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QUANTIZATION COMMUTES WITH REDUCTION, A SURVEY*

Dedicated to the memory of Professor Jiarong YU

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Abstract We review the themes relating to the proposition that "quantization commutes with reduction" ($[Q, R] = 0$), from symplectic manifolds to Cauchy-Riemann manifolds.

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From September 1989 until July 1993, I was a student at the Sino-French mathematics class in Wuhan University founded by Professor Jiarong Yu. As a young person from the countryside, it was the most precious chance of my life, and the starting point of my mathematical career. I am very lucky to have witnessed Professor Yu's great contribution to the development of modern mathematical education in China.

In this note, we will review some recent progress on the idea "Quantization commutes with reduction", or briefly, that "$[Q, R] = 0$", which first appeared as the famous Guillemin-Sternberg conjecture for compact symplectic manifolds.

Note that the phase space of a classical mechanical system is a symplectic manifold. Geometric quantization, introduced in the 1960's by Kostant and Souriau, gives a geometric method to properly quantize classical mechanical systems. To quantize a compact symplectic manifold, i.e., to associate a Hilbert space, we need a (prequantum) line bundle whose first Chern form equals the symplectic form.

Bott suggested that the Hilbert space appearing in the quantization should be the kernel of the Dirac operator acting on spinor bundles twisted by the line bundle. The way in which symmetries of the classical systems are reflected in the quantization has been formulated into the principle that "quantization commutes with reduction".

Let (M, ω) be a symplectic manifold with a prequantum line bundle L. Assume that a compact connected Lie group G acts on M, and that the action lifts to L. Then the quantization of M should be a G-virtual representation, and it is interesting to determine the multiplicity of the irreducible representations of G.

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The Guillemin-Sternberg conjecture that "quantization commutes with reduction" gives a precise geometric answer to this problem. By using the associated moment map when M is compact, roughly, they conjectured that the following diagram commutes:

$$\begin{tikzcd} (M,L) \arrow[r, "\text{quantization}"] \arrow[d, "\text{reduction}"] & Q(L) \arrow[d, "\text{reduction}"] \\ (M_\nu, L_\nu) \arrow[r, "\text{quantization}"] & Q(L_\nu). \end{tikzcd} \tag{0.1}$$

New difficulties appear when the manifold M is no longer supposed to be compact, since in this case the index of the Dirac operator is not well defined. In her ICM 2006 plenary lecture, Michèle Vergne proposed to replace this by a certain transversal index introduced by Atiyah, under the natural hypothesis that the moment map is proper, and that the zero-set of the vector field induced by the moment map is compact. She conjectured that the idea that "quantization commutes with reduction" still holds in this case.

This note is organized as follows. In Section 1, we review the principle that "quantization commutes with reduction" in the symplectic case; in particular, we discuss our solution with Zhang [11, 12] on Vergne's conjecture. In Section 2, we review our recent work with Hsiao and Marinescu [7], on the principle "quantization commutes with reduction" for Cauchy-Riemann (CR) manifolds. An important difference between the CR setting and the symplectic setting is that the quantum spaces in the case of compact symplectic manifolds are finite dimensional, whereas for the compact strictly pseudoconvex CR manifolds that we consider, the quantum spaces consist of CR functions and are infinite dimensional.

Due to space limitations, we only cite few references. One can find more comments, references and motivations in [9, 10] and [25].

1 Quantization Commutes with Reduction on Symplectic Manifolds

This Section is organized as follows. In Section 1.1, we recall the definition of the Dirac operator on an almost complex manifold and the Atiyah-Singer index theorem. In Section 1.2, we review the Guillemin-Sternberg conjecture for compact symplectic manifolds. In Section 1.3, we explain our solution to Vergne's conjecture regarding noncompact symplectic manifolds. In Section 1.4, we give the refinement of $[Q,R]=0$ in the compact Kähler case.

1.1 Dirac operators

Let M be a manifold of real dimension $2n$ with a compatible almost complex structure J. We endow M with a Riemannian metric g^{TM} compatible with J, i.e., $g^{TM}(J\cdot, J\cdot) = g^{TM}(\cdot,\cdot)$. Let (E, h^E) be a Hermitian vector bundle on M with Hermitian connection ∇^E and curvature $R^E = (\nabla^E)^2$.

The almost complex structure J induces a splitting of the complexification of the tangent bundle $TM \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}M \oplus T^{(0,1)}M$, where $T^{(1,0)}M$ and $T^{(0,1)}M$ are the eigenbundles of J corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. Let $T^{*\,(0,1)}M$ be the dual space of $T^{(0,1)}M$.

Let dv_M be the Riemannian volume form of (TM, g^{TM}). The L^2–Hermitian product induced by g^{TM}, h^E on the space $\Omega^{0,\bullet}(M,E)$ of smooth sections of $\Lambda^\bullet(T^{*\,(0,1)}M) \otimes E$ is given by

$$\langle s_1, s_2\rangle = \int_M \langle s_1(x), s_2(x)\rangle\, dv_M(x)\,. \tag{1.1}$$

For any $v \in T^{(1,0)}M$, let $v^* \in T^{*\,(0,1)}M$ be the metric dual of v, so

$$\mathbf{c}(v) = \sqrt{2}\, v^*\wedge, \quad \mathbf{c}(\overline{v}) = -\sqrt{2}\, i_{\overline{v}} \tag{1.2}$$

define the Clifford actions of v and $\overline{v}$ on $\Lambda^{0,\bullet} := \Lambda^{\bullet}(T^{*\,(0,1)}M)$, where $\wedge$ and i denote the exterior and interior multiplications, respectively. In particular, for any $U, V \in TM$, we have

$$\mathbf{c}(U)\mathbf{c}(V) + \mathbf{c}(V)\mathbf{c}(U) = -2\,\langle U, V\rangle\,, \tag{1.3}$$

where $\langle U, V\rangle$ is the scalar product of U, V in (TM, g^{TM}).

Consider the Levi–Civita connection ∇^{TM} of (TM, g^{TM}) with associated curvature R^{TM}. Let $\nabla^{T^{(1,0)}M}$ be the connection on $T^{(1,0)}M$ induced by projecting ∇^{TM}; $\nabla^{T^{(1,0)}M}$ induces the connection $\nabla^{\det}$ on $\det(T^{(1,0)}M) := \Lambda^n(T^{(1,0)}M)$. The Clifford connection ∇^{Cl} on $\Lambda^{0,\bullet}$ is induced canonically by ∇^{TM} and $\nabla^{\det}$ (cf. [10, §1.3]). Finally, let $\nabla^{\Lambda^{0,\bullet}\otimes E}$ be the connection on $\Lambda^{0,\bullet}\otimes E$ induced by ∇^{Cl} and ∇^E.

We recall briefly the construction of the Clifford connection ∇^{Cl} here. Let $\{w_j\}_{j=1}^n$ be a local orthonormal frame of $T^{(1,0)}M$ with dual frame $\{w^j\}_{j=1}^n$. Then

$$e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j) \quad \text{and} \quad e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j)\,, \quad j = 1, \cdots, n \tag{1.4}$$

form an orthonormal frame of TM. Let $\Gamma^{TM} \in T^*M \otimes \mathrm{End}(TM)$, $\Gamma^{\det}$ be the connection forms of ∇^{TM}, $\nabla^{\det}$ associated with the frames $\{e_j\}$, $w_1\wedge\cdots\wedge w_n$, i.e.,

$$\nabla^{TM}_{e_i} e_j = \Gamma^{TM}(e_i)e_j, \quad \Gamma^{\det} = \textstyle\sum_j \langle \Gamma^{TM} w_j, \overline{w}_j\rangle. \tag{1.5}$$

The Clifford connection ∇^{Cl} on $\Lambda(T^{*(0,1)}M)$ is defined for the frame $\{\overline{w}^{j_1}\wedge\cdots\wedge\overline{w}^{j_k}, 1 \leqslant j_1 < \cdots < j_k \leqslant n\}$ by the local formula

$$\nabla^{\mathrm{Cl}} = d + \frac{1}{4}\sum_{i,j} \langle \Gamma^{TM} e_i, e_j\rangle \mathbf{c}(e_i)\mathbf{c}(e_j) + \frac{1}{2}\Gamma^{\det}. \tag{1.6}$$

Definition 1.1 The spin^c Dirac operator D^E is defined by

$$D^E := \sum_j \mathbf{c}(e_j)\nabla^{\Lambda^{0,\bullet}\otimes E}_{e_j} : \Omega^{0,\bullet}(M,E) \longrightarrow \Omega^{0,\bullet}(M,E)\,, \quad D^E_{\pm} := D^E|_{\Omega^{0,\frac{\text{even}}{\text{odd}}}}\,. \tag{1.7}$$

The operator D^E is a formally self–adjoint, first order elliptic differential operator on $\Omega^{0,\bullet}(M,E)$, which interchanges $\Omega^{0,\mathrm{even}}(M,E)$ and $\Omega^{0,\mathrm{odd}}(M,E)$ (cf. [10, §1.3]).

If M is compact, then $\mathrm{Ker}\left(D^E_+\right)$ and $\mathrm{Ker}\left(D^E_-\right)$ are finite dimensional Hilbert spaces and the quantization space of E is defined as their formal difference,

$$Q(E) := \mathrm{Ind}(D^E_+) := \mathrm{Ker}\left(D^E_+\right) - \mathrm{Ker}\left(D^E_-\right). \tag{1.8}$$

To explain the Atiyah-Singer index theorem which computes the virtual dimension of $Q(E)$ by using characteristic numbers, we need to introduce first some characteristic classes. For any Hermitian (complex) vector bundle (F, h^F) with Hermitian connection ∇^F and curvature R^F

on M, set

$$\begin{aligned} \mathrm{ch}(F,\nabla^F) &:= \mathrm{Tr}\left[\exp\left(\frac{-R^F}{2\pi\sqrt{-1}}\right)\right], \\ c_1(F,\nabla^F) &:= \mathrm{Tr}\left[\frac{-R^F}{2\pi\sqrt{-1}}\right], \\ \mathrm{Td}(F,\nabla^F) &:= \det\left(\frac{R^F/(2\pi\sqrt{-1})}{\exp(R^F/(2\pi\sqrt{-1}))-1}\right). \end{aligned} \tag{1.9}$$

These are closed real differential forms on M and their cohomology classes do not depend on the choice of the metric h^F and connection ∇^F. The corresponding cohomology classes are called the Chern character of F, the first Chern class of F, and the Todd class of F, respectively, and we denote them by $\mathrm{ch}(F)$, $c_1(F)$ and $\mathrm{Td}(F) \in H^{2\bullet}(M,\mathbb{R})$.

Theorem 1.2 (Atiyah-Singer index theorem cf. [2, §4.1], [10, Th. 1.3.9]) If M is compact, we have

$$\dim Q(E) = \int_M \mathrm{Td}(T^{(1,0)}M)\mathrm{ch}(E). \tag{1.10}$$

In particular, the virtual dimension of $Q(E)$ does not depend on the choice of g^{TM} or the metric and connection on E. If $\mathrm{Ker}(D^E_-)=0$, then the quantization space $Q(E)$ is an ordinary vector space.

1.2 Quantization commutes with reduction

We explain now the idea of the geometric quantization introduced by Kostant [8] and Souriau [21].

Let (M,J,ω) be a compact symplectic manifold of real dimension $2n$ with a compatible almost complex structure J, i.e., $g^{TM}=\omega(\cdot,J\cdot)$ is a J-invariant metric on TM.

Let (L,h^L) be a Hermitian line bundle over M endowed with a Hermitian connection ∇^L with curvature $R^L=(\nabla^L)^2$. We assume that (L,h^L,∇^L) satisfies the prequantization condition, that is, that

$$\omega = \frac{\sqrt{-1}}{2\pi}R^L. \tag{1.11}$$

In this case, we say that (L,h^L,∇^L) is a prequantum line bundle on M.

Let G be a compact connected Lie group with Lie algebra $\mathfrak{g}$. We assume that G acts on the left on M and that this action lifts to L. Moreover, we assume that G preserves g^{TM}, J, h^L and ∇^L.

Thus the G-action commutes with the Dirac operator D^L, and $\mathrm{Ker}\left(D^L_\pm\right)$ are finite dimensional G-representations. The quantization space $Q(L)$ of L (cf. (1.8)) is an element in the representation ring $R(G)$ of G.

For $K\in\mathfrak{g}$, let K^M be the vector field on M generated by K, and let L_K be the corresponding Lie derivative. Let $\Lambda^*_+\subset\mathfrak{g}^*$ be the set of dominant weights, and let V^G_γ be the irreducible representation of G with highest weight $\gamma\in\Lambda^*_+$. Let $Q(L)_\gamma\in\mathbb{Z}$ be the multiplicity of V^G_γ in $Q(L)$. Then we have

$$Q(L)=\bigoplus_{\gamma\in\Lambda^*_+} Q(L)_\gamma\cdot V^G_\gamma \in R(G), \tag{1.12}$$

and there are only finitely many $\gamma\in\Lambda^*_+$ such that $Q(L)_\gamma\neq 0$.

It is not easy to read off $Q(L)_\gamma$ directly from the Atiyah-Bott-Segal-Singer equivariant index theorem for its character. Guillemin and Sternberg [4] suggested a geometric way to compute $Q(L)_\gamma$ by using the associated moment map.

Definition 1.3 The moment map $\mu : M \to \mathfrak{g}^*$ is defined by the Kostant formula [8]

$$2\sqrt{-1}\pi\mu(K) = \nabla^L_{K^M} - L_K, \quad \text{for } K \in \mathfrak{g}. \tag{1.13}$$

Then μ is G-equivariant and one has that $i_{K^M}\omega = d\,\mu(K)$.

For a regular value $\nu \in \mathfrak{g}^*$ of μ, the Marsden-Weinstein symplectic reduction $M_\nu := \mu^{-1}(G\cdot \nu)/G$ is a compact symplectic orbifold with the symplectic form ω_ν induced by ω. Moreover, L (resp. J) induces a prequantum orbifold line bundle L_ν (resp. an almost complex structure J_ν) over (M_ν, ω_ν). One can then construct the associated spin^c Dirac operator (twisted by L_ν), $D^{L_\nu}_+$ on M_ν, of which we have the index $Q\left(L_\nu\right) \in \mathbb{Z}$ (identified as the virtual dimension of $Q\left(L_\nu\right)$ in (1.8)).

If $\gamma \in \Lambda^*_+$ is not a regular value of μ, then by [16] (cf. [17, §7.4] for a standard perturbation definition), $Q(L_\gamma)$ is still well defined. Now we can state the

Guillemin-Sternberg conjecture For any $\gamma \in \Lambda^*_+$,

$$Q(L)_\gamma = Q\left(L_\gamma\right). \tag{1.14}$$

By the classical shifting trick (i.e., by working on $M \times \mathcal{O}_\gamma$, where $\mathcal{O}_\gamma = G\cdot\gamma$ is the orbit of the co-adjoint action of G on $\mathfrak{g}^*$), we only need to prove (1.14) for $\gamma = 0$.

This conjecture was proved by Meinrenken [14] and Vergne [24] when G is abelian, and by Meinrenken [15] and Meinrenken-Sjamaar[16] for non-abelian groups G, by using the symplectic cut technique of Lerman.

Tian and Zhang [22] gave an analytic proof of the Guillemin-Sternberg conjecture using a deformation of the Dirac operator which is associated with the function $|\mu|^2$, and also the analytic localization technique in the local index theory developed by Bismut-Lebeau [3]. Their approach works for a general vector bundle E satisfying certain positivity conditions [22, (4.2)] (used afterwards by Paradan [17, p. 445]), and also for manifolds with boundary [23]. Paradan [17] developed later a K-theoretic approach by making use of the theory of transversally elliptic operators; see [25] for a survey and complete references on this subject.

1.3 $[Q,R]=0$: the noncompact case

We use the same notation and assumption as in Section 1.2, but we now assume that M is noncompact.

Then the quantization space $Q(L) = \mathrm{Ind}(D^L_+)$ of L is not well defined, since usually D^L is not a Fredholm operator, and we need to make precise the self-adjoint extension of $D^L|_{\Omega^{0,\bullet}_0(M,L)}$, where $\Omega^{0,\bullet}_0(M,L)$ denotes the space of sections with compact support.

Let $\tau : TM \to M$ be the natural projection. Following [1, p. 7] (cf. [17, §3]), set

$$T_G M = \left\{(x,v) \in T_x M : \left\langle v, K^M(x)\right\rangle = 0 \text{ for all } K \in \mathfrak{g}\right\}. \tag{1.15}$$

We suppose that the moment map $\mu : M \to \mathfrak{g}^*$ is proper. Then the right hand side of (1.14) is well defined.

We identify $\mathfrak{g}$ with $\mathfrak{g}^*$ by using an Ad_G-invariant metric on $\mathfrak{g}$. Let $\mu^M(x) := (\mu(x))^M(x)$ $(x \in M)$ be the vector field induced by $\mu : M \to \mathfrak{g}$.

We suppose for the moment that $\{x \in M : \mu^M(x) = 0\}$ is compact.

Recall that $\mathbf{c}(\cdot)$ is the Clifford action defined in (1.2). For $x \in M,\ \xi \in T_xM$, set

$$\begin{aligned}\sigma^M_{L,\mu}(x,\xi) = \tau^*\left(\sqrt{-1}\mathbf{c}(\xi+\mu^M)\otimes \mathrm{Id}_L\right)\big|_{(x,\xi)}\\ : \tau^*(\Lambda^{\mathrm{even}}(T^{*(0,1)}M)\otimes L) \to \tau^*(\Lambda^{\mathrm{odd}}(T^{*(0,1)}M)\otimes L).\end{aligned} \tag{1.16}$$

Then $\sigma^M_{L,\mu}$ is a transversally elliptic symbol[1] on T_GM in the sense of Atiyah [1, §1, §3] and Paradan [17, §3], [18, §3], which determines a transversal index $\mathrm{Ind}\left(\sigma^M_{L,\mu}\right)$ in the formal representation ring $R[G]$ of G,

$$\mathrm{Ind}\left(\sigma^M_{L,\mu}\right) = \bigoplus_{\gamma\in\Lambda^*_+} \mathrm{Ind}_\gamma\left(\sigma^M_{L,\mu}\right)\cdot V^G_\gamma \in R[G]. \tag{1.17}$$

The index $\mathrm{Ind}\left(\sigma^M_{L,\mu}\right)$ does not depend on g^{TM}, h^L, ∇^L; it depends only on the homotopy classes of J, μ^M. The set $\{\gamma \in \Lambda^*_+ : \mathrm{Ind}_\gamma\left(\sigma^M_{L,\mu}\right) \neq 0\}$ can be infinite. Michèle Vergne suggested to use $\mathrm{Ind}_\gamma\left(\sigma^M_{L,\mu}\right)$ to replace the left hand side of (1.14).

Vergne's conjecture (ICM 2006 plenary lecture [26, §4.3]) If $\mu : M \to \mathfrak{g}^*$ is proper and if $\{x \in M : \mu^M(x) = 0\}$ is compact, then for any $\gamma \in \Lambda^*_+$,

$$\mathrm{Ind}_\gamma\left(\sigma^M_{L,\mu}\right) = Q\left(L_\gamma\right). \tag{1.18}$$

Special cases of this conjecture, related to the discrete series of semi-simple Lie groups, have been proved by Paradan [18].

For $a > 0$, set $M_a = \{x \in M : |\mu|^2(x) \leqslant a\}$. If a is a regular value of $|\mu|^2$, then M_a is a compact manifold with boundary ∂M_a, and μ^M is nowhere zero on ∂M_a. Thus $\sigma^{M_a}_{L,\mu}$ is a transversally elliptic symbol on M_a.

Theorem 1.4 (Quantization commutes with reduction, Ma-Zhang [11, 12]) Suppose that $\mu : M \to \mathfrak{g}^*$ is proper. For any $\gamma \in \Lambda^*_+$, there exists $a_\gamma > 0$ such that the function $a \mapsto \mathrm{Ind}_\gamma\left(\sigma^{M_a}_{L,\mu}\right)$ is constant on $\{a > a_\gamma : a \text{ is regular value of } |\mu|^2\}$. Denote by $Q(L)_\gamma$ this constant. Then, for any $\gamma \in \Lambda^*_+$, we have

$$Q(L)_\gamma = Q(L_\gamma). \tag{1.19}$$

If $\{x \in M : \mu^M(x) = 0\}$ is compact, then $Q(L)_\gamma = \mathrm{Ind}_\gamma\left(\sigma^M_{L,\mu}\right)$. Therefore Theorem 1.4 implies Vergne's conjecture. Note that Paradan [19] gives a new proof of Theorem 1.4 by using symplectic cuts and the wonderful compactifications of the complexification of G of de Concini-Procesi.

Note that the most difficult part of the proof of Theorem 1.4 is to show that the shifting trick (i.e., by working on $M\times\mathcal{O}_\gamma$ to reduce to the case $\gamma = 0$) still works in the current situation.

A new twist was introduced by Paradan and Vergne [20], who considered so-called spin quantization and established a version of $[Q,R] = 0$ in the compact setting. Hochs and Song [6] then established a version of $[Q,R] = 0$ in the noncompact setting along the lines of [12].

Thus the next natural step is to consider $[Q,R] = 0$ for noncompact groups and manifolds. Such a generalization is relevant to physics, since most classical mechanical phase spaces are noncompact, and to representation theory, since the representation theory for noncompact

[1] The symbol $\sigma^M_{L,\mu}$ is the (semi-classical) symbol of Tian-Zhang's [22] deformed Dirac operator $D^L_T = D^L + \sqrt{-1}T\mathbf{c}\left(\mu^M\right)$ in their approach to the Guillemin-Sternberg geometric quantization conjecture.

groups is much more intricate than for compact groups. Besides the problem of how to define the index, we need to work on the multiplicities of an infinite dimensional irreducible representation of G (Cf. [5, 13] and their recent works for the progress in this direction).

1.4 $[Q,R]=0$: the Kähler case

In this subsection, we will explain the refinement of Section 1.2 in the Kähler case.

Let E be a holomorphic vector bundle over a complex manifold M. The operator $\bar\partial^E : \mathscr{C}^\infty(M,E)\to\Omega^{0,1}(M,E)$ is well-defined. Any section $s\in\mathscr{C}^\infty(M,E)$ has the local form $s=\sum_l\varphi_l\xi_l$, where $\{\xi_l\}_{l=1}^r$ is a local holomorphic frame of E and φ_l are smooth functions. In holomorphic coordinates $(z_1,\cdots,z_n)$, we set

$$\bar\partial^E s=\sum_l(\bar\partial\varphi_l)\,\xi_l \qquad \text{with } \bar\partial\varphi_l=\sum_j d\overline{z}_j\frac{\partial}{\partial\overline{z}_j}\varphi_l. \tag{1.20}$$

Since E is holomorphic, the operator $\bar\partial^E$ extends naturally to $\bar\partial^E:\Omega^{0,\bullet}(M,E)\to\Omega^{0,\bullet+1}(M,E)$, verifying that for $\alpha\in\Omega^{0,q}(M), s\in\Omega^{0,\bullet}(M,E)$, we have

$$\bar\partial^E(\alpha\wedge s)=\bar\partial\alpha\wedge s+(-1)^q\alpha\wedge\bar\partial^E s. \tag{1.21}$$

Then, from $\bar\partial^2=0$, we verify that $(\bar\partial^E)^2=0$.

The complex $(\Omega^{0,\bullet}(M,E),\overline{\partial}^E)$ is called the Dolbeault complex and its cohomology, called the Dolbeault cohomology of M with values in E, is denoted by $H^\bullet(M,E)$, i.e., for $q\in\mathbb{N}$,

$$H^q(M,E):=\frac{\operatorname{Ker}(\bar\partial^E|_{\Omega^{0,q}(M,E)})}{\operatorname{Im}(\bar\partial^E|_{\Omega^{0,q-1}(M,E)})}. \tag{1.22}$$

From now on, we assume that (M,ω,J) is a compact Kähler manifold, and that (L,h^L) is a holomorphic Hermitian line bundle with Chern connection ∇^L verifying (1.11).

Let $\overline{\partial}^{L,*}$ be the adjoint of the Dolbeault operator $\overline{\partial}^L$ on $\Omega^{0,\bullet}(M,L)$. In this case, D^L in (1.7) is given by

$$D^L=\sqrt{2}\,(\overline{\partial}^L+\overline{\partial}^{L,*}). \tag{1.23}$$

Thus

$$(D^L)^2=2\,(\overline{\partial}^L\overline{\partial}^{L,*}+\overline{\partial}^{L,*}\overline{\partial}^L) \tag{1.24}$$

preserves the $\mathbb{Z}$-grading on $\Omega^{0,\bullet}(M,L)$. By the Hodge theory, we have that

$$\operatorname{Ker}(D^L|_{\Omega^{0,q}(M,L)})\simeq H^q(M,L). \tag{1.25}$$

Let a compact Lie group G act holomorphically on M, and let the G-action lift on a holomorphic action on L which preserves the metric h^L.

For a G-representation F, we will denote F^G as its G-invariant part.

Let $\mu: M\to\mathfrak{g}^*$ be the associated moment map in (1.13). Assume that G acts freely on $\mu^{-1}(0)$, so the line bundle L_G on $M_G=\mu^{-1}(0)/G$ is defined by

$$\mathscr{C}^\infty(M_G,L_G)=\mathscr{C}^\infty(\mu^{-1}(0),L)^G. \tag{1.26}$$

Let $J_G,\omega_G,h^{L_G},\nabla^{L_G}$ be the objets on M_G induced by J,ω,h^L,∇^L on M as in Section 1.2.

Theorem 1.5 If G acts freely on $\mu^{-1}(0)$, then (M_G,J_G,ω_G) is also a Kähler manifold, and (L_G,h^{L_G}) is a holomorphic Hermitian line bundle over M_G, and ∇^{L_G} is the Chern connection on (L_G,h^{L_G}).

Let N be the normal bundle to $\mu^{-1}(0)$ in M. Then one verifies that JN is the vertical tangent vector bundle of the G-principal bundle $\pi : \mu^{-1}(0) \to M_G$. Let $T^H\mu^{-1}(0)$ be the orthogonal complement of JN in $T\mu^{-1}(0)$. Thus one has the canonical orthogonal splittings

$$TM|_{\mu^{-1}(0)} = N \oplus JN \oplus T^H\mu^{-1}(0), \tag{1.27}$$

and J preserves $N \oplus JN$ and $T^H\mu^{-1}(0)$, and $T^H\mu^{-1}(0)$ is isomorphic to $\pi^*(TM_G)$. Thus we have a canonical splitting of Hermitian vector bundles

$$\begin{aligned} T^{*(0,1)}M|_{\mu^{-1}(0)} &= N^{*(0,1)} \oplus \pi^*(T^{*(0,1)}M_G), \\ \Lambda(T^{*(0,1)}M)|_{\mu^{-1}(0)} &= \Lambda(N^{*(0,1)})\widehat{\otimes}\pi^*(\Lambda(T^{*(0,1)}M_G)), \end{aligned} \tag{1.28}$$

where $N^{(0,1)}$ is the eigenbundle of J associated with the eigenvalue $-\sqrt{-1}$ in $(N \oplus JN) \otimes_{\mathbb{R}} \mathbb{C}$. Let q be the canonical orthogonal projection on $\mu^{-1}(0)$:

$$\begin{aligned} q : \Lambda(T^{*(0,1)}M) \otimes L &= \Lambda(N^{*(0,1)})\widehat{\otimes}\pi^*(\Lambda(T^{*(0,1)}M_G)) \otimes L \\ &\to \Lambda^0(N^{*(0,1)})\widehat{\otimes}\pi^*(\Lambda(T^{*(0,1)}M_G)) \otimes L = \pi^*(\Lambda(T^{*(0,1)}M_G)) \otimes L. \end{aligned} \tag{1.29}$$

The following result of Zhang refines and extends an earlier result of Teleman and Braverman.

Theorem 1.6 (Zhang [27, Theorems 0.1 and 1.2]) If G acts freely on $\mu^{-1}(0)$, then we have the map

$$\psi : (\Omega^{0,\bullet}(M, L)^G, \bar{\partial}^L) \to (\Omega^{0,\bullet}(M_G, L_G), \bar{\partial}^{L_G}), \tag{1.30}$$

by the restriction first on $\mu^{-1}(0)$, and then using (1.26) and (1.29) to induce a section on M_G, this is a morphism of complexes and induces an isomorphism

$$H^j(M, L)^G \simeq H^j(M_G, L_G) \quad \text{ for any } j \geq 0. \tag{1.31}$$

For $j = 0$, (1.31) is the main result of [4], and based on this result, Guillemin-Sternberg made the famous conjecture that "quantization commutes with reduction", explained in Section 1.2 for compact symplectic manifolds.

2 Quantization Commutes with Reduction on CR Manifolds

This Section is organized as follows. In Section 2.1, we recall in detail the definition of Cauchy-Riemann manifolds. In Section 2.2, we explain an important example of CR manifolds: the circle bundle of a holomorphic line bundle on a complex manifold. In Section 2.3, we present our recent work [7] on $[Q, R] = 0$ for CR manifolds.

2.1 CR manifolds and CR functions

Let $(X, T^{1,0}X)$ be a compact, connected and orientable Cauchy-Riemann (CR) manifold of dimension $2n + 1$, $n \geq 1$, where $T^{1,0}X$ is a CR structure of X; that is, $T^{1,0}X$ is a complex vector sub-bundle of rank n of the complexified tangent bundle $\mathbb{C}TX := TX \otimes_{\mathbb{R}} \mathbb{C}$ satisfying

$$T^{1,0}X \cap T^{0,1}X = \{0\}, \ [\mathcal{V}, \mathcal{V}] \subset \mathcal{V}, \text{ with } T^{0,1}X = \overline{T^{1,0}X}, \mathcal{V} = \mathscr{C}^\infty(X, T^{1,0}X). \tag{2.1}$$

Denote by $T^{*1,0}X$ and $T^{*0,1}X$ the dual bundles of $T^{1,0}X$ and $T^{0,1}X$, respectively. Define the vector bundle of $(0, q)$-forms by

$$T^{*0,q}X := \Lambda^q T^{*0,1}X. \tag{2.2}$$

The Levi distribution (or holomorphic tangent space) HX of the CR manifold X is the real part of $T^{1,0}X \oplus T^{0,1}X$, i.e., the unique sub-bundle HX of TX such that

$$\mathbb{C}HX = T^{1,0}X \oplus T^{0,1}X. \tag{2.3}$$

Let $J : HX \to HX$ be the complex structure given by $J(u + \overline{u}) = \sqrt{-1}u - \sqrt{-1}\overline{u}$ for every $u \in T^{1,0}X$. If we extend the J complex linearly to $\mathbb{C}HX$, we have

$$T^{1,0}X = \left\{ V \in \mathbb{C}HX : JV = \sqrt{-1}V \right\}. \tag{2.4}$$

Thus the CR structure $T^{1,0}X$ is determined by the Levi distribution and we shall also write (X, HX, J) to denote the CR manifold $(X, T^{1,0}X)$.

The annihilator $(HX)^0 \subset T^*X$ of HX is called the characteristic bundle of the CR manifold. Since X is orientable, the characteristic bundle $(HX)^0$ is a trivial real line sub-bundle. We fix a global frame of $(HX)^0$, that is, a real non-vanishing 1-form $\omega_0 \in \mathscr{C}^\infty(X, T^*X)$ such that $(HX)^0 = \mathbb{R}\omega_0$: this is called a characteristic 1-form. We have

$$\langle \omega_0(x), u \rangle = 0, \quad \text{for any } u \in H_xX,\ x \in X. \tag{2.5}$$

Then, by (2.1), the restriction of $d\omega_0$ on HX is a $(1,1)$-form. The Levi form of X at $x \in X$ is the Hermitian quadratic form on $T_x^{1,0}X$ given by

$$\mathscr{L}_x(u, \overline{v}) = -\frac{1}{2\sqrt{-1}} \langle d\omega_0(x), u \wedge \overline{v} \rangle = -\frac{1}{2\sqrt{-1}} d\omega_0(u, \overline{v}), \quad \text{for } u, v \in T_x^{1,0}X. \tag{2.6}$$

A CR manifold X is said to be strictly pseudoconvex if, for every $x \in X$, the Levi form $\mathscr{L}_x$ is positive definite (negative definite). By (2.6) we see that the definition does not depend on the choice of the characteristic 1-form ω_0. By a change of sign of ω_0 we can and shall assume in the sequel that the Levi form is positive definite. If X is strictly pseudoconvex, then ω_0 is a contact form and the Levi distribution HX is a contact structure.

Let $T \in \mathscr{C}^\infty(X, TX)$ be a vector field, called characteristic vector field, such that

$$\mathbb{C}TX = T^{1,0}X \oplus T^{0,1}X \oplus \mathbb{C}T, \tag{2.7}$$

and

$$i_T\, \omega_0 = 1. \tag{2.8}$$

Let $g^{\mathbb{C}TX}$ be a Hermitian metric on $\mathbb{C}TX$ such that the decomposition (2.7) is orthogonal. For $u, v \in \mathbb{C}TX$, we denote by $\langle u, v \rangle_g$ the inner product given by $g^{\mathbb{C}TX}$, and for $u \in \mathbb{C}TX$, we write $|u|_g^2 := \langle u, u \rangle_g$.

The Hermitian metric $g^{\mathbb{C}TX}$ on $\mathbb{C}TX$ induces, by duality, a Hermitian metric on $\mathbb{C}T^*X$ and also on the bundles of $(0, q)$ forms $T^{*0,q}X$, $q = 1, 2, \cdots, n$. We shall also denote the inner product given by these metrics by $\langle \cdot, \cdot \rangle_g$. The metric $g^{\mathbb{C}TX}$ induces a Riemannian metric g^{TX} on TX and g^{TX} induces in turn a Riemannian volume form dv_X on X.

The natural global L^2 inner product $\langle \cdot\,, \cdot \rangle$ on $\Omega^{0,q}(X)$, induced by dv_X and $\langle \cdot\,, \cdot \rangle_g$, is given by

$$\langle u, v \rangle := \int_X \langle u(x), v(x) \rangle_g \, dv_X(x), \quad u, v \in \Omega^{0,q}(X). \tag{2.9}$$

We denote by $(L^2_{(0,q)}(X), \langle \cdot\,, \cdot \rangle)$ the completion of $\Omega^{0,q}(X)$ with respect to $\langle \cdot\,, \cdot \rangle$. We set $L^2(X) := L^2_{(0,0)}(X)$.

Let $\overline{\partial}_b : \Omega^{0,q}(X) \to \Omega^{0,q+1}(X)$ be the tangential Cauchy-Riemann operators on X, which is the composition of the exterior differential d and the projection $\pi^{0,q+1} : \Lambda^{q+1}(\mathbb{C}T^*X) \to T^{*0,q+1}X$. We extend $\overline{\partial}_b$ to L^2 spaces by taking the weak maximal extension

$$\begin{aligned} &\mathrm{Dom}\overline{\partial}_b = \left\{ u \in L^2_{(0,q)}(X) : \overline{\partial}_b u \in L^2_{(0,q+1)}(X) \right\}, \\ &\mathrm{Dom}\overline{\partial}_b \ni u \longmapsto \overline{\partial}_b u \in L^2_{(0,q+1)}(X). \end{aligned} \tag{2.10}$$

The space of L^2 CR functions on X is given by

$$H^0_b(X) := \left\{ u \in L^2(X) = L^2_{(0,0)}(X) : \overline{\partial}_b u = 0 \right\}. \tag{2.11}$$

Note that in contrast to holomorphic functions, a CR function does not even need to be continuous. Here is a trivial example: consider a compact complex manifold M, such that $X = S^1 \times M$ is a compact CR manifold with the CR structure defined by $T^{(1,0)}M$. Now a function f on X is CR if and only if there is a function h on the circle S^1 such that $f(t,m) = h(t)$ for any $t \in S^1$, $m \in M$.

2.2 An important example: Grauert tube

Let (L, h^L) be a Hermitian holomorphic line bundle over a connected compact complex manifold (M, J). Let h^{L^*} be the Hermitian metric on L^* induced by h^L. Let

$$X := \left\{ v \in L^* : |v|^2_{h^{L^*}} = 1 \right\} \tag{2.12}$$

be the circle bundle of L^* (Grauert tube); this is isomorphic to the S^1 principal bundle associated to L. Since X is a hypersurface in the complex manifold L^*, it a has a CR structure inherited from the complex structure of L^* by setting $T^{1,0}X = TX \cap T^{(1,0)}L^*$.

In this situation, S^1 acts on X by fiberwise multiplication, denoted $(x, \mathrm{e}^{\sqrt{-1}\theta}) \mapsto x\mathrm{e}^{\sqrt{-1}\theta}$. A point $x \in X$ is a pair $x = (p, \lambda)$, where λ is a linear functional on L_p, and the S^1 action is $x\mathrm{e}^{\sqrt{-1}\theta} = (p, \lambda)\mathrm{e}^{\sqrt{-1}\theta} = (p, \mathrm{e}^{\sqrt{-1}\theta}\lambda)$.

On X, we have a globally defined vector field ∂_θ, the generator of the S^1 action. The span of ∂_θ defines a rank one subbundle $T^V X \cong TS^1 \subset TX$, the vertical subbundle of the fibration $\pi : X \to M$. Moreover, (2.7) holds for $T = \partial_\theta$.

For $m \in \mathbb{Z}$, the space $\mathscr{C}^\infty(X, L^m)$ of smooth sections of L^m can be identified with the space m-equivariant smooth functions

$$\mathscr{C}^\infty(X)_m = \{ f \in \mathscr{C}^\infty(X, \mathbb{C}) : f(x\mathrm{e}^{\sqrt{-1}\theta}) = \mathrm{e}^{\sqrt{-1}m\theta} f(x), \text{ for } \mathrm{e}^{\sqrt{-1}\theta} \in S^1,\, x \in X \}$$

by

$$\mathscr{C}^\infty(X, L^m) \ni s \mapsto f \in \mathscr{C}^\infty(X)_m, \quad f(x) = f(p, \lambda) = \lambda^{\otimes m}(s(p)), \tag{2.13}$$

where $\lambda^m = \lambda^{\otimes m}$ for $m \geq 0$ and $\lambda^m = (\lambda^{-1})^{\otimes(-m)}$ for $m < 0$. Through the identification (2.13), the holomorphic sections correspond to CR functions as follows:

$$H^0(X, L^m) \cong H^0_{b,m}(X) := \{ f \in \mathscr{C}^\infty(X)_m : \overline{\partial}_b f = 0 \}. \tag{2.14}$$

We construct now a Riemannian metric on X. Let g^{TM} be a J-invariant metric on M. The Chern connection ∇^L on L induces a connection on the S^1-principal bundle $\pi : X \to M$, and we let $T^H X \subset TX$ be the corresponding horizontal bundle. Let $g^{TX} = \pi^* g^{TM} \oplus \frac{d\theta^2}{4\pi^2}$ be the metric on $TX = T^H X \oplus TS^1$, with $d\theta^2$ the standard metric on $S^1 = \mathbb{R}/2\pi\mathbb{Z}$.

Pertaining to g^{TX}, we construct the L^2 inner product $\langle\cdot,\cdot\rangle_X$ given by (2.9) on X. The metric g^{TM} induces a Riemannian volume form dv_M on M, which, together with the fiber metric h^{L^m}, gives rise to an L^2 inner product $\langle\cdot,\cdot\rangle_m$ on $\mathscr{C}^\infty(X,L^m)$. Then the isomorphism (2.13) becomes an isometry $(\mathscr{C}^\infty(M,L^m),\langle\cdot,\cdot\rangle_m)\cong(\mathscr{C}^\infty(X)_m,\langle\cdot,\cdot\rangle_X)$, and accordingly, an isometry $L^2(M,L^m)\cong L^2(X)_m$, where the latter space is the completion of $(\mathscr{C}^\infty(X)_m,\langle\cdot,\cdot\rangle_X)$. Moreover, (2.13) induces an isometry

$$(H^0(M,L^m),\langle\cdot,\cdot\rangle_m)\cong(H^0_{b,m}(X),\langle\cdot,\cdot\rangle_X). \tag{2.15}$$

The S^1-action gives rise to a Fourier decomposition $L^2(X)\cong\widehat{\bigoplus}_{m\in\mathbb{Z}}L^2(X)_m$ and this induces the following decomposition at the level of CR functions:

$$H^0_b(X)\cong\widehat{\bigoplus}_{m\in\mathbb{Z}}H^0_{b,m}(X)\cong\widehat{\bigoplus}_{m\in\mathbb{Z}}H^0(M,L^m). \tag{2.16}$$

Let ω_0 be the connection 1-form on X associated to the Chern connection ∇^L. Then $\omega_0(\partial_\theta)=1$, and thus (2.7) and (2.8) are fullfiled and $T=\partial_\theta$ is a characteristic vector field on X and ω_0 is a characteristic 1-form for the CR structure on X. Moreover,

$$d\omega_0=\pi^*(\sqrt{-1}R^L), \tag{2.17}$$

where R^L is the curvature of ∇^L. On account of (2.6), X is strictly pseudoconvex at $x\in X$ if and only if (L,h^L) is positive at $\pi(x)\in M$. In particular, if (L,h^L) is positive on M, X is a strictly pseudoconvex CR manifold, ω_0 is a contact form, and ∂_θ is the associated Reeb vector field. Note also that in this case, by the Kodaira vanishing theorem, $H^0(X,L^m)=0$ for $m<0$, so the decomposition (2.16) becomes

$$H^0_b(X)\cong\widehat{\bigoplus}_{m\in\mathbb{N}}H^0_{b,m}(X)\cong\widehat{\bigoplus}_{m\in\mathbb{N}}H^0(M,L^m). \tag{2.18}$$

2.3 $[Q,R]=0$ on CR manifolds

Let (X,HX,J) be a compact connected and orientable CR manifold of dimension $2n+1$, $n\geq 1$, and let ω_0 be a characteristic 1-form.

Let G be a d-dimensional compact Lie group with Lie algebra $\mathfrak{g}$. We assume that G acts smoothly on X and that the G-action preserves J and ω_0.

Definition 2.1 The moment map associated to the characteristic 1-form ω_0 is the map $\mu: X\to\mathfrak{g}^*$ defined by

$$\langle\mu(x),K\rangle=\omega_0(K^X(x)),\quad x\in X,\ K\in\mathfrak{g}. \tag{2.19}$$

Let $\iota: Y:=\mu^{-1}(0)\to X$ be the natural inclusion and let $\iota^*:\Omega^\bullet(X)\to\Omega^\bullet(Y)$ be the pull-back of differential forms by ι. Let $\pi: Y\to Y/G$ be the natural projection.

Theorem 2.2 If G acts freely on $Y=\mu^{-1}(0)$ and the Levi form is positive on $\mu^{-1}(0)$, then the reduced space $X_G=Y/G$ is a strictly pseudoconvex manifold with contact form $\omega_{0,G}$ satisfying $\iota^*\omega_0=\pi^*\omega_{0,G}$. Moreover, we can choose the characteristic vector field T (cf. (2.7), (2.8)) such that $T|_Y\in\mathscr{C}^\infty(Y,TY)$ and T is G-invariant.

The space X_G is called the CR reduction. Under our hypotheses, if $\dim X_G\geq 3$, X_G is a strictly pseudoconvex CR manifold with characteristic 1-form (in this case also the contact form) $\omega_{0,G}$ induced canonically by ω_0. If $\dim X_G=1$, then each of the finitely many components of X_G is diffeomorphic to a circle.

We will mainly work in the following setting:

Assumption 2.3 The G-action preserves the complex structure J on HX and the characteristic 1-form ω_0, is free on $\mu^{-1}(0)$, and one of the following conditions are fulfilled:

(i) $\dim X \geq 5$, and the Levi form of X is positive definite near $\mu^{-1}(0)$;

(ii) $\dim X = 3$, and the Levi form of X is positive definite everywhere and $\overline{\partial}_b$ has closed range in L^2 on X.

If $\dim X_G \geq 3$, let $\overline{\partial}_{b,X_G} : \mathscr{C}^\infty(X_G) \to \Omega^{0,1}(X_G)$ be the tangential Cauchy-Riemann operators on X_G. We consider the spaces of L^2 CR functions

$$H^0_b(X_G) := \left\{u \in L^2(X_G) : \overline{\partial}_{b,X_G} u = 0\right\}. \tag{2.20}$$

If $\dim X_G = 1$, X_G is a finite union of circles, and we set $H^0_b(X_G)$ to be the direct sum of the Hardy spaces of the components, that is, the L^2 subspaces of functions with vanishing Fourier coefficients of negative degree. The common feature of the spaces $H^0_b(X_G)$ for $\dim X_G \geq 3$ and $\dim X_G = 1$ is the fact that they are boundary values of holomorphic functions in a filling of X_G by a complex manifold.

Note that $H^0_b(X)$ is a (possible infinite dimensional) G-representation, so its G-invariant part is formed by the G-invariant L^2 CR functions on X,

$$H^0_b(X)^G := \left\{u \in H^0_b(X) : h^*u = u, \quad \text{for any } h \in G\right\}. \tag{2.21}$$

For every $s \in \mathbb{R}$, let $H^s(X)$ and $H^s(X_G)$ denote the Sobolev space of X of order s and the Sobolev space of X_G of order s, respectively. For every $s \in \mathbb{R}$, put

$$\begin{aligned} H^0_b(X)_s &:= \left\{u \in H^s(X) : \overline{\partial}_b u = 0 \text{ in the sense of distributions}\right\}, \\ H^0_b(X_G)_s &:= \left\{u \in H^s(X_G) : \overline{\partial}_b u = 0 \text{ in the sense of distributions}\right\}. \end{aligned} \tag{2.22}$$

If $\dim X_G = 1$, we set $H^0_b(X_G)_s$ to be the direct sum of the Hardy-Sobolev spaces of the components, that is, the subspaces of $H^s(S^1)$ of distributions with vanishing Fourier coefficients of negative degree.

Let $\iota_G : \mathscr{C}^\infty(Y)^G \to \mathscr{C}^\infty(X_G)$ be the natural identification. Let

$$\sigma_G : H^0_b(X)^G \cap \mathscr{C}^\infty(X)^G \to H^0_b(X_G), \quad \sigma_G = \iota_G \circ \iota^*. \tag{2.23}$$

The map (2.23) is well defined; see the construction of the CR reduction in Section 2.3. The map σ_G does not extend to a bounded operator on L^2, so it necessary to consider its extension to Sobolev spaces.

Theorem 2.4 ([7]) Let X be a compact orientable CR manifold and let G be a compact Lie group acting on X such that the G-action preserves J and ω_0 and that Assumption 2.3 holds. Suppose that $\overline{\partial}_{b,X_G}$ has closed range in L^2. Then, for every $s \in \mathbb{R}$, the σ_G extends by density to a bounded operator

$$\sigma_G = \sigma_{G,s} : H^0_b(X)^G_s \to H^0_b(X_G)_{s-\frac{d}{4}}, \quad \text{for every } s \in \mathbb{R}, \tag{2.24}$$

and it is Fredholm. Actually, $\mathrm{Ker}(\sigma_{G,s})$ and $(\mathrm{Im}(\sigma_{G,s}))^\perp$ are finite dimensional subspaces of $\mathscr{C}^\infty(X) \cap H^0_b(X)^G$ and $\mathscr{C}^\infty(X_G) \cap H^0_b(X_G)$, respectively, and $\mathrm{Ker}(\sigma_{G,s})$ and the index $\dim \mathrm{Ker}(\sigma_{G,s}) - \dim\left(\mathrm{Im}(\sigma_{G,s})\right)^\perp$ are independent of s.

This operator σ_G can be thought as a Guillemin-Sternberg map in the CR setting. It maps the "first quantize and then reduce" space (the space of G-invariant Sobolev CR functions on X) to the "first reduce and then quantize" space (the space Sobolev CR functions on X_G).

Under Assumption 2.3 (i), the hypothesis that $\dim X \geq 5$ is used in order to have local subelliptic Sobolev estimates on the set where the Levi form is positive definite and leads to the fact that the G-invariant Kohn Laplacian has closed range in L^2. Note also that the Kohn Laplacian on strictly pseudoconvex CR manifolds of dimension greater than or equal to five always has closed range in L^2 but this is not true for all three dimensional strictly pseudoconvex CR manifolds.

We turn now our attention to Sasakian manifolds. Let $(X, T^{1,0}X)$ be a compact connected Sasakian manifold, i.e., the metric cone $(C(X) = \mathbb{R}_+ \times X, dr^2 + r^2 g_{\omega_0})$ is a Kähler manifold. We fix a contact form ω_0 and an associated Reeb vector field R (defined by $i_R\,\omega_0 = 1,\ i_R\,d\omega_0 = 0$).

We assume that

$$R \text{ is } G\text{-invariant.} \tag{2.25}$$

Then the R-action preserves HX, J and the natural metric g_{ω_0} on TX. As X is compact, this implies that the R-action generates a compact torus $\mathbb{T}$-action on X and this $\mathbb{T}$-action commutes with the G-action. This naturally induces a $\mathbb{T}$-action on X_G and the generator R induces the Reeb vector field $\widehat{R}$ on X_G. It is clear that X_G is also a compact Sasakian manifold and that $\widehat{R}$ preserves the CR structure $T^{1,0}X_G$, and that $\widehat{R}$, $T^{1,0}X_G \oplus T^{0,1}X_G$ generate the complex tangent bundle of X_G.

Now $H^0_b(X)^G$ and $H^0_b(X_G)$ are both $\mathbb{T}$-Hilbert spaces, so we have the decomposition of Hilbert spaces via the weight $\alpha \in \mathbb{T}^*(\simeq \mathbb{Z}^{\dim \mathbb{T}})$ of $\mathbb{T}$-action:

$$H^0_b(X)^G = \bigoplus_{\alpha \in \mathbb{T}^*} H^0_{b,\alpha}(X)^G, \qquad H^0_b(X_G) = \bigoplus_{\alpha \in \mathbb{T}^*} H^0_{b,\alpha}(X_G). \tag{2.26}$$

Both $H^0_{b,\alpha}(X)^G$ and $H^0_{b,\alpha}(X_G)$ are finite dimensional subspaces of $\mathscr{C}^\infty(X)^G$ and $\mathscr{C}^\infty(X_G)$, respectively, as subspaces of the eigenspaces of the elliptic operators $\overline{\partial}^*_{b,X}\overline{\partial}_{b,X} - R^2$, $\overline{\partial}^*_{b,X_G}\overline{\partial}_{b,X_G} - \widehat{R}^2$, respectively, of eigenvalues $|\alpha(R)|^2$. From (2.23), we see that

$$\sigma_G R u = \widehat{R}\sigma_G u, \quad \text{for any } u \in H^0_b(X)^G, \tag{2.27}$$

and hence σ_G maps $H^0_{b,\alpha}(X)^G$ to $H^0_{b,\alpha}(X_G)$. From this observation, Theorem 2.4, and the fact that $\overline{\partial}_b$ has closed range in L^2 on Sasakian manifolds, we deduce:

Theorem 2.5 (Quantization commutes with reduction for Sasakian manifolds) Let X be a Sasakian manifold. Assume that the Reeb vector field is G-invariant. Then with the exception of finitely many α, the map

$$\sigma_G : H^0_{b,\alpha}(X)^G \to H^0_{b,\alpha}(X_G) \tag{2.28}$$

is an isomorphism.

We now apply Theorem 2.4 to the case of complex manifolds.

Theorem 2.6 Let M be a compact connected complex manifold, $\dim_{\mathbb{C}} M \geq 2$, and let (L, h^L) be a Hermitian holomorphic line bundle over M. Let G be a compact Lie group acting holomorphically on M, whose action lifts to (L, h^L). Suppose that $\sqrt{-1}R^L$ is positive near $\mu^{-1}(0)$ and that G acts freely on $\mu^{-1}(0)$ with $\mu : M \to \mathfrak{g}^*$ in (1.13). Then, for m large enough, the canonical map between $H^0(M, L^m)^G$ and $H^0(M_G, L^m_G)$ by restriction is an isomorphism, in particular,

$$\dim H^0(M, L^m)^G = \dim H^0(M_G, L^m_G). \tag{2.29}$$

If L is positive on the whole M, Theorem 2.6 is a weaker version of Theorem 1.6, which holds for any $m \in \mathbb{N}^*$.

Inspired by Theorem 1.6, we expect that σ_G in Theorem 2.4 is in fact an isomorphism if X is a compact strictly pseudoconvex CR manifold.

References

[1] Atiyah M F. Elliptic Operators and Compact Groups. Lecture Notes in Mathematics, Vol 401. Berlin: Springer-Verlag, 1974

[2] Berline N, Getzler E, Vergne M. Heat Kernels and Dirac Operators. Grundl Math Wiss Band 298. Berlin: Springer-Verlag, 1992

[3] Bismut J -M, Lebeau G. Complex immersions and Quillen metrics. Inst Hautes Études Sci Publ Math, 1991, **74**: 1–298

[4] Guillemin V, Sternberg S. Geometric quantization and multiplicities of group representations. Invent Math, 1982, **67**(3): 515–538

[5] Hochs P, Mathai V. Geometric quantization and families of inner products. Adv Math, 2015, **282**: 362–426

[6] Hochs P, Song Y. Equivariant indices of Spin-Dirac operators for proper moment maps. Duke Math J, 2017, **166**: 1125–1178

[7] Hsiao C -Y, Ma X, Marinescu G. Geometric quantization on CR manifolds. arXiv:1906.05627

[8] Kostant B. Quantization and unitary representations//Lect Notes in Math, **170**. Springer, 1970: 87–207

[9] Ma X. Geometric quantization on Kähler and symplectic manifolds//Proceedings of the International Congress of Mathematicians, Volume II (New Delhi). Hindustan Book Agency, 2010: 785–810

[10] Ma X, Marinescu G. Holomorphic Morse Inequalities and Bergman Kernels. Progress in Mathematics, Vol 254. Boston, MA: Birkhäuser Boston Inc, 2007, 422 pp

[11] Ma X, Zhang W. Geometric quantization for proper moment maps. C R Math Acad Sci Paris, 2009, **347**: 389–394

[12] Ma X, Zhang W. Geometric quantization for proper moment maps: the Vergne conjecture. Acta Math, 2014, **212**(1): 11–57

[13] Mathai V, Zhang W. Geometric quantization for proper actions. Adv Math, 2010, **225**: 1224–1247

[14] Meinrenken E. On Riemann-Roch formulas for multiplicities. J Amer Math Soc, 1996, **9**(2): 373–389

[15] Meinrenken E. Symplectic surgery and the Spin^c-Dirac operator. Adv Math, 1998, **134**(2): 240–277

[16] Meinrenken E, Sjamaar R. Singular reduction and quantization. Topology, 1999, **38**: 699–762

[17] Paradan P -É. Localization of the Riemann-Roch character. J Funct Anal, 2001, **187**(2): 442–509

[18] Paradan P -É. Spin^c-quantization and the K-multiplicities of the discrete series. Ann Sci Ecole Norm Sup (4), 2003, **36**(5): 805–845

[19] Paradan P -É. Formal geometric quantization II. Pacific J Math, 2011, **253**: 169–211

[20] Paradan P -É, Vergne M. Equivariant Dirac operators and differentiable geometric invariant theory. Acta Math, 2017, **218**: 137–199

[21] Souriau J -M. Structure des Systèmes Dynamiques. Maîtrises de Mathématiques. Paris: Dunod, 1970

[22] Tian Y, Zhang W. An analytic proof of the geometric quantization conjecture of Guillemin-Sternberg. Invent Math, 1998, **132**(2): 229–259

[23] Tian Y, Zhang W. Quantization formula for symplectic manifolds with boundary. Geom Funct Anal, 1999, **9**(3): 596–640

[24] Vergne M. Multiplicities formula for geometric quantization. I, II. Duke Math J, 1996, **82**(1): 143–179, 181–194

[25] Vergne M. Quantification géométrique et réduction symplectique. Astérisque, 2002, (282): 249–278

[26] Vergne M. Applications of equivariant cohomology//International Congress of Mathematicians. Vol I. Zürich: Eur Math Soc, 2007: 635–664

[27] Zhang W. Holomorphic quantization formula in singular reduction. Commun Contemp Math, 1999, **1**(3): 281–293