^*_X\beta) \in T_gG \times \mathfrak{g}^*. \tag{5.3.27}$$

Thus the vector field on $G \times \mathfrak{g}^*$ induced by $(X,Y) \in T_{(e,e)}(G \times G) = \mathfrak{g} \oplus \mathfrak{g}$, for the action (5.3.25) is

$$(X,Y)^{G\times\mathfrak{g}^*}_{(g,\beta)} = X^L_{(g,\beta)} - X^R_{(g,\beta)}. \tag{5.3.28}$$

Let $\mu_L : G \times \mathfrak{g}^* \to \mathfrak{g}^*$, $\mu_R : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ be defined by

$$\mu_L(g,\beta) = \mathrm{Ad}_g^*\beta, \quad \mu_R(g,\beta) = -\beta. \tag{5.3.29}$$

By (2.3.46), (5.3.27)–(5.3.29), the moment map $\mu : G \times \mathfrak{g}^* \to (\mathfrak{g} \oplus \mathfrak{g})^*$ for the $G \times G$-action (5.3.25) on T^*G is given by

$$\begin{aligned}\Big(\mu, (X,Y)\Big)_{(g,\beta)} &= (L_{g^{-1}}^*\beta, d\pi(X,Y)^{T^*G}) = (\beta, dL_{g^{-1}} d\pi_L(X,Y)^{G\times\mathfrak{g}^*}) \\ &= (\beta, \mathrm{Ad}_{g^{-1}}X - Y) = (\mu_L, X)_{(g,\beta)} + (\mu_R, Y)_{(g,\beta)}.\end{aligned} \tag{5.3.30}$$

Thus the moment map μ is given by

$$\mu = (\mu_L, \mu_R) : G \times \mathfrak{g}^* \to (\mathfrak{g} \oplus \mathfrak{g})^*. \tag{5.3.31}$$

By Proposition 2.3.19 and (5.3.24), we know μ_L (resp. μ_R) is the moment map of the left (resp. R^{-1}) action of G on T^*G.

Now we consider the symplectic reduction with respect to μ_R. For $\gamma \in \mathfrak{g}^*$, by (5.3.29),

$$\mu_R^{-1}(-\gamma) = G \times \{\gamma\} \subset G \times \mathfrak{g}^* \quad \text{and} \quad R_h^{-1}(g,\gamma) = (gh^{-1},\gamma) \text{ for } h \in G_\gamma. \tag{5.3.32}$$

Let $\pi_R : \mu_R^{-1}(-\gamma) \to M_{-\gamma} = \mu_R^{-1}(-\gamma)/G_\gamma$ be the natural projection with R^{-1}-action of G_γ on $\mu_R^{-1}(-\gamma)$. Then we can identify $M_{-\gamma}$ with the coadjoint orbit by

$$\varphi_1 : M_{-\gamma} \ni [g,\gamma] \to \mathrm{Ad}_g^*\gamma \in \mathcal{O}_\gamma = G\cdot\gamma. \tag{5.3.33}$$

Now by (5.3.33), the left action on G induces the Ad^*-action on $\mathcal{O}_\gamma$, thus from (2.2.3),

$$\begin{aligned}d(\varphi_1\circ\pi_R)X^L_{(g,\gamma)} &= \frac{\partial}{\partial t}|_{t=0}\varphi_1\circ\pi_R(e^{tX}g,-\gamma) = \frac{\partial}{\partial t}|_{t=0}\mathrm{Ad}^*_{e^{tX}}\mathrm{Ad}_g^*\gamma \\ &= \mathrm{ad}_X^*\mathrm{Ad}_g^*\gamma = X^{\mathcal{O}_\gamma}_{\mathrm{Ad}_g^*\gamma}.\end{aligned} \tag{5.3.34}$$

By Theorem 2.3.18, (2.1.20), (5.3.18), (5.3.23) and (5.3.27), we get for $X, Y \in \mathfrak{g}$,

$$\begin{aligned}\omega_\gamma(X^{\mathcal{O}_\gamma}, Y^{\mathcal{O}_\gamma})_{\mathrm{Ad}_g^*\gamma} &= \omega(X^L, Y^L)_{(g,\gamma)} \\ &= (\gamma, [\theta(dR_gX), \theta(dR_gY)]) = (\gamma, [\mathrm{Ad}_{g^{-1}}X, \mathrm{Ad}_{g^{-1}}Y]) \\ &= (\mathrm{Ad}_g^*\gamma, [X,Y]).\end{aligned} \tag{5.3.35}$$

From (5.3.35), ω_γ is exactly the symplectic form on $\mathcal{O}_\gamma$ defined in (2.3.51). By Theorem 2.3.18, the associated moment map on $(\mathcal{O}_\gamma, \omega_\gamma)$ is induced by μ_L :

$$(\mu^{\mathcal{O}_\gamma}, X)_{\mathrm{Ad}_g^*\gamma} = (\mu_L, X)_{(g,\gamma)} = (\mathrm{Ad}_g^*\gamma, X). \tag{5.3.36}$$

Thus we recover Proposition 2.3.9 by using the reduction from $G \times \mathfrak{g}^*$.

Now we suppose that $2\pi\gamma \in \Lambda^* \cap \mathfrak{t}^*_+$. Then $G_\gamma = \mathbb{T}$. Let ρ_γ be the representation of $\mathbb{T}$ defined by

$$\rho_\gamma : \mathbb{T} \ni g = \exp(\tau) \to e^{2i\pi\langle\gamma,\tau\rangle} \in \mathbb{S}^1. \tag{5.3.37}$$

The $G \times G$-action on $L = \mathbb{C}$ is defined by $I_{g_1,g_2}\cdot\mathbf{1} = \mathbf{1}$ as explained after (5.1.21). Thus $s \in L_{\gamma,y}$ means that $s \in \mathscr{C}^\infty(\pi^{-1}(y),\mathbb{C})$ and $s(xh^{-1}) = \rho_\gamma(h^{-1})s(x)$ for $x \in \pi^{-1}(y)$, $h \in \mathbb{T}$. In other words, L_γ is the quotient space of $G\times\mathbb{C}$ by the $\mathbb{T}$-action defined by: for $h \in \mathbb{T}$, $(g,v) \in G\times\mathbb{C}$,

$$h(g,v) = (gh, \rho_\gamma(h)v). \tag{5.3.38}$$

Thus L_γ is the line bundle $G \times_{\rho_\gamma} \mathbb{C}$ on $G/\mathbb{T}$ associated with the principal $\mathbb{T}$-bundle $G \to G/\mathbb{T}$ and the representation ρ_γ. Let ∇^{L_γ} be the connection constructed in (5.3.14). By Theorem 5.3.2, we have

$$c_1(L_\gamma, \nabla^{L_\gamma}) = \omega_\gamma. \tag{5.3.39}$$

In fact, we will show in Chapter 6 that $(\mathcal{O}_\gamma, \omega_\gamma)$ is a Kähler manifold, and L_γ is a holomorphic line bundle on $\mathcal{O}_\gamma$.

Theorem 5.3.5 (Borel-Weil-Bott). *If $2\pi\gamma \in \Lambda^*_+$, we have*

$$H^{0,j}(\mathcal{O}_\gamma, L_\gamma) = 0, \quad \text{for } j \geq 1, \tag{5.3.40}$$

and $H^{0,0}(\mathcal{O}_\gamma, L_\gamma)$, the space of holomorphic sections of L_γ on $\mathcal{O}_\gamma$, is the irreducible representation associated with highest weight $2\pi\gamma$.

We know that $(M \times \mathcal{O}_\gamma, \omega + \omega_\gamma)$ is prequantized by $L \otimes L_\gamma$, the tensor product of the natural lifts of L and L_γ on $M \times \mathcal{O}_\gamma$. By applying Theorem 5.3.1 for $M \times \mathcal{O}_\gamma$, we then recover the symplectic reduction at $-\gamma$ and its prequantization line bundle.

5.4 Kähler prequantizations and reductions

Let (M, ω) be a symplectic manifold, and let G be a compact Lie group with Lie algebra $\mathfrak{g}$. We suppose that G acts on M Hamiltonianly with moment map $\mu : M \to \mathfrak{g}^*$.

We assume that G acts freely on $Y = \mu^{-1}(0)$. For $x \in M$, set

$$\mathfrak{g}^M_x = \{K^M_x \in T_xM : K \in \mathfrak{g}\}. \tag{5.4.1}$$

Then it forms a vector bundle $\mathfrak{g}^M$ on Y. For simplify, we denote $\mathfrak{g}^Y = \mathfrak{g}^M|_Y$.

Let J be a G-invariant, compatible almost complex structure on (M, ω). Then $g^{TM} = \omega(\cdot, J\cdot)$ is a G-invariant metric on TM, and we denote by $\perp$ to mean the orthogonality with respect to g^{TM}. Let T^HY be the orthogonal complement of $\mathfrak{g}^Y$ in TY with respect to g^{TM}.

Proposition 5.4.1. *The following G-invariant orthogonal decomposition of vector bundles on Y with respect to g^{TM} holds:*

$$TM|_Y = T^HY \oplus \mathfrak{g}^Y \oplus J\mathfrak{g}^Y. \tag{5.4.2}$$

Moreover, T^HY is J-invariant, and $T^HY \perp_\omega (\mathfrak{g}^Y \oplus J\mathfrak{g}^Y)$.

Proof. By (2.3.76), we get

$$\mathfrak{g}^Y \subset TY, \qquad\qquad (TY)^{\perp_\omega} = \mathfrak{g}^Y. \tag{5.4.3}$$

By (5.4.3), $g^{TM} = \omega(\cdot, J\cdot)$ and $\mathfrak{g}^Y \perp T^HY$, we get

$$J\mathfrak{g}^Y = (TY)^\perp, \quad \mathfrak{g}^Y \perp T^HY, \quad \text{i.e., } \mathfrak{g}^Y \perp_\omega TY, \quad J\mathfrak{g}^Y \perp_\omega T^HY. \tag{5.4.4}$$

As $\mathfrak{g}^Y$ is G-invariant, thus T^HY and $J\mathfrak{g}^Y$ are G-invariant. From (5.4.4) and $TY = T^HY \oplus \mathfrak{g}^Y$, we get the G-invariant orthogonal decomposition (5.4.2).

If $u \in T^HY$, then $u \perp (\mathfrak{g}^Y \oplus J\mathfrak{g}^Y)$, thus $Ju \perp (\mathfrak{g}^Y \oplus J\mathfrak{g}^Y)$. This and (5.4.2) imply $J(T^HY) \subset T^HY$. By (5.4.4), we know $T^HY \perp_\omega (\mathfrak{g}^Y \oplus J\mathfrak{g}^Y)$.

The proof of Proposition 5.4.1 is completed. □

For a G-vector bundle E on Y, set

$$\begin{aligned} E_{G,y} &= \{s \in \mathscr{C}^\infty(\pi^{-1}(y), E) : g \cdot s = s \quad \text{for any } g \in G\}, \quad \text{for } y \in M_G, \\ F &= Y \times_G \mathfrak{g}. \end{aligned} \tag{5.4.5}$$

Let $x \in Y$, $y \in M_G$ such that $\pi(x) = y$, for $U \in T_y M_G$, let $U^H \in T_x^H Y$ be the lifting of U, i.e., the unique $U^H \in T_x^H Y$ such that $d\pi(U^H) = U$, then automatically, $U^H \in \mathscr{C}^\infty(Y, T^H Y)^G$. This gives the canonical isomorphism $(T^H Y)_G \to TM_G$. From (5.4.2), we get on M_G,

$$(TM|_Y)_G \simeq TM_G \oplus (F \oplus F^*). \tag{5.4.6}$$

Let $J_{G,y} : T_y M_G \to T_y M_G$ be defined by

$$J_G U = d\pi(JU^H). \tag{5.4.7}$$

As $JU^H \in \mathscr{C}^\infty(Y, T^H Y)^G$, we know that $d\pi(JU^H)_x$ does not depend on the choice of $x \in \pi^{-1}(y)$, thus $J_G U$ is well-defined.

Theorem 5.4.2. *The J_G in (5.4.7) is a compatible almost complex structure on (M_G, ω_G). Moreover, if J is integrable, J_G is also integrable. In particular, if (M, J, ω) is a Kähler manifold, then (M_G, J_G, ω_G) is also a Kähler manifold.*

Proof. For $y \in M_G$, $U \in T_y M_G$, since J preserves $T^H Y$, $JU^H \in T_x^H Y$ for $x \in \pi^{-1}(y)$. Thus $JU^H = \big(d\pi(JU^H)\big)^H = (J_G U)^H$. Then by (5.4.7), we have

$$J_G^2 U = J_G \, d\pi(JU^H) = d\pi(J^2 U^H) = -U. \tag{5.4.8}$$

This means J_G is an almost complex structure on M_G.

On the other hand, for $U, V \in T_y M_G$, we have

$$\begin{aligned} &\omega_G(J_G U, J_G V) = \omega(JU^H, JV^H) = \omega(U^H, V^H) = \omega_G(U, V), \\ &\omega_G(U, J_G U) = \omega(U^H, JU^H) > 0 \quad \text{if } U \neq 0. \end{aligned} \tag{5.4.9}$$

Thus, J_G is a compatible almost complex structure on (M_G, ω_G).

We suppose now J is integrable. If $u, v \in \mathscr{C}^\infty(M_G, T^{(1,0)} M_G)$, then there exist $U, V \in \mathscr{C}^\infty(M_G, TM_G)$ such that

$$u = U - \sqrt{-1} J_G U, \quad v = V - \sqrt{-1} J_G V. \tag{5.4.10}$$

By (5.4.7), $(J_G U)^H = JU^H$, thus

$$u^H = U^H - \sqrt{-1} JU^H, \quad v^H = V^H - \sqrt{-1} JV^H \in T^{(1,0)} M \cap (TY \otimes_{\mathbb{R}} \mathbb{C}). \tag{5.4.11}$$

As both $T^{(1,0)} M$, $TY \otimes_{\mathbb{R}} \mathbb{C}$ are integrable, we get

$$[u^H, v^H] \in T^{(1,0)} M \cap (TY \otimes_{\mathbb{R}} \mathbb{C}), \tag{5.4.12}$$

i.e., there exists $W \in TM$ such that $[u^H, v^H] = W - \sqrt{-1} JW$. Let P^{TY} be the orthogonal projection from TM onto TY via (5.4.2). Then from (5.4.2),

$$\begin{aligned} [u, v] = d\pi[u^H, v^H] &= d\pi P^{TY} W - \sqrt{-1} d\pi P^{TY} JW \\ &= d\pi P^{TY} W - \sqrt{-1} J_G d\pi P^{TY} W \in T^{(1,0)} M_G. \end{aligned} \tag{5.4.13}$$

From the Newlander-Nirenberg theorem, (5.4.13) means J_G is integrable. The proof of Theorem 5.4.2 is completed. □

In the rest of this section, let (M, J, ω) be a Kähler manifold prequantized by a holomorphic Hermitian line bundle (L, h^L) with the Chern connection ∇^L. Let a compact Lie group G act holomorphically and symplectically on M, and the G-action can be lifted on a holomorphic action on L. Then by Theorem 5.1.10, the G-action preserves the metric h^L. Let $\mu : M \to \mathfrak{g}^*$ be the associated moment map.

Theorem 5.4.3. *If G acts freely on $\mu^{-1}(0)$, then (L_G, h^{L_G}) in Theorem 5.3.1 is a holomorphic Hermitian line bundle over M_G and ∇^{L_G} is the Chern connection on (L_G, h^{L_G}).*

Proof. We decompose ∇^{L_G} into holomorphic part and anti-holomorphic part,

$$\nabla^{L_G} = (\nabla^{L_G})^{1,0} + (\nabla^{L_G})^{0,1}. \tag{5.4.14}$$

From Theorem 5.4.2, ω_G, and so R^{L_G} is a $(1,1)$-form, thus

$$((\nabla^{L_G})^{0,1})^2 = 0. \tag{5.4.15}$$

Now, for $s \in \mathscr{C}^\infty(M_G, L_G)$, we define

$$\overline{\partial}^{L_G} s := (\nabla^{L_G})^{0,1} s. \tag{5.4.16}$$

Let s_0 be a local frame of L_G near $x_0 \in M_G$. Then there is a $(0,1)$-form a near x_0 such that $(\nabla^{L_G})^{0,1} s_0 = a s_0$. From (5.4.15),

$$0 = ((\nabla^{L_G})^{0,1})^2 s_0 = (\overline{\partial} a) s_0. \tag{5.4.17}$$

Thus $\overline{\partial} a = 0$ near x_0. By the $\overline{\partial}$-Lemma, there exists a function b near x_0 such that $\overline{\partial} b = -a$, i.e., $(\overline{\partial} b) s_0 + (\nabla^{L_G})^{0,1} s_0 = 0$. This means that (5.4.16) defines a holomorphic structure on L_G and $e^b s_0$ is a holomorphic frame of L_G.

The proof of Theorem 5.4.3 is completed. □

As the G-action commutes with the Dolbeault operator $\overline{\partial}^L$, we know for $j \geq 0$, the j-th Dolbeault cohomology group $H^{0,j}(M, L)$ (cf. (4.1.38)) is a G-representation. As in (4.2.80), we denote by V^G the G-invariant part of a G-representation V.

Now we can state the "quantization commutes with reduction" in the holomorphic case which was established by Guillemin-Sternberg for $j = 0$, and Teleman, Zhang for $j > 0$.

Theorem 5.4.4. *If M is compact and if G acts freely on $\mu^{-1}(0)$, then the map $\psi : \Omega^{0,\bullet}(M, L)^G \to \Omega^{0,\bullet}(M_G, L_G)$, by the restriction first on $\mu^{-1}(0)$, then using (5.3.2) and (5.4.6) to induce a section on M_G, induces an isomorphism*

$$H^{0,j}(M, L)^G \simeq H^{0,j}(M_G, L_G) \quad \textit{for any } j \geq 0. \tag{5.4.18}$$

Exercise 5.4.1. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Let $\pi : Y \to G\backslash Y = B$ be a G-principal bundle with fiberwise tangent bundle TZ. Let (E, h^E, ∇^E) be a Hermitian vector bundle with Hermitian connection ∇^E and curvature R^E. Let $T^H Y$ be a horizontal subbundle of TY such that

$$TY = T^H Y \oplus TZ. \tag{5.4.19}$$

Let $P^{TZ} : TY = T^H Y \oplus TZ \to TZ$ be the projection. Let $\theta \in \Omega^1(Y, \mathfrak{g})$ be defined as $\theta(T^H Y) = 0$, and $\theta(K^Y) = K$ for $K \in \mathfrak{g}$.

We suppose that all geometric objects are G-equivariant. Then the horizontal bundle $T^H X$ defines a connection θ of the principal G-bundle $\pi : Y \to B$. The curvature of the fibration $\pi : Y \to B$ associated with θ is defined by: for $U, V \in \mathscr{C}^\infty(B, TB)$,

$$\Omega(U^H, V^H) = -\theta([U^H, V^H]), \tag{5.4.20}$$

here $U^H \in \mathscr{C}^\infty(Y, T^H Y)$ is the unique lift of U such that $d\pi(U^H) = U$.

As in (5.1.24), we define $\mu^E \in \mathscr{C}^\infty(Y, \mathfrak{g}^* \otimes \mathrm{End}(E))$ by: for $K \in \mathfrak{g}$,

$$2\pi\sqrt{-1}\mu^E(K) = \nabla^E_{K^X} - L_K. \tag{5.4.21}$$

We still call μ^E the moment map associated to the G-action on E.

1. Verify that E_G in (5.4.5) defines a vector bundle on B, and h^E induces a Hermitian metric h^{E_G} on E_G.
2. For $s \in \mathscr{C}^\infty(B, E_G) = \mathscr{C}^\infty(Y, E)^G$, $U \in \mathscr{C}^\infty(B, TB)$, we define

$$\nabla^{E_G}_U s := \nabla^E_{U^H} s. \tag{5.4.22}$$

Verify that ∇^{E_G} is a Hermitian connection on (E_G, h^{E_G}). Its curvature R^{E_G} is given by

$$R^{E_G}(U, V) = R^E(U^H, V^H) - 2\pi\sqrt{-1}\mu^E(\Omega(U^H, V^H)). \tag{5.4.23}$$

Exercise 5.4.2. We put in the context of Theorem 5.4.3. Let E be a holomorphic vector bundle on M.

1. For the fibration $\pi : \mu^{-1}(0) \to M_G$, (5.4.5) defines a holomorphic vector vector bundle E_G on M_G. (Hint: We only need to verify $(\overline{\partial}^{E_G})^2 = 0$ by the Koszul-Malgrange integrability theorem.)
2. Let h^E be a G-invariant metric on E, and ∇^E be the Chern connection on (E, h^E). Then (5.4.22) induces the Chern connection on (E_G, h^{E_G}).
3. Verify that $(T^{(1,0)}M)_G$ is a holomorphic vector bundle on M_G, and the map $d\pi : (T^{(1,0)}M)_G \to T^{(1,0)}M_G$ is holomorphic and surjective.

Exercise 5.4.3 (Symplectic cut).

1. In Section 5.3.1, if $G \times H$ acts Hamiltonianly on (M, ω) and (L, ∇^L), then H acts Hamiltonianly on $(M_G, \omega_G, L_G, \nabla^{L_G})$.
2. Assume the Hamiltonian $\mathbb{S}^1$-manifold (M, ω) is prequantizable by the prequantum line bundle (L, h^L, ∇^L).

Let $\mathbf{1}$ be the canonical section of the trivial line bundle $F = \mathbb{C}$ on $\mathbb{C}$. We define $\mathbb{S}^1$ action on $(\mathbb{C}, F)$ by $g \cdot \mathbf{1}_y = \mathbf{1}_{g \cdot y}$. Then $G = \mathbb{S}^1$-acts on $L \otimes F$ on $M \times \mathbb{C}$ by $g \cdot (\sigma_x \otimes \mathbf{1}_y) = (g \cdot \sigma_x) \otimes \mathbf{1}_{g \cdot y}$ and the $H = \mathbb{S}^1$-action on $L \otimes F$ on $M \times \mathbb{C}$ by $g \cdot (\sigma_x \otimes \mathbf{1}_y) = (g \cdot \sigma_x) \otimes \mathbf{1}_y$ for $g \in \mathbb{S}^1$, $x \in M, y \in \mathbb{C}$, $\sigma_x \in L_x$.

Conclude that $M_{\geq 0}$ as defined in Section 2.4 is prequantized by $(L_{\geq 0}, h^{L_{\geq 0}}, \nabla^{L_{\geq 0}})$ such that

$$(L_{\geq 0}, h^{L_{\geq 0}}, \nabla^{L_{\geq 0}})|_{M_{\mathbb{S}^1}} = (L_{\mathbb{S}^1}, h^{L_{\mathbb{S}^1}}, \nabla^{L_{\mathbb{S}^1}}). \tag{5.4.24}$$

3. We can define $(M_{\geq n}, L_{\geq n})$ by using the prequantum line $L \otimes \mathbb{C}_{[-n]}$, here $\mathbb{C}_{[-n]}$ is the trivial line bundle with trivial connection and it's a $\mathbb{S}^1$-representation $\mathbb{C}$ with weight $-n$. Then normal bundle $N_{M_{\mathbb{S}^1}/M_{\geq 0}}$ of $M_{\mathbb{S}^1}$ in $M_{\geq 0}$ is given by $\mu^{-1}(0) \times_{\mathbb{S}^1} \mathbb{C}_{[-1]}$, and $H = \mathbb{S}^1$ acts on $N_{M_{\mathbb{S}^1}/M_{\geq 0}}$ by $g \cdot [x, y] = [g \cdot x, y]$ for $x \in \mu^{-1}(0)$, $y \in \mathbb{C}$, thus $H = \mathbb{S}^1$ acts as identity on M_n and L_n as the fiberwise representation with weight n.

5.5 An infinite dimensional example: moduli spaces of flat connections

Let Σ be a compact oriented surface, and let (E, h^E) be a Hermitian vector bundle on Σ. Let $\mathcal{A}$ be the affine space of all Hermitian connection on (E, h^E). Note that $\mathcal{A}$ is of infinite dimension. For a fixed Hermitian connection ∇_0^E on (E, h^E), $\mathcal{A}$ can be expressed as

$$\mathcal{A} = \nabla_0^E + \Omega^1(\Sigma, \mathrm{End}^{\mathrm{anti}}(E)). \tag{5.5.1}$$

Formally, the tangent space $T\mathcal{A}$ takes the form

$$T_{\nabla_0^E}\mathcal{A} = \Omega^1(\Sigma, \mathrm{End}^{\mathrm{anti}}(E)). \tag{5.5.2}$$

For $X, Y \in \Omega^1(\Sigma, \mathrm{End}^{\mathrm{anti}}(E))$, we define a 2-form ω by

$$\omega(X, Y) = \int_\Sigma \mathrm{Tr}^E[X \wedge Y], \tag{5.5.3}$$

where $\mathrm{Tr}^E[X \wedge Y]$ is understood as follows: if $X = \alpha \otimes A$, $Y = \beta \otimes B$ with $\alpha, \beta \in \Omega^1(\Sigma)$ and $A, B \in \mathrm{End}(E)$, then $X \wedge Y = \alpha \wedge \beta \otimes AB$ and

$$\mathrm{Tr}^E[X \wedge Y] = \alpha \wedge \beta \;\; \mathrm{Tr}^E[AB]. \tag{5.5.4}$$

One verifies directly that ω defines a symplectic form on $\mathcal{A}$.

Set $G = U(n)$ and $E = P(U(n)) \times_\rho \mathbb{C}^n$ with $P(U(n))$ the principle $U(n)$-bundle on Σ. Let ΣG be the space of all smooth maps from Σ to G. Then ΣG is an infinite dimensional Lie group which we call the gauge group of E, and it acts smoothly on $\mathcal{A}$ by: for $g \in \Sigma G$ and $\nabla^A \in \mathcal{A}$,

$$g \cdot \nabla^A := g\nabla^A g^{-1} = \nabla^A - (\nabla^A g)g^{-1}. \tag{5.5.5}$$

For $g \in \Sigma G$ and $X, Y \in \Omega^1(\Sigma, \mathrm{End}^{\mathrm{anti}}(E))$, we have

$$\begin{aligned}\omega(g \cdot X, g \cdot Y) =& \omega(gXg^{-1}, gYg^{-1}) = \int_\Sigma \mathrm{Tr}^E[gXg^{-1} \wedge gYg^{-1}] \\ =& \int_\Sigma \mathrm{Tr}^E[X \wedge Y] = \omega(X, Y).\end{aligned} \tag{5.5.6}$$

That is, ΣG preserves the symplectic form ω. Denote $\mathfrak{g}$ by the Lie algebra of G. Then the Lie algebra of ΣG is $\Sigma\mathfrak{g} = C^\infty(\Sigma, \mathfrak{g})$. Take $X \in \Sigma\mathfrak{g}$, by (5.5.5), the induced vector field $X^{\mathcal{A}} \in C^\infty(\mathcal{A}, T\mathcal{A})$ is given by

$$X^{\mathcal{A}}_{\nabla^A} = \frac{d}{dt}\Big|_{t=0} e^{tX} \cdot \nabla^A = -\nabla^A X \in \Omega^1(\Sigma, \mathrm{End}^{\mathrm{anti}}(E)). \tag{5.5.7}$$

Denote by F^A the curvature of ∇^A, i.e.,

$$F^A = (\nabla^A)^2 \in \Omega^2(\Sigma, \mathrm{End}^{\mathrm{anti}}(E)). \tag{5.5.8}$$

We can identify $\Omega^2(\Sigma, \mathrm{End}^{\mathrm{anti}}(E))$ with $(\Sigma\mathfrak{g})^*$ as follows:

$$\begin{aligned}\Omega^2(\Sigma, \mathrm{End}^{\mathrm{anti}}(E)) \simeq& (\Sigma\mathfrak{g})^* \\ \alpha \otimes A \longmapsto& \langle \alpha \otimes A, B\rangle = \int_\Sigma \alpha \;\; \mathrm{Tr}^E[AB] \quad \text{for } B \in \Sigma\mathfrak{g}.\end{aligned} \tag{5.5.9}$$

Under the identification, the curvature F^A can be viewed as an element of $(\Sigma\mathfrak{g})^*$. We now define

$$\mu : \mathcal{A} \to (\Sigma\mathfrak{g})^*, \ \nabla^A \longmapsto F^A. \tag{5.5.10}$$

That is,

$$\big\langle \mu(\nabla^A), X \big\rangle = \int_\Sigma \mathrm{Tr}^E[F^A X] \ \text{ for } X \in \Sigma\mathfrak{g}. \tag{5.5.11}$$

Theorem 5.5.1. *μ is a moment map for the ΣG-action on $(\mathcal{A}, \omega)$.*

Proof. Denote by $F^{g\cdot A}$ the curvature of the connection $g \cdot \nabla^A$, i.e.,

$$F^{g\cdot A} = (g\nabla^A g^{-1})^2 = g\big(\nabla^A\big)^2 g^{-1} = gF^A g^{-1}. \tag{5.5.12}$$

Then

$$\begin{aligned} \big\langle \mu(g\cdot\nabla^A), X \big\rangle =& \big\langle F^{g\cdot A}, X \big\rangle = \int_\Sigma \mathrm{Tr}^E[F^{g\cdot A} \wedge X] = \int_\Sigma \mathrm{Tr}^E[F^A \wedge g^{-1}Xg] \\ =& \big\langle \mu(\nabla^A), \mathrm{Ad}_{g^{-1}} X \big\rangle = \big\langle \mathrm{Ad}_g^* \mu(\nabla^A), X \big\rangle. \end{aligned} \tag{5.5.13}$$

That is

$$\mu(g\cdot\nabla^A) = \mathrm{Ad}_g^* \mu(\nabla^A). \tag{5.5.14}$$

For $Y \in T_{\nabla^A}\mathcal{A} = \Omega^1(\Sigma, \mathrm{End}^{\mathrm{anti}}(E))$, we have

$$(Y \cdot F^A)_{\nabla^A} = \frac{\partial}{\partial t}\Big|_{t=0} (\nabla^A + tY)^2 = \nabla^A_\bullet Y. \tag{5.5.15}$$

By (5.5.7) and (5.5.15),

$$\begin{aligned} Y\big\langle \mu, X \big\rangle =& \int_\Sigma \mathrm{Tr}^E[\nabla^A_\bullet Y \wedge X] = \int_\Sigma \mathrm{Tr}^E[Y \wedge \nabla^A_\bullet X] \\ =& -\int_\Sigma \mathrm{Tr}^E[\nabla^A_\bullet X \wedge Y]. = \omega(X^{\mathcal{A}}, Y) = (i_{X^{\mathcal{A}}}\omega)(Y). \end{aligned} \tag{5.5.16}$$

That is

$$d\big\langle \mu, X \big\rangle = i_{X^{\mathcal{A}}}\omega. \tag{5.5.17}$$

By (5.5.14) and (5.5.17), we know that μ is a moment map. The proof of Theorem 5.5.1 is completed. □

Set

$$\mathcal{A}_0 := \mu^{-1}(0) = \big\{A \in \mathcal{A} : \ \mu(A) = 0\big\} = \big\{A \in \mathcal{A} : \ F^A = 0\big\}. \tag{5.5.18}$$

Theorem 5.5.2. *The quotient space $\mathcal{A}_0/\Sigma G$ is isomorphic to the space of equivariant class of flat Hermitian connections on E.*

Fix $x_0 \in \Sigma$, set

$$\Sigma_0 G = \big\{ g \in \Sigma G : g(x_0) = e \in G \big\}. \tag{5.5.19}$$

Then $\Sigma_0 G$ acts freely on $\mu^{-1}(0)$. Clearly,

$$\Sigma G / \Sigma_0 G = G, \quad \mu^{-1}(0)/\Sigma G = \big(\mu^{-1}(0)/\Sigma_0 G\big)/G. \tag{5.5.20}$$

Take $A \in \mathcal{A}_0$, then

$$\nabla^A = d + A, \quad (\nabla^A)^2 = 0. \tag{5.5.21}$$

We have the following complex with the differential operator ∇^A:

$$0 \longrightarrow \Omega^0(\Sigma, \mathrm{End}(E)) \longrightarrow \Omega^1(\Sigma, \mathrm{End}^{\mathrm{anti}}(E)) \longrightarrow \Omega^2(\Sigma, \mathrm{End}^{\mathrm{anti}}(E)) \longrightarrow 0. \tag{5.5.22}$$

For $X \in \Omega^0(\Sigma, \mathrm{End}(E)) = \Sigma\mathfrak{g}$, we have $\nabla^A X = -X^{\mathcal{A}}$.

Definition 5.5.3. The j-th cohomology of the complex $\big(\Omega^\bullet(\Sigma, \mathrm{End}^{\mathrm{anti}}(E)), \nabla^A\big)$ is defined by

$$H^j_{\nabla^A}(\Sigma, \mathrm{End}^{\mathrm{anti}}(E)) = \frac{\ker\ \nabla^A\big|_{\Omega^j(\Sigma, \mathrm{End}^{\mathrm{anti}}(E))}}{\mathrm{Im}\ \nabla^A\big|_{\Omega^{j-1}(\Sigma, \mathrm{End}^{\mathrm{anti}}(E))}}. \tag{5.5.23}$$

If G acts locally freely on $\mathcal{A}_0/\Sigma_0 G$, then $X \to X^{\mathcal{A}}$ is injective, i.e.,

$$H^0_{\nabla^A}(\Sigma, \mathrm{End}^{\mathrm{anti}}(E)) = 0. \tag{5.5.24}$$

By (5.5.15),

$$d\mu_{\nabla^A} : T_{\nabla^A}\mathcal{A} \to (\Sigma\mathfrak{g})^*, \ X \longmapsto \nabla^A X. \tag{5.5.25}$$

Then

$$T_{\nabla^A}\mathcal{A}_0 = \ker\ d\mu_{\nabla^A} = \big\{ X \in \Omega^1(\Sigma, \mathrm{End}^{\mathrm{anti}}(E)) : \nabla^A X = 0 \big\}. \tag{5.5.26}$$

Moreover,

$$T\big(\mu^{-1}(0)/\Sigma G\big) = \frac{T_{\nabla^A}\mathcal{A}_0}{\Sigma\mathfrak{g}\cdot\nabla^A} = \frac{\ker\ \nabla^A\big|_{\Omega^1(\Sigma, \mathrm{End}^{\mathrm{anti}}(E))}}{\mathrm{Im}\ \nabla^A\big|_{\Omega^0(\Sigma, \mathrm{End}^{\mathrm{anti}}(E))}} = H^1_{\nabla^A}(\Sigma, \mathrm{End}^{\mathrm{anti}}(E)). \tag{5.5.27}$$

If $X, Y \in H^1_{\nabla^A}(\Sigma, \mathrm{End}^{\mathrm{anti}}(E))$, then

$$\omega_{\mathcal{A}_0/\Sigma G}(X^0, Y^0) = \int_\Sigma \mathrm{Tr}^E[X \wedge Y] \tag{5.5.28}$$

defines a symplectic form on $\mathcal{A}_0/\Sigma G$.

Remark 5.5.4. The distinct flat vector bundles on Σ are in one-to-one correspondence to the equivariant class of the conjugate representation of $\pi_1(\Sigma) \to U(n)$, here $\pi_1(\Sigma)$, the fundamental group of Σ, is the group with $2r$ generators with r the genus of Σ satisfying

$$\prod_{j=1}^r u_j v_j u_j^{-1} v_j^{-1} = 1. \tag{5.5.29}$$

Moreover, the space of equivariant classes of flat connection on Σ is isomorphic to the following space:

$$\left\{ (u_j, v_j) \in U(n) \times \cdots \times U(n) : \prod_{j=1}^r u_j v_j u_j^{-1} v_j^{-1} = I \right\} / U(n) \tag{5.5.30}$$

5.6 Bibliographic notes

For Remark 5.1.2, cf. [28], [41, Example A.5].

Basic reference for Section 5.2 is [16]. Corollary 5.2.2 is [16, Lemma 4.2.5], Theorem 5.2.3 is [16, Theorem 4.2.9]. Theorem 5.2.5 is [16, Theorem 5.8.1]. Theorem 5.2.6 is a combination of Theorem 2.1.15 and [16, §5.7]. Theorem 5.2.8 is [16, Theorem 5.4.5]. Theorem 5.2.9: first part is [16, Lemma 5.4.3, Prop. 5.4.12], b) is [16, Prop. 5.2.16], c) is [16, Prop. 5.7.1].

About the Koszul-Malgrange integrability theorem in Exercise 5.4.2, we can deduce it from Newlander-Nirenberg theorem for integrable almost complex structure as in [40, Proposition 1.3.7], also a direct proof in [24, Theorem 2.1.53, §2.2.2].