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Toeplitz Quantization and Symplectic Reduction

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Dedicated to the memory of Professor Shiing-Shen Chern

In [9], we announced the asymptotic expansion of the G-invariant Bergman kernel of the spin^c Dirac operator associated with high tensor powers of a positive line bundle on a symplectic manifold. In this note, we describe several consequences of our asymptotic expansion of the G-invariant Bergman kernel in the Kähler case, especially, we study the Toeplitz quantization in the framework of the symplectic reduction. The full details can be found in [10].

1. Toeplitz quantization

Let (X,ω) be a compact Kähler manifold with Kähler form ω, and $\dim_{\mathbb{C}} X = n$. Let J be the almost complex structure on the real tangent bundle TX. Let $g^{TX}(v,w) := \omega(v, Jw)$ be the corresponding Riemannian metric on TX.

Let L be a holomorphic line bundle over X with Hermitian metric h^L. Let ∇^L be the holomorphic Hermitian connection on (L, h^L) with curvature $R^L := (\nabla^L)^2$. We suppose that (L, h^L) is a pre-quatum line bundle of (X,ω), i.e.

$$\frac{\sqrt{-1}}{2\pi} R^L = \omega. \tag{1.1}$$

According the geometric quantization introduced by Kostant and

Souriau, the Kähler manifold (X,ω) is the classical phase space and $H^0(X,L)$, the space of holomorphic sections of L on X, is the quantum space. The set of classical observables is the Poisson algebra $\mathscr{C}^\infty(X)$, the quantum observables are the linear operators on $H^0(X,L)$. The semi-classical limit is a way to relate the classical and quantum observables, basically, for any $p \in \mathbb{N}$, we replace L by L^p, then we obtain a sequence of spaces $H^0(X,L^p)$, the semi-classical limit is the process of $p \to \infty$. In this note, we will restrict ourself to a family of quantum observables : Toeplitz operators.

Let $\{\,,\}$ be the Poisson bracket on $(X, 2\pi\omega)$: for $f_1, f_2 \in \mathscr{C}^\infty(X)$, if ξ_{f_2} is the Hamiltonian vector field generated by f_2 which is defined by $2\pi i_{\xi_{f_2}}\omega = df_2$, then

$$\{f_1, f_2\}(x) = (\xi_{f_2}(df_1))(x). \tag{1.2}$$

Let dv_X be the Riemannian volume form of (X, g^{TX}), then $dv_X = \omega^n/n!$. We define the L^2-scalar product $\langle\ \rangle$ on $\mathscr{C}^\infty(X,L^p)$ by

$$\langle s_1, s_2\rangle = \int_X \langle s_1, s_2\rangle_{L^p}(x)\, dv_X(x)\,. \tag{1.3}$$

Let Π_p denote the orthogonal projection from $(L^2(X,L^p), \langle\ \rangle)$, the space of L^2 sections of L^p on X, to $H^0(X,L^p)$, the space of holomorphic sections of L^p on X.

For any $f \in \mathscr{C}^\infty(X)$, consider the Toeplitz operators

$$T_p(f) = \Pi_p f \Pi_p : H^0(X,L^p) \to H^0(X,L^p). \tag{1.4}$$

We denote by $\|T_p(f)\|$ the operator norm of $T_p(f)$ with respect to the scalar product $\langle\ \rangle$.

We now state two results of Bordemann-Meinrenken-Schlichenmaier[2], concerning the asymptotic behavior of $T_p(f)$ as $p \to +\infty$.

Theorem **1.1.** *As $p \to +\infty$, one has*

$$\lim_{p\to+\infty} \|T_p(f)\| = \|f\|_\infty, \tag{1.5a}$$

$$[T_p(f), T_p(g)] = \frac{1}{\sqrt{-1}p} T_p(\{f,g\}) + O(p^{-2}). \tag{1.5b}$$

2. Hamiltonian action and symplectic reduction

Let E be a holomorphic vector bundle on X with Hermitian metric h^E. Let ∇^E be the holomorphic Hermitian connection on (E, h^E). Let G be a compact connected Lie group. Let $\mathfrak{g}$ be the Lie algebra of G.

Suppose that G acts holomorphically on X, and the action of G lifts holomorphically on L, E and preserves the metrics h^L, h^E. Then the action of G preserves ω, the connections ∇^L, ∇^E.

For $K \in \mathfrak{g}$, we denote by K^X the vector field on X generated by K, and by L_K the infinitesimal action induced by K on the corresponding vector bundles. Let $\mu : X \to \mathfrak{g}^*$ be defined by

$$2\pi\sqrt{-1}\mu(K) := \nabla^L_{K^X} - L_K,\ K \in \mathfrak{g}. \tag{2.1}$$

Then μ is the corresponding **moment map**, i.e. for any $K \in \mathfrak{g}$,

$$d\mu(K) = i_{K^X}\omega. \tag{2.2}$$

Definition 2.1. The Marsden-Weinstein **symplectic reduction** space X_G is defined to be

$$X_G = \mu^{-1}(0)/G. \tag{2.3}$$

Basic assumption: $0 \in \mathfrak{g}^*$ is a regular value of the moment map $\mu : X \to \mathfrak{g}^*$.

Then $\mu^{-1}(0)$ is a closed manifold. For simplicity, also assume that G acts on $\mu^{-1}(0)$ freely, then X_G is a compact smooth manifold and carries an induced symplectic form ω_G.

Moreover, J induces a complex structure J_G on TX_G such that $\omega_G(\cdot, J_G\cdot)$ determines a Riemannian metric g^{TX_G} on TX_G. Thus (X_G, ω_G, J_G) is also Kähler.

The line bundle (L, h^L) induces a Hermitian line bundle (L_G, h^{L_G}) on X_G by identifying G-invariant sections of L on $\mu^{-1}(0)$. In fact (L_G, h^{L_G}) is a pre-quantized holomorphic line bundle over (X_G, ω_G), cf. [5].

In the same way, (E, h^E) induces a holomorphic Hermitian vector bundle (E_G, h^{E_G}) on X_G.

3. Toeplitz quantization and symplectic reduction

We now assume that a connected compact Lie group acts on (X, ω, J, L) in a Hamiltonian way as before.

Let $i : \mu^{-1}(0) \hookrightarrow X$ denote the canonical embedding. We assume as before that 0 is a regular value of μ and G acts on $\mu^{-1}(0)$ freely. Then

$$\pi : \mu^{-1}(0) \to X_G$$

is a principal fibration with fiber G.

Let $H^0(X, L^p \otimes E)^G$ be the G-invariant part of $H^0(X, L^p \otimes E)$, the space of holomorphic sections of $L^p \otimes E$ on X. Let $\mathscr{C}^\infty(X, L^p \otimes E)^G$ (resp.

$\mathscr{C}^\infty(\mu^{-1}(0), L^p \otimes E)^G)$ be the G-invariant smooth sections of $L^p \otimes E$ on X (resp. $\mu^{-1}(0)$). Let $\pi_G : \mathscr{C}^\infty(\mu^{-1}(0), L^p \otimes E)^G \to \mathscr{C}^\infty(X_G, L^p_G \otimes E_G)$ be the natural identification. By a result of Zhang[13], for p large enough, the map

$$\pi_G \circ i^* : \mathscr{C}^\infty(X, L^p \otimes E)^G \to \mathscr{C}^\infty(X_G, L^p_G \otimes E_G)$$

induces a natural isomorphism

$$\sigma_p = \pi_G \circ i^* : H^0(X, L^p \otimes E)^G \to H^0(X_G, L^p_G \otimes E_G). \tag{3.1}$$

(When $E = \mathbb{C}$, this result was first proved by Guillemin-Sternberg[5].)

Let dv_{X_G} be the Riemannian volume form on (X_G, g^{TX_G}). Let $\Pi_{G,p}$ be the orthogonal projection from $\mathscr{C}^\infty(X_G, L^p_G \otimes E_G)$ (with the scalar product $\langle\ \rangle$ induced by h^{L_G}, h^{E_G} and dv_{X_G} as in (1.3)), onto $H^0(X_G, L^p_G \otimes E_G)$.

Definition 3.1. A family of operators $T_p : H^0(X_G, L^p_G \otimes E_G) \to H^0(X_G, L^p_G \otimes E_G)$ is a Toeplitz operator if there exists a sequence of sections $g_l \in \mathscr{C}^\infty(X_G, \mathrm{End}(E_G))$ with an asymptotic expansion $g(\cdot, p)$ of the form $\sum_{l=0}^\infty p^{-l} g_l(x) + \mathscr{O}(p^{-\infty})$ in the $\mathscr{C}^\infty$ topology such that

$$T_p = \Pi_{G,p} g(\cdot, p) \Pi_{G,p} + \mathscr{O}(p^{-\infty}). \tag{3.2}$$

We call $g_0(x)$ the principal symbol of T_p.

For any $x \in X_G$, let $\mathrm{vol}(\pi^{-1}(x))$ be the volume of the orbit $\pi^{-1}(x)$ equipped with the metric induced by g^{TX}. We define the potential function

$$h(x) = \sqrt{\mathrm{vol}(\pi^{-1}(x))}. \tag{3.3}$$

For any $p > 0$, let P^G_p denote the orthogonal projection from $(\mathscr{C}^\infty(X, L^p \otimes E), \langle\ \rangle)$ to $H^0(X, L^p \otimes E)^G$. Set

$$\sigma^G_p = \sigma_p P^G_p : \mathscr{C}^\infty(X, L^p \otimes E) \to H^0(X_G, L^p_G \otimes E_G). \tag{3.4}$$

Let

$$(\sigma^G_p)^* : H^0(X_G, L^p_G \otimes E_G) \to \mathscr{C}^\infty(X, L^p \otimes E)$$

denote the adjoint of σ_p.

Theorem 3.1. *For any $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$, let $f^G \in \mathscr{C}^\infty(X_G, \mathrm{End}(E_G))$ denote the associated G-invariant section defined by $f^G(x) = \int_G g f(g^{-1}x) dg$, here dg is a Haar measure on G. Then*

$$T_p(f) = p^{-\frac{\dim G}{2}} \sigma^G_p f (\sigma^G_p)^* : H^0(X_G, L^p_G \otimes E_G) \to H^0(X_G, L^p_G \otimes E_G) \tag{3.5}$$

is a Toeplitz operator with principal symbol $2^{\frac{\dim G}{2}} \frac{f^G}{h^2}(x)$. Especially,

$$T_p(f) = \Pi_{G,p} 2^{\frac{\dim G}{2}} \frac{f^G}{h^2} \Pi_{G,p} + \mathscr{O}(1/p) \tag{3.6}$$

as $p \to +\infty$. In particular, $p^{-\dim G/2} \sigma^G_p (\sigma^G_p)^*$ is a Toeplitz operator with principal symbol $2^{\dim G/2}/h^2$.

Corollary 3.1. *For any $f_1, f_2 \in \mathscr{C}^\infty(X)$, we identify them as sections of $\mathrm{End}(E)$ by multiplications, then one has*

$$[T_p(f_1), T_p(f_2)] = \frac{2^{\dim G}}{\sqrt{-1}p} \Pi_{G,p} \left\{ \frac{f_1^G}{h^2}, \frac{f_2^G}{h^2} \right\} \Pi_{G,p} + \mathscr{O}(p^{-2}). \tag{3.7}$$

One can view this corollary as a generalization of the Bordemann-Meinrenken-Schlichenmaier theorem, Theorem 1.1, in the framework of geometric quantization. If $E = \mathbb{C}$ and $G = \{1\}$, Corollary 3.1 is (1.5b). If $G = \{1\}$ and general E, Corollary 3.1 was obtained in [7], [8].

On the other hand, if one defines the unitary operator

$$\Sigma_p = (\sigma^G_p)^* (\sigma^G_p (\sigma^G_p)^*)^{-1/2} : H^0(X_G, L^p_G \otimes E_G) \to \mathscr{C}^\infty(X, L^p \otimes E), \tag{3.8}$$

then one has the following result:

Theorem 3.2. *For any $f \in \mathscr{C}^\infty(X, \mathrm{End}(E))$,*

$$T^G_p(f) = \Sigma^*_p f \Sigma_p : H^0(X_G, L^p_G \otimes E_G) \to H^0(X_G, L^p_G \otimes E_G) \tag{3.9}$$

is a Toeplitz operator on X_G with principal symbol f^G.

Remark 3.1. If $E = \mathbb{C}$, Paoletti[11] also claimed that $p^{-\frac{\dim G}{2}} \sigma^G_p (\sigma^G_p)^*$ is a Toeplitz operator. When $G = T^k$ is a torus, and $E = \mathbb{C}$, Theorem 3.2 was first proved by Charles[3].

Let $\langle\, , \rangle_{L^p_G \otimes E_G}$ be the metric on $L^p_G \otimes E_G$ induced by h^{L_G} and h^{E_G}. In view of Tian and Zhang's analytic approach (cf. [12], (3.54)) of geometric quantization conjecture of Guillemin-Sternberg, the natural Hermitian product on $\mathscr{C}^\infty(X_G, L^p_G \otimes E_G)$ is the following weighted Hermitian product $\langle\, , \rangle_h$:

$$\langle s_1, s_2 \rangle_h = \int_{X_G} \langle s_1, s_2 \rangle_{L^p_G \otimes E_G}(x_0) h^2(x_0)\, dv_{X_G}(x_0). \tag{3.10}$$

Theorem 3.3. *The isomorphism $(2p)^{-\frac{\dim G}{4}} \sigma_p$ is an asymptotic isometry from $(H^0(X, L^p \otimes E)^G, \langle\, , \rangle)$ onto $(H^0(X_G, L^p_G \otimes E_G), \langle\, , \rangle_h)$: i.e. if $\{s^p_i\}_{i=1}^{d_p}$*

is an orthonormal basis of $(H^0(X, L^p \otimes E)^G, \langle\, , \rangle)$, *then*

$$(2p)^{-\frac{\dim G}{2}} \langle \sigma_p s_i^p, \sigma_p s_j^p \rangle_h = \delta_{ij} + \mathscr{O}(\frac{1}{p}). \tag{3.11}$$

4. The asymptotic expansion of the G-invariant Bergman kernel

Definition 4.1. The G-invariant Bergman kernel $P_p^G(x,x')$ with $x,\ x' \in X$ is the smooth kernel of the orthogonal projection $P_p^G : \mathscr{C}^\infty(X, L^p \otimes E) \to H^0(X, L^p \otimes E)^G$ with respect to $dv_X(x')$.

Our proof of the results in Section 3 relies on the asymptotic behavior as $p \to +\infty$ of the G-invariant Bergman kernel $P_p^G(x,x')$. We now describe some behavior of $P_p^G(x,y)$, as $p \to +\infty$.

Let U be an arbitrary (fixed) small open G-invariant neighborhood of $\mu^{-1}(0)$. At first, we have that for any $x,\ x' \in X \setminus U$, as $p \to +\infty$,

$$|P_p^G(x,x')|_{\mathscr{C}^\infty} = \mathscr{O}(p^{-\infty}). \tag{4.1}$$

This result shows that when $p \to +\infty$, $P_p^G(x,x')$ "localizes" near $\mu^{-1}(0)$ (and thus close to X_G). The main technical result of [9]. Theorem 2.2, and [10]. Theorem 0.2 is the asymptotic expansion of $P_p^G(x,x')$ for $x,\ x' \in U$ when $p \to \infty$ whose proofs use techniques adapting from the works of Bismut-Lebeau [1], Dai-Liu-Ma[4] and Ma-Marinescu[6]. One key step is to deform the Laplacian of the spin^c Dirac operator by a Casimir type operator. We refer the readers to [9], [10] for the details.

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