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An arithmetic Lefschetz–Riemann–Roch theorem

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with an appendix by Xiaonan Ma

Abstract

In this article, we consider regular projective arithmetic schemes in the context of Arakelov geometry, any of which is endowed with an action of the diagonalizable group scheme associated to a finite cyclic group and with an equivariant very ample invertible sheaf. For any equivariant morphism between such arithmetic schemes, which is smooth over the generic fiber, we define a direct image map between corresponding higher equivariant arithmetic K-groups and we discuss its transitivity property. Then we use the localization sequence of higher arithmetic K-groups and the higher arithmetic concentration theorem developed in Tang (*Math. Z.* 290 (2018) 307–346) to prove an arithmetic Lefschetz-Riemann-Roch theorem. This theorem can be viewed as a generalization, to the higher equivariant arithmetic K-theory, of the fixed-point formula of Lefschetz type proved by Köhler and Roessler (*Invent. Math.* 145 (2001) 333–396).

Contents

1. *Introduction*

The aim of this article is to prove an arithmetic Riemann–Roch theorem of Lefschetz type for the higher equivariant arithmetic K-theory of regular arithmetic schemes in the context of Arakelov geometry. This theorem is an arithmetic analogue of a special case of Köck's Lefschetz theorem in higher equivariant K-theory (cf. [**23**]), and it also generalizes Köhler–Roessler's Lefschetz fixed point formula [**24**, Theorem 4.4] to the case where higher arithmetic K-groups are concerned. To make things more explicit, let us first recall the study of such Lefschetz–Riemann–Roch problems.

Let X be a smooth projective variety over an algebraically closed field k, and suppose that X is endowed with an action of a cyclic group $\langle g\rangle$ of finite order n such that n is prime to the characteristic of k. A $\langle g\rangle$-equivariant coherent sheaf on X is a coherent $\mathcal{O}_X$-module F on X together with an automorphism $\varphi : g^*F \to F$ such that φ^n is equal to the identity map. Then the classical Lefschetz trace formula gives an expression of the alternating sum of the trace of $H^i(\varphi)$ on the cohomology space $H^i(X, F)$, as a sum of the contributions from the components of the fixed-point subvariety X_g. For $k = \mathbb{C}$, the field of complex numbers, such a Lefschetz

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trace formula was presented via index theory and topological K-theory in [1, III]. While for general k, a Grothendieck-type generalization to the scheme theoretic algebraic geometry is very natural to expect. Precisely, denote by $K_0(X,g)$ the Grothendieck group of the category of equivariant locally free coherent sheaves on X, then $K_0(\mathrm{Pt},g)$ is isomorphic to the group ring $\mathbb{Z}[g] \cong \mathbb{Z}[T]/(1-T^n)$ and $K_0(X,g)$ has a natural $K_0(\mathrm{Pt},g)$-algebra structure (Pt stands for the point Spec(k)). Let Y be another $\langle g\rangle$-equivariant smooth projective variety, let $f: X \to Y$ be a projective morphism compatible with both $\langle g\rangle$-actions on X and on Y, then we have a direct image map $f_*: K_0(X,g) \to K_0(Y,g)$ given by

$$E \mapsto \sum_{i\geqslant 0}(-1)^i R^i f_*(E).$$

Unsurprisingly, the direct image map f_* does not commute with the restriction map $\tau: K_0(\cdot,g) \to K_0((\cdot)_g,g)$ from the equivariant K_0-group of an equivariant variety to the equivariant K_0-group of its fixed-point subvariety. Namely, the restriction map τ is not a natural transformation between the covariant functors $K_0(\cdot,g)$ and $K_0((\cdot)_g,g)$. Like the other Riemann–Roch problems, the Lefschetz–Riemann–Roch theorem makes a correction of τ such that it becomes a natural transformation. In fact, for any $\langle g\rangle$-equivariant smooth projective variety X, let N_{X/X_g} stand for the normal bundle associated to the regular immersion $X_g \hookrightarrow X$ and let $\lambda_{-1}(N^\vee_{X/X_g})$ be the alternating sum $\sum(-1)^j \wedge^j N^\vee_{X/X_g}$, then $\lambda_{-1}(N^\vee_{X/X_g})$ is an invertible element in $K_0(X_g,g)\otimes_{\mathbb{Z}[g]}\mathcal{R}$, where $\mathcal{R}$ is any $\mathbb{Z}[g]$-algebra in which $1-T^k$ is invertible for $k=1,\dots,n-1$. We formally define $L_X: K_0(X,g) \to K_0(X_g,g)\otimes_{\mathbb{Z}[g]}\mathcal{R}$ as $\lambda^{-1}_{-1}(N^\vee_{X/X_g})\cdot\tau$, the Lefschetz–Riemann–Roch theorem reads: the following diagram

$$\begin{array}{ccc} K_0(X,g) & \xrightarrow{L_X} & K_0(X_g,g)\otimes_{\mathbb{Z}[g]}\mathcal{R} \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_{g*}} \\ K_0(Y,g) & \xrightarrow{L_Y} & K_0(Y_g,g)\otimes_{\mathbb{Z}[g]}\mathcal{R} \end{array} \tag{1}$$

is commutative.

This commutative diagram (1) was presented by Donovan in [19], and later it was generalized to singular varieties by Baum, Fulton and Quart in [2]. Note that the settings in [2, 19] are more general than that in this introduction. The reasoning in the first paper runs similarly to the technique used in Borel–Serre's paper [15], while the reasoning in the second paper relies on the deformation to the normal cone construction. These two processes are both traditional for producing the Grothendieck-type Riemann–Roch theorem.

After Quillen and other mathematicians' work, algebraic K-groups are extended to higher degrees and the higher (equivariant) algebraic K-groups of X are defined as the higher homotopy groups of the K-theory space associated to the category of (equivariant) locally free coherent sheaves on X. There are many methods to construct this 'K-theory space', but no matter which construction we choose, the tensor product of locally free coherent sheaves always induces a graded ring structure on $K_\bullet(X,g)$. In particular, each $K_m(X,g)$ is a $K_0(X,g)$-module. Moreover, the functor $K_\bullet(\cdot,g)$ is again covariant with respect to equivariant proper morphisms. Then, for any $m\geqslant 1$, the following diagram for higher algebraic K-groups which is similar to (1) does make sense:

$$\begin{array}{ccc} K_m(X,g) & \xrightarrow{L_X} & K_m(X_g,g)\otimes_{\mathbb{Z}[g]}\mathcal{R} \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_{g*}} \\ K_m(Y,g) & \xrightarrow{L_Y} & K_m(Y_g,g)\otimes_{\mathbb{Z}[g]}\mathcal{R}. \end{array} \tag{2}$$

The commutativity of diagram (2), which is named the Lefschetz–Riemann–Roch theorem for higher equivariant algebraic K-theory, was proved by Köck in [**23**]. The main ingredient is an excess intersection formula whose proof also relies on the deformation to the normal cone construction. Moreover, it is worth indicating that the commutative diagram (2), combined with the Gillet's Riemann–Roch theorem for higher algebraic K-theory (cf. [**20**]), implies a higher Lefschetz trace formula.

In the field of arithmetic geometry, one considers those Noetherian and separated schemes $f : X \to \mathrm{Spec}(\mathbb{Z})$ over the ring of integers (actually over any excellent regular Noetherian domain). In this context, it is possible to produce an analogue of the commutative diagram (1), by endowing X with an action of the diagonalizable group scheme $\mu_n = \mathrm{Spec}(\mathbb{Z}[\mathbb{Z}/n\mathbb{Z}])$ of nth roots of unity rather than with the action of an automorphism of order n. Here, a μ_n-action on X is a morphism of schemes $m_X : \mu_n \times X \to X$ which satisfies the usual associativity property. The reason for this choice is that the fixed-point subscheme X_{μ_n} of a regular scheme X equipped with an action of μ_n is still regular and the natural inclusion $i_X : X_{\mu_n} \hookrightarrow X$ is a regular immersion, while the fixed-point subscheme of a regular scheme under an automorphism of order n can be very singular over the fibers lying above the primes dividing n. By a μ_n-equivariant coherent sheaf F on X, we understand a coherent $\mathcal{O}_X$-module F together with an isomorphism

$$m_F : m_X^* F \to \mathrm{pr}_X^* F$$

of $\mathcal{O}_{\mu_n \times X}$-modules which satisfies the following associativity property:

$$(\mathrm{pr}_{2,3}^* m_F) \circ ((1 \times m_X)^* m_F) = (m_{\mu_n} \times 1)^* m_F.$$

Here, m_{μ_n} denotes the multiplication $\mu_n \times \mu_n \to \mu_n$, $\mathrm{pr}_X : \mu_n \times X \to X$ and $\mathrm{pr}_{2,3} : \mu_n \times \mu_n \times X \to \mu_n \times X$ denote the obvious projections. Under this situation, Baum–Fulton–Quart method still works, so that the commutative diagram (1) holds for regular μ_n-equivariant schemes over $\mathbb{Z}$.

In [**35**], Thomason used another way to do the same thing and he even got a generalization of the commutative diagram (2) for regular μ_n-equivariant schemes. Thomason's strategy was to use Quillen's localization sequence for higher equivariant algebraic K-groups to show a concentration theorem. This theorem states that, after a suitable localization, the equivariant algebraic K-group $K_m(X_{\mu_n}, \mu_n)_\rho$ is isomorphic to $K_m(X, \mu_n)_\rho$ for any $m \geqslant 0$, and the inverse map is exactly given by $\lambda_{-1}^{-1}(N^\vee_{X/X_{\mu_n}}) \cdot i_X^*$. Here, ρ is any prime ideal in $R(\mu_n) := K_0(\mathrm{Spec}\mathbb{Z}, \mu_n) \cong \mathbb{Z}[T]/(1 - T^n)$ which does not contain the elements $1 - T^k$ for $k = 1, \ldots, n-1$. For instance, ρ can be chosen to be the kernel of the natural morphism $\mathbb{Z}[T]/(1 - T^n) \to \mathbb{Z}[T]/(\Phi_n)$, where Φ_n stands for the nth cyclotomic polynomial. Then the Lefschetz-Riemann–Roch theorem for regular μ_n-equivariant schemes

$$\begin{array}{ccc} K_m(X, \mu_n) & \xrightarrow{L_X} & K_m(X_{\mu_n}, \mu_n)_\rho \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_{\mu_n *}} \\ K_m(Y, \mu_n) & \xrightarrow{L_Y} & K_m(Y_{\mu_n}, \mu_n)_\rho \end{array} \tag{3}$$

follows from the covariant functoriality of $K_\bullet(\cdot, \mu_n)$ with respect to proper morphisms.

Now, let us turn to Arakelov geometry. Let X be an arithmetic scheme over an arithmetic ring (D, Σ, F_∞) in the sense of Gillet–Soulé (cf. [**21**]), then X is quasi-projective over D with smooth generic fiber. We denote $\mu_n := \mathrm{Spec}(D[\mathbb{Z}/n\mathbb{Z}])$ the diagonalizable group scheme over D associated to a cyclic group $\mathbb{Z}/n\mathbb{Z}$. By saying X is μ_n-projective, we understand that X is endowed with a projective μ_n-action. That means X is projective and there exists a very ample invertible μ_n-sheaf on X.

For each regular μ_n-projective arithmetic scheme X, K. Köhler and D. Roessler have defined an equivariant arithmetic K_0-group $\widehat{K_0}(X,\mu_n)$ in [24]. This arithmetic K_0-group is a modified Grothendieck group of the category of equivariant hermitian vector bundles on X, it contains some smooth form class on $X_{\mu_n}(\mathbb{C})$ as analytic datum. The same as the algebraic K_0-group $K_0(X,\mu_n)$, $\widehat{K_0}(X,\mu_n)$ has a ring structure and it is an $R(\mu_n)$-algebra. Moreover, direct image maps between equivariant arithmetic K_0-groups can be defined for an equivariant morphism which is smooth over the generic fiber, by using Bismut–Köhler–Ma analytic torsion forms. Choose a Kähler metric for $X(\mathbb{C})$, and let $\overline{N}_{X/X_{\mu_n}}$ be the normal bundle endowed with the quotient metric, then the main theorem in [24] reads: the element $\lambda_{-1}(\overline{N}^{\vee}_{X/X_{\mu_n}})$ is a unit in $\widehat{K_0}(X_{\mu_n},\mu_n)_\rho$ and the following diagram

$$\begin{array}{ccc} \widehat{K_0}(X,\mu_n) & \xrightarrow{\Lambda_R\cdot\tau} & \widehat{K_0}(X_{\mu_n},\mu_n)_\rho \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_{\mu_n *}} \\ \widehat{K_0}(D,\mu_n) & \xrightarrow{\tau} & \widehat{K_0}(D,\mu_n)_\rho \end{array} \tag{4}$$

is commutative, where ρ is any prime ideal in $R(\mu_n)$ which does not contain the elements $1-T^k$ for $k=1,\dots,n-1$, Λ_R is defined as $(1-R_g(N_{X/X_{\mu_n}}))\cdot\lambda_{-1}^{-1}(\overline{N}^{\vee}_{X/X_{\mu_n}})$, and $R_g(\cdot)$ is the equivariant R-genus due to Bismut (see below).

Later, two refinements of (4) were presented by the author in [33] and in [32], respectively. In [33], D was replaced by a general regular μ_n-projective scheme Y. In [32], X was allowed to have singularities on its finite fibers. The aim of this article is to show an Arakelovian analogue of a special case of (3), in which the higher equivariant algebraic K-groups are replaced by the higher equivariant arithmetic K-groups. Hence, our work is a generalization of Köhler–Roessler's Lefschetz fixed-point formula to the higher equivariant arithmetic K-theory.

Let us describe the main result more precisely. First, note that we have constructed a group endomorphism $\otimes\lambda_{-1}(\overline{N}^{\vee}_{X/X_{\mu_n}}):\widehat{K}_m(X_{\mu_n},\mu_n)\to\widehat{K}_m(X_{\mu_n},\mu_n)$ and its formal inverse $\otimes\lambda_{-1}^{-1}(\overline{N}^{\vee}_{X/X_{\mu_n}}):\widehat{K}_m(X_{\mu_n},\mu_n)_\rho\to\widehat{K}_m(X_{\mu_n},\mu_n)_\rho$ in [34, Section 5]. As what we stated before, ρ is any prime ideal in $R(\mu_n):=K_0(\mathrm{Spec}\mathbb{Z},\mu_n)\cong\mathbb{Z}[T]/(1-T^n)$ which does not contain the elements $1-T^k$ for $k=1,\dots,n-1$. For instance, ρ can be chosen to be the kernel of the natural morphism $\mathbb{Z}[T]/(1-T^n)\to\mathbb{Z}[T]/(\Phi_n)$, where Φ_n stands for the nth cyclotomic polynomial. In this article, we shall further construct a group endomorphism $R_g(\overline{N}_{X/X_{\mu_n}}):\widehat{K}_m(X_{\mu_n},\mu_n)\to\widehat{K}_m(X_{\mu_n},\mu_n)$ and we shall prove that this endomorphism $R_g(\overline{N}_{X/X_{\mu_n}})$ is independent of the choice of the metric over $N_{X/X_{\mu_n}}$ after tensoring by $\mathbb{Q}$. So, the expression $\Lambda_R=(1-R_g(N_{X/X_{\mu_n}}))\cdot\lambda_{-1}^{-1}(\overline{N}^{\vee}_{X/X_{\mu_n}})$ still makes sense as an endomorphism of $\widehat{K}_m(X_{\mu_n},\mu_n)_\rho\otimes\mathbb{Q}$. Moreover, for any equivariant morphism $f:X\to Y$ between regular μ_n-projective arithmetic schemes, which is smooth over the generic fiber, we shall prove that there exists a reasonable direct image map $f_*:\widehat{K}_m(X,\mu_n)\to\widehat{K}_m(Y,\mu_n)$ with $m\geqslant 1$ and we discuss the transitivity property of the direct image maps up to torsion. Assume that the μ_n-action on Y is trivial and still use the notation τ to denote the morphism

$$\widehat{K}_m((\cdot),\mu_n)\to\widehat{K}_m((\cdot)_{\mu_n},\mu_n)_\rho\otimes\mathbb{Q}$$

$$x\mapsto\tau(x)\otimes 1,$$

Our main theorem reads: the following diagram

$$\begin{array}{ccc} \widehat{K}_m(X,\mu_n) & \xrightarrow{\Lambda_R\cdot\tau} & \widehat{K}_m(X_{\mu_n},\mu_n)_\rho\otimes\mathbb{Q} \\ {\scriptstyle f_*}\downarrow & & \downarrow{\scriptstyle f_{\mu_n *}} \\ \widehat{K}_m(Y,\mu_n) & \xrightarrow{\tau} & \widehat{K}_m(Y,\mu_n)_\rho\otimes\mathbb{Q} \end{array} \tag{5}$$

is commutative. In such a formulation, the equivariant R-genus again plays a crucial role.

To this aim, the definition of higher equivariant arithmetic K-groups and some reasonable technique that can be carried out for higher equivariant arithmetic K-theory should be clarified. We have settled these in [**34**]. In fact, we have defined the higher equivariant arithmetic K-groups via the simplicial description of the Beilinson's regulators (cf. [**17**]) and we have developed a localization sequence as well as an arithmetic concentration theorem. So, principally, we shall follow Thomason's approach to prove the commutativity of (5), but the fact that the direct image maps are only defined for the morphisms which are smooth over the generic fibers will lead to a big gap comparing with the purely algebraic case. Some highly non-trivial analytic machinery should be involved, such as the transitivity property of analytic torsion forms and the Bismut–Ma immersion formula.

The Köhler–Roessler arithmetic Lefschetz fixed-point formula has fruitful applications in number theory and in arithmetic geometry. One important reason is that the equivariant R-genus is closely related to the logarithmic derivative of certain L-functions. Köhler–Roessler and Maillot–Roessler have shown in [**25**] and in [**29**] that the Faltings heights and the periods of C.M. abelian varieties can be expressed as a formula in terms of the special value of logarithmic derivative of L-functions at 0. Further, in [**28**], Maillot–Roessler presented a series of conjectures about the relation between several invariants of arithmetic varieties and the special values of logarithmic derivative of Artin L-functions at negative integers. We hope that our Lefschetz–Riemann–Roch theorem for higher equivariant arithmetic K-groups would be helpful to understand these conjectures.

The structure of this article is as follows. In Section 2, we define the direct image maps between higher equivariant arithmetic K-groups. As an opportunity, we recall the analytic torsion for cubes of hermitian vector bundles introduced by Roessler in [**30**], actually our construction is slightly different to Roessler's construction. In Section 3, we discuss certain transitivity property of the direct image maps, the relation of equivariant analytic torsion forms with respect to families of submersions will be presented. In Section 4, we formulate and prove the commutativity of the diagram (5), an accurate computation via the deformation to the normal cone construction is given. In the last section, we attach an Appendix on some properties of equivariant analytic torsion forms and immersion formula. These purely analytic properties are crucial for the main arguments in this article, the author is very grateful to Prof. Ma Xiaonan for writing this Appendix.

2. *Higher equivariant arithmetic K-theory*

2.1. *Bott–Chern forms and arithmetic K-groups*

Suggested by Soulé (cf. [**31**]), and also by Deligne (cf. [**18**]), the higher arithmetic K-groups of an arithmetic scheme X can be defined as the homotopy groups of the homotopy fiber of Beilinson's regulator map, so that one obtains a long exact sequence

$$\cdots \longrightarrow \widehat{K}_m(X) \longrightarrow K_m(X) \xrightarrow{\text{ch}} \bigoplus_{p\geq 0} H_{\mathcal{D}}^{2p-m}(X,\mathbb{R}(p)) \longrightarrow \widehat{K}_{m-1}(X) \longrightarrow \cdots,$$

where $H^*_{\mathcal{D}}(X,\mathbb{R}(p))$ is the real Deligne–Beilinson cohomology and ch is the Beilinson's regulator map. In order to do this, a simplicial description of Beilinson's regulator map is necessary. In [**17**], such a simplicial description was given by Burgos and Wang by using the higher Bott–Chern forms. Recently, in [**34**], we followed Burgos–Wang approach to define the higher equivariant Bott–Chern forms and further the higher equivariant arithmetic K-theory. In this subsection, we shall recall some relevant constructions and definitions, for more details the reader is referred to [**17, 34**].

At first, let X be a smooth algebraic variety over $\mathbb{C}$. In this subsection, we shall work with the analytic topology of X. Denote by $E^*_{\log}(X)$ the complex of differential forms on X with logarithmic singularities along infinity (cf. [**34**, Definition 2.1]), then $E^*_{\log}(X)$ has a natural bigrading $E^n_{\log}(X) = \bigoplus_{p+q=n} E^{p,q}_{\log}(X)$ and this grading induces a Hodge filtration $F^pE^n_{\log}(X) = \bigoplus_{\substack{p'\geqslant p\\ p'+q'=n}} E^{p',q'}_{\log}(X)$. Write $E^*_{\log,\mathbb{R}}(X,p) := (2\pi i)^p E^*_{\log,\mathbb{R}}(X)$ with $E^*_{\log,\mathbb{R}}(X)$ the subcomplex of $E^*_{\log}(X)$ consisting of real forms, then we have a decomposition $E^*_{\log}(X) = E^*_{\log,\mathbb{R}}(X,p) \oplus E^*_{\log,\mathbb{R}}(X,p-1)$ and the projection $\pi_p : E^*_{\log}(X) \to E^*_{\log,\mathbb{R}}(X,p)$ is given by $\pi_p(x) = \frac{1}{2}(x + (-1)^p\overline{x})$. Moreover, for any $x \in E^n_{\log}(X)$, we define two filtered functions

$$F^{k,k}x = \sum_{l\geqslant k, l'\geqslant k} x^{l,l'} \quad \text{and} \quad F^kx = \sum_{l\geqslant k} x^{l,l'}.$$

Then we set $\pi(x) := \pi_{p-1}(F^{n-p+1,n-p+1}x)$.

The main result in [**16**, Section 2] states that the following Deligne complex

$$\mathfrak{D}^n(E_{\log}(X),p) = \begin{cases} E^{n-1}_{\log,\mathbb{R}}(X,p-1) \bigcap \bigoplus_{\substack{p'+q'=n-1\\ p'<p, q'<p}} E^{p',q'}_{\log}(X), & n < 2p;\\ E^n_{\log,\mathbb{R}}(X,p) \bigcap \bigoplus_{\substack{p'+q'=n\\ p'\geqslant p, q'\geqslant p}} E^{p',q'}_{\log}(X), & n \geqslant 2p, \end{cases}$$

with differential

$$d_{\mathcal{D}}x = \begin{cases} -\pi(dx), & n < 2p-1;\\ -2\partial\overline{\partial}x, & n = 2p-1;\\ dx, & n > 2p-1. \end{cases}$$

computes the real Deligne–Beilinson cohomology of X. Namely, one has

$$H^n_{\mathcal{D}}(X,\mathbb{R}(p)) = H^n(\mathfrak{D}^*(E_{\log}(X),p)).$$

We shall write $D^*(X,p) := \mathfrak{D}^*(E_{\log}(X),p)$ for short.

Remark 2.1. (i) According to the definition, the real Deligne–Beilinson cohomology of X at degrees $2p$ and $2p-1$ are given by

$$H^{2p}(\mathfrak{D}^*(E_{\log}(X),p)) = \{x \in E^{p,p}_{\log}(X) \cap E^{2p}_{\log,\mathbb{R}}(X,p) \mid dx = 0\}/\mathrm{Im}(\partial\overline{\partial})$$

and

$$H^{2p-1}(\mathfrak{D}^*(E_{\log}(X),p)) = \{x \in E^{p-1,p-1}_{\log}(X) \cap E^{2p-2}_{\log,\mathbb{R}}(X,p-1) \mid \partial\overline{\partial}x = 0\}/(\mathrm{Im}\,\partial + \mathrm{Im}\,\overline{\partial}).$$

(ii) Let $x \in D^n(X,p)$ and $y \in D^m(X,q)$, we write $l = n+m$ and $r = p+q$. Then

$$x \bullet y = \begin{cases} (-1)^n r_p(x) \wedge y + x \wedge r_q(y), & n < 2p, m < 2q\\ \pi(x \wedge y), & n < 2p, m \geqslant 2q, l < 2r\\ F^{r,r}(r_p(x) \wedge y) + 2\pi_r\partial\big((x \wedge y)^{r-1,l-r}\big), & n < 2p, m \geqslant 2q, l \geqslant 2r\\ x \wedge y, & n \geqslant 2p, m \geqslant 2q \end{cases}$$

induces a product on $\bigoplus_p D^*(X,p)$ which is graded commutative and is associative up to chain homotopy. Here, $r_px = 2\pi_p(F^pdx)$ if $n \leqslant 2p-1$ and $r_px = x$ otherwise. At the level of cohomology groups, this product coincides with the product defined by Beilinson. Note that if $x \in D^{2p}(X,p)$ is a cocycle, then for all y, z we have $x \bullet y = y \bullet x$ and $y \bullet (x \bullet z) = (y \bullet x) \bullet z = x \bullet (y \bullet z)$.

In order to introduce the higher Bott–Chern form, let us construct a new complex $\widetilde{D}^*(X,p)$ using the cocubical structure of the cartesian product of projective lines $(\mathbb{P}^1)^{\cdot}$. This complex

$\widetilde{D}^*(X,p)$ has the same cohomology groups as $D^*(X,p)$. First, one note that $D^*(X\times(\mathbb{P}^1)^{\cdot},p)$ form a cubical complex with face and degeneracy maps

$$d_i^j=(\mathrm{Id}\times d_j^i)^* \quad\text{and}\quad s_i=(\mathrm{Id}\times s^i)^*,$$

where

$$d_j^i:(\mathbb{P}^1)^k\to(\mathbb{P}^1)^{k+1},\quad i=1,\dots,k,j=0,1,$$
$$s^i:(\mathbb{P}^1)^k\to(\mathbb{P}^1)^{k-1},\quad i=1,\dots,k,$$

which are given by

$$d_0^i(x_1,\dots,x_k)=(x_1,\dots,x_{i-1},(0:1),x_i,\dots,x_k),$$
$$d_1^i(x_1,\dots,x_k)=(x_1,\dots,x_{i-1},(1:0),x_i,\dots,x_k),$$
$$s^i(x_1,\dots,x_k)=(x_1,\dots,x_{i-1},x_{i+1},\dots,x_k)$$

are the coface and the codegeneracy maps of $(\mathbb{P}^1)^{\cdot}$. Then we write $D_{\mathbb{P}}^{r,k}(X,p)=D^r(X\times(\mathbb{P}^1)^{-k},p)$ and denote by $D_{\mathbb{P}}^{*,*}(X,p)$ the associated double complex with differentials

$$d'=d_{\mathcal{D}}\quad\text{and}\quad d''=\sum(-1)^{i+j-1}d_i^j.$$

Next, let $(x:y)$ be the homogeneous coordinates of $\mathbb{P}^1$, and let $\omega=\partial\overline{\partial}\log\frac{x\overline{x}+y\overline{y}}{x\overline{x}}\in(2\pi i)E^2_{\mathbb{P}^1,\mathbb{R}}$ be a Kähler form over $\mathbb{P}^1$. We shall write $\omega_i=p_i^*\omega\in E^*_{\log}(X\times(\mathbb{P}^1)^k)$, where $p_i:X\times(\mathbb{P}^1)^k\to\mathbb{P}^1,i=1,\dots,k$ is the projection over the ith projective line. The complex $\widetilde{D}^*(X,p)$ is constructed by killing the degenerate classes and the classes coming from the projective spaces.

Definition 2.2. We define $\widetilde{D}^*(X,p)$ as the associated simple complex of the double complex $\widetilde{D}^{*,*}(X,p)$ which is given by

$$\widetilde{D}^{r,k}(X,p)=D_{\mathbb{P}}^{r,k}(X,p)/\sum_{i=1}^{-k}s_i\Big(D_{\mathbb{P}}^{r,k+1}(X,p)\Big)\oplus\omega_i\wedge s_i\Big(D_{\mathbb{P}}^{r-2,k+1}(X,p-1)\Big).$$

The differential of this complex will be denoted by d.

A repetition of the proofs of [17, Proposition 1.2, Lemma 1.3] gives that the natural morphism of complexes

$$\iota:D^*(X,p)=\widetilde{D}^{*,0}(X,p)\to\widetilde{D}^*(X,p)$$

is a quasi-isomorphism.

Now, let X be a smooth μ_n-projective variety over $\mathbb{C}$ and denote by $\mathcal{U}:=\widehat{\mathcal{P}}(X,\mu_n)$ the exact category of μ_n-equivariant vector bundles on X equipped with μ_n-invariant smooth hermitian metrics. We consider the exact cubes in the category $\mathcal{U}$. By definition, an exact k-cube in $\mathcal{U}$ is a functor $\mathcal{F}$ from $\langle-1,0,1\rangle^k$, the kth power of the ordered set $\langle-1,0,1\rangle$, to $\mathcal{U}$ such that for any $\alpha\in\langle-1,0,1\rangle^{k-1}$ and $1\leqslant i\leqslant k$, the 1-cube ∂_i^α defined by

$$\mathcal{F}_{\alpha_1,\dots,\alpha_{i-1},-1,\alpha_i,\dots,\alpha_{k-1}}\to\mathcal{F}_{\alpha_1,\dots,\alpha_{i-1},0,\alpha_i,\dots,\alpha_{k-1}}\to\mathcal{F}_{\alpha_1,\dots,\alpha_{i-1},1,\alpha_i,\dots,\alpha_{k-1}}$$

which is called an edge of $\mathcal{F}$ is a short exact sequence. From now on, we shall write cubes instead of exact cubes for short. Let $\mathcal{F}$ be a k-cube in $\mathcal{U}$, for $1\leqslant i\leqslant k$ and $j\in\langle-1,0,1\rangle$, the $(k-1)$-cube $\partial_i^j\mathcal{F}$ defined by $(\partial_i^j\mathcal{F})_{\alpha_1,\dots,\alpha_{k-1}}=\mathcal{F}_{\alpha_1,\dots,\alpha_{i-1},j,\alpha_i,\dots,\alpha_{k-1}}$ is called a face of $\mathcal{F}$. On the other hand, for any $1\leqslant i\leqslant k+1$, we denote by $S_i^1\mathcal{F}$ the $(k+1)$-cube

$$(S_i^1\mathcal{F})_{\alpha_1,\dots,\alpha_{k+1}}=\begin{cases}0, & \alpha_i=1;\\ \mathcal{F}_{\alpha_1,\dots,\alpha_{i-1},\alpha_{i+1},\dots,\alpha_{k+1}}, & \alpha_i\neq1,\end{cases}$$

such that the morphisms $(S_i^1\mathcal{F})_{\alpha_1,\dots,\alpha_{i-1},-1,\alpha_{i+1},\dots,\alpha_{k+1}} \to (S_i^1\mathcal{F})_{\alpha_1,\dots,\alpha_{i-1},0,\alpha_{i+1},\dots,\alpha_{k+1}}$ are the identities of $(S_i^1\mathcal{F})_{\alpha_1,\dots,\alpha_{i-1},\alpha_{i+1},\dots,\alpha_{k+1}}$. Similarly, we have $(k+1)$-cube $S_i^{-1}\mathcal{F}$.

Denote by $C_k\mathcal{U}$ the set of all k-cubes in $\mathcal{U}$, then we have the face maps $\partial_i^j : C_k\mathcal{U} \to C_{k-1}\mathcal{U}$ and the degeneracy maps $S_i^j : C_k\mathcal{U} \to C_{k+1}\mathcal{U}$. The cubes in the image of S_i^j are said to be degenerate. Let $\mathbb{Z}C_k\mathcal{U}$ be the free abelian group generated by $C_k\mathcal{U}$ and D_k be the subgroup of $\mathbb{Z}C_k\mathcal{U}$ generated by all degenerate k-cubes. Set $\widetilde{\mathbb{Z}}C_k\mathcal{U} = \mathbb{Z}C_k\mathcal{U}/D_k$ and

$$d = \sum_{i=1}^{k}\sum_{j=-1}^{1}(-1)^{i+j-1}\partial_i^j : \widetilde{\mathbb{Z}}C_k\mathcal{U} \to \widetilde{\mathbb{Z}}C_{k-1}\mathcal{U}.$$

Then $\widetilde{\mathbb{Z}}C_*\mathcal{U} = (\widetilde{\mathbb{Z}}C_k\mathcal{U}, d)$ is a homological complex.

Assume that $\overline{E}$ is a hermitian k-cube in the category $\mathcal{U} = \widehat{\mathcal{P}}(X, \mu_n)$. If $\overline{E}$ is an emi-cube, namely the metrics on the quotient terms in all edges of $\overline{E}$ are induced by the metrics on the middle terms (cf. [17, Definition 3.5]), one can follow [17, (3.7)] to associate a hermitian locally free sheaf $\mathrm{tr}_k(\overline{E})$ on $X \times (\mathbb{P}^1)^k$. This $\mathrm{tr}_k(\overline{E})$ is called the k-transgression bundle of $\overline{E}$. If $k = 1$, as an emi-1-cube, $\overline{E}$ is a short exact sequence

$$0 \longrightarrow \overline{E}_{-1} \xrightarrow{\ i\ } \overline{E}_0 \longrightarrow \overline{E}_1 \longrightarrow 0\ ,$$

where the metric of $\overline{E}_1$ is induced by the metric of $\overline{E}_0$. Then $\mathrm{tr}_1(\overline{E})$ is the cokernel with quotient metric of the map $E_{-1} \to E_{-1} \otimes \mathcal{O}(1) \oplus E_0 \otimes \mathcal{O}(1)$ by the rule $e_{-1} \mapsto e_{-1} \otimes \sigma_\infty \oplus i(e_{-1}) \otimes \sigma_0$. Here, σ_0 (respectively, σ_∞) is the section of the tautological bundle $\mathcal{O}(1)$ on $\mathbb{P}^1$ which vanishes only at 0 (respectively, ∞), and $\mathcal{O}(1)$ is endowed with the Fubini-Study metric. If $k > 1$, suppose that the transgression bundle is defined for $k-1$. Let $\mathrm{tr}_1(\overline{E})$ be the emi-$(k-1)$-cube over $X \times \mathbb{P}^1$ given by $\mathrm{tr}_1(\overline{E})_\alpha = \mathrm{tr}_1(\partial_1^\alpha(\overline{E}))$, then $\mathrm{tr}_k(\overline{E})$ is defined as $\mathrm{tr}_{k-1}(\mathrm{tr}_1(\overline{E}))$.

Moreover, according to [17, Proposition 3.6], for any hermitian cube $\overline{E}$ in the category $\mathcal{U}$, there is a unique way to change the metrics on E_α for $\alpha \nleqslant 0$ such that the obtained new hermitian cube is emi. In fact, for $i = 1, \dots, k$, define $\lambda_i^1\overline{E}$ to be

$$(\lambda_i^1\overline{E})_\alpha = \begin{cases} (E_\alpha, h_\alpha), & \text{if } \alpha_i = -1, 0; \\ (E_\alpha, h'_\alpha), & \text{if } \alpha_i = 1, \end{cases}$$

where h'_α is the metric induced by $h_{\alpha_1,\dots,\alpha_{i-1},0,\alpha_{i+1},\dots,\alpha_k}$. Thus, $\lambda_i^1\overline{E}$ has the same locally free sheaves as $\overline{E}$, but the metrics on the face $\partial_i^1 E$ are induced by the metrics of the face $\partial_i^0\overline{E}$. To measure the difference between $\overline{E}$ and $\lambda_i^1\overline{E}$, let $\lambda_i^2(\overline{E})$ be the hermitian k-cube determined by $\partial_i^{-1}\lambda_i^2(\overline{E}) = \partial_i^1\overline{E}$, $\partial_i^0\lambda_i^2(\overline{E}) = \partial_i^1\lambda_i^1(\overline{E})$, and $\partial_i^1\lambda_i^2(\overline{E}) = 0$. Set $\lambda_i = \lambda_i^1 + \lambda_i^2$, $\lambda = \lambda_k \circ \cdots \circ \lambda_1$ if $k \geqslant 1$ and $\lambda = \mathrm{Id}$ otherwise. Then the map λ induces a morphism of complexes

$$\widetilde{\mathbb{Z}}C_*\mathcal{U} \to \widetilde{\mathbb{Z}}C_*^{\mathrm{emi}}\mathcal{U}$$

which is the quasi-inverse of the inclusion $\widetilde{\mathbb{Z}}C_*^{\mathrm{emi}}\mathcal{U} \hookrightarrow \widetilde{\mathbb{Z}}C_*\mathcal{U}$. To specify the μ_n-equivariant variety X, we shall write $\widetilde{\mathbb{Z}}C_*(X, \mu_n) := \widetilde{\mathbb{Z}}C_*\mathcal{U}$.

Definition 2.3. Fix a primitive nth root of unity ζ_n, the restriction of an equivariant hermitian vector bundle $\overline{F}\,|_{X_{\mu_n}}$ over the fixed-point subvariety splits into a direct sum $\oplus_{l=1}^n \overline{F}_l$, where F_l is the eigenbundle of $F\,|_{X_{\mu_n}}$ corresponding to the eigenvalue ${\zeta_n}^l$. Let K_l be the curvature form with respect to the unique connection on $\overline{F}_l$ compatible with both the hermitian and the complex structure, the equivariant Chern–Weil form associated to $\overline{F}$ is defined as

$$\mathrm{ch}_g^0(\overline{F}) := \sum_{l=1}^{n} {\zeta_n}^l \mathrm{Tr}(\exp(-K_l)).$$

Define $R_n = \mathbb{R}$ if $n = 1$ and $R_n = \mathbb{C}$ otherwise, denote $V \otimes_{\mathbb{R}} R_n$ by V_{R_n} for any real vector space V, the equivariant higher Bott–Chern form associated to hermitian k-cube $\overline{E}$ is defined as

$$\mathrm{ch}_g^k(\overline{E}) := \mathrm{ch}_g^0\big(\mathrm{tr}_k\big(\lambda(\overline{E})\big)\big) \in \bigoplus_{p\geqslant 0} \widetilde{D}^*(X_{\mu_n}, p)[2p]_{R_n}.$$

Definition 2.4. Let $\overline{F}\,|_{X_{\mu_n}} = \oplus_{l=1}^n \overline{F}_l$ be the restriction of an equivariant hermitian vector bundle over the fixed-point subvariety, where F_l is the eigenbundle of $F\,|_{X_{\mu_n}}$ corresponding to the eigenvalue ${\zeta_n}^l$ and K_l is the curvature form of $\overline{F}_l$. The equivariant Todd form is defined as

$$\mathrm{Td}_g(\overline{F}) = \det\left(\frac{-K_n}{1-e^{K_n}}\right) \prod_{l\neq n} \det\left(\frac{1}{1-\zeta_n^{-l}e^{K_l}}\right).$$

When X is proper, Burgos and Wang gave in [17, Section 6] a quasi-inverse $\varphi : \widetilde{D}^*(X,p) \to D^*(X,p)$ of the quasi-isomorphism $\iota : D^*(X,p) \to \widetilde{D}^*(X,p)$. By means of this quasi-inverse, the equivariant higher Bott–Chern form has another expression with value in $\bigoplus_{p\geqslant 0} D^*(X_{\mu_n}, p)[2p]_{R_n}$. To see this expression, let us set $z = x/y$ which defines the coordinate map $\mathbb{C} \to \mathbb{P}^1_{\mathbb{C}}$ by sending $z \to [z,1]$. Then $\log \mid z \mid$ defines an L^1 function on $\mathbb{P}^1_{\mathbb{C}}$, which can be considered as a current. We shall denote by $\log \mid z_1 \mid, \ldots, \log \mid z_k \mid$ the corresponding currents on $(\mathbb{P}^1_{\mathbb{C}})^k$. These currents can be formally considered as elements in $D^1((\mathbb{P}^1_{\mathbb{C}})^k, 1)$, and they satisfy the following differential equation

$$d_{\mathcal{D}} \log \mid z_j \mid = -2\partial\overline{\partial} \log \mid z_j \mid = -2i\pi(\delta_{\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}\times\cdots\times\{\infty\}\times\cdots\times\mathbb{P}^1_{\mathbb{C}}} - \delta_{\mathbb{P}^1_{\mathbb{C}}\times\mathbb{P}^1_{\mathbb{C}}\times\cdots\times\{0\}\times\cdots\times\mathbb{P}^1_{\mathbb{C}}}),$$

where ∞ and 0 stand at the jth place. Let $u_1, \ldots, u_k$ be k elements in $\bigoplus_{p\geqslant 0} D^{2p-1}(\cdot, p)$, we define an element in $\bigoplus_{p\geqslant 0} D^{2p-k}(\cdot, p)$ by the formula

$$C_k(u_1, \ldots, u_k) := -\left(-\frac{1}{2}\right)^{k-1} \sum_{\sigma\in\mathfrak{S}_k} (-1)^{\sigma} u_{\sigma(1)} \bullet (u_{\sigma(2)} \bullet (\cdots u_{\sigma(k)}) \cdots),$$

where $\mathfrak{S}_k$ stands for the kth symmetric group. Then we have

$$\begin{aligned} d_{\mathcal{D}} C_k(u_1, \ldots, u_k) &= \left(-\frac{1}{2}\right) k \sum_{j=1}^k (-1)^{j-1} d_{\mathcal{D}}(u_j) \bullet C_{k-1}(u_1, \ldots, \widehat{u_j}, \ldots, u_k) \\ &= \left(-\frac{1}{2}\right) k \sum_{j=1}^k (-1)^{j-1} d_{\mathcal{D}}(u_j) \wedge C_{k-1}(u_1, \ldots, \widehat{u_j}, \ldots, u_k). \end{aligned} \tag{6}$$

We refer to [30, Lemma 2.9] for a proof of these identities. With the above notations, the equivariant higher Bott–Chern form associated to a hermitian k-cube $\overline{E}$ with $k \geqslant 0$ is given by the expression

$$\varphi\Big(\mathrm{ch}_g^k(\overline{E})\Big) = \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \mathrm{ch}_g^k(\overline{E}) \wedge C_k(\log \mid z_1 \mid^2, \ldots, \log \mid z_k \mid^2).$$

Theorem 2.5. *The equivariant higher Bott–Chern forms induce a morphism of complexes*

$$\widetilde{\mathbb{Z}}C^*(X,\mu_n) \xrightarrow{\lambda} \widetilde{\mathbb{Z}}C^*_{\mathrm{emi}}(X,\mu_n) \xrightarrow{\mathrm{ch}_g^0\circ\mathrm{tr}_*} \bigoplus_{p\geq 0} \widetilde{D}^*(X_{\mu_n},p)[2p]_{R_n} \xrightarrow{\varphi} \bigoplus_{p\geq 0} D^*(X_{\mu_n},p)[2p]_{R_n},$$

which is denoted by ch_g. *Here,* $\widetilde{\mathbb{Z}}C^*(X,\mu_n)$ *and* $\widetilde{\mathbb{Z}}C^*_{\mathrm{emi}}(X,\mu_n)$ *are the (cohomological) complexes associated to the homological complexes* $\widetilde{\mathbb{Z}}C_*(X,\mu_n)$ *and* $\widetilde{\mathbb{Z}}C_*^{\mathrm{emi}}(X,\mu_n)$.

Specify to the case $k=1$, let $\bar{\varepsilon}: 0 \to \bar{E}_{-1} \to \bar{E}_0 \to \bar{E}_1 \to 0$ be a hermitian 1-cube, then

$$\begin{aligned} d_{\mathcal{D}}\mathrm{ch}_g(\bar{\varepsilon}) &= d_{\mathcal{D}}\left(\frac{1}{4\pi i}\int_{\mathbb{P}^1} \mathrm{ch}_g^0(\mathrm{tr}_1(\lambda(\bar{\varepsilon})))\log \mid z \mid^2\right) \\ &= \mathrm{ch}_g(\bar{E}_0) - \mathrm{ch}_g(\bar{E}_{-1}) - \mathrm{ch}_g(\bar{E}_1). \end{aligned}$$

If $\bar{\varepsilon}$ is split, by replacing z by $1/z$, we know that

$$\begin{aligned} \mathrm{ch}_g(\bar{\varepsilon}) &= \frac{1}{4\pi i}\int_{\mathbb{P}^1} \mathrm{ch}_g^0(\mathrm{tr}_1(\lambda(\bar{\varepsilon})))\log \mid z \mid^2 \\ &= \frac{1}{4\pi i}\int_{\mathbb{P}^1} \mathrm{ch}_g^0\big(\bar{E}_1(1)\big)\log \mid z \mid^2 + \frac{1}{4\pi i}\int_{\mathbb{P}^1} \mathrm{ch}_g^0\left(\frac{\bar{E}_{-1}(1)\oplus \bar{E}_{-1}(1)}{\bar{E}_{-1}}\right)\log \mid z \mid^2 \\ &= 0. \end{aligned}$$

Let ch'_g denote the usual equivariant Chern–Weil forms with the factor $2\pi i$ inside

$$\mathrm{ch}'_g(\bar{E}) = \sum_{l=1}^{n} {\zeta_n}^l \mathrm{Tr}\left(\exp(\frac{-K_l}{2\pi i})\right),$$

and let Φ be an operator acting on $2n$-forms by $\Phi(\alpha) = (2\pi i)^{-n}\alpha$. Then

$$\Phi\big(\mathrm{ch}_g(\bar{E})\big) = \mathrm{ch}'_g(\bar{E})$$

and

$$\frac{\bar{\partial}\partial}{2\pi i}(2\Phi(\mathrm{ch}_g(\bar{\varepsilon}))) = \mathrm{ch}'_g(\bar{E}_0) - \mathrm{ch}'_g(\bar{E}_{-1}) - \mathrm{ch}'_g(\bar{E}_1).$$

This means, after a rescaling, $\mathrm{ch}_g(\bar{\varepsilon})$ satisfies the axiomatic conditions for a theory of unique equivariant secondary Bott–Chern classes [24, Theorem 3.4] (see [9, §1, (f)] for the non-equivariant case). Note that in [9], the authors used the supertraces of Quillen's superconnections to define the non-equivariant secondary Bott–Chern form $\widetilde{\mathrm{ch}}$. Split $\bar{\varepsilon}\mid_{X_{\mu_n}}$ into a direct sum of short exact sequences of its eigenbundles $\oplus_{l=1}^n \bar{\varepsilon}_l$ and define

$$\widetilde{\mathrm{ch}}_g(\bar{\varepsilon}) := \sum_{l=1}^{n} {\zeta_n}^l \widetilde{\mathrm{ch}}(\bar{\varepsilon}_l).$$

Then we get another way, using the supertraces of Quillen's superconnections, to define the equivariant secondary Bott–Chern form $\widetilde{\mathrm{ch}}_g(\bar{\varepsilon})$ which satisfies the equation

$$\frac{\bar{\partial}\partial}{2\pi i}\widetilde{\mathrm{ch}}_g(\bar{\varepsilon}) = \mathrm{ch}'_g(\bar{E}_0) - \mathrm{ch}'_g(\bar{E}_{-1}) - \mathrm{ch}'_g(\bar{E}_1).$$

So, $2\Phi(\mathrm{ch}_g(\bar{\varepsilon}))$ must be equal to $\widetilde{\mathrm{ch}}_g(\bar{\varepsilon})$ modulo $\mathrm{Im}\,\partial + \mathrm{Im}\,\bar{\partial}$. Let us write $2\Phi(\mathrm{ch}_g(\bar{\varepsilon})) - \widetilde{\mathrm{ch}}_g(\bar{\varepsilon}) = \partial\Delta_\partial(\bar{\varepsilon}) + \bar{\partial}\Delta_{\bar{\partial}}(\bar{\varepsilon})$, the following theorem states that $\Delta_\partial(\bar{\varepsilon})$ and $\Delta_{\bar{\partial}}(\bar{\varepsilon})$ can be chosen to admit some functorial property.

THEOREM 2.6. *Let notations and assumptions be as above. There is a functorial choice of the differential forms $\Delta_\partial(\bar{\varepsilon})$ and $\Delta_{\bar{\partial}}(\bar{\varepsilon})$ such that*

$$2\Phi(\mathrm{ch}_g(\bar{\varepsilon})) - \widetilde{\mathrm{ch}}_g(\bar{\varepsilon}) = \partial\Delta_\partial(\bar{\varepsilon}) + \bar{\partial}\Delta_{\bar{\partial}}(\bar{\varepsilon})$$

and that $\Delta_\partial(j^*\bar{\varepsilon}) = j^*\Delta_\partial(\bar{\varepsilon}), \Delta_{\bar{\partial}}(j^*\bar{\varepsilon}) = j^*\Delta_{\bar{\partial}}(\bar{\varepsilon})$ for any equivariant morphism $j: X' \to X$.

Proof. For hermitian 1-cube $\bar{\varepsilon}: 0 \to \bar{E}_{-1} \to \bar{E}_0 \to \bar{E}_1 \to 0$, we divide it into two emi-1-cubes

$$\bar{\varepsilon}_1 : 0 \to \bar{E}_{-1} \to \bar{E}_0 \to \bar{E}'_1 \to 0$$

and

$$\bar{\varepsilon}_2 : 0 \to \bar{E}_1 \to \bar{E}_1' \to 0 \to 0,$$

where $\bar{E}_1'$ is E_1 endowed with the quotient metric. According to the definition of the morphism λ, the higher Bott–Chern form is additive $\mathrm{ch}_g(\bar{\varepsilon}) = \mathrm{ch}_g(\bar{\varepsilon}_1) + \mathrm{ch}_g(\bar{\varepsilon}_2)$. To study the secondary Bott–Chern form constructed by the supertraces of Quillen's superconnections, we write down a double complex

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bar{E}_{-1} & \longrightarrow & \bar{E}_0 & \longrightarrow & \bar{E}_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bar{E}_{-1} & \longrightarrow & \bar{E}_0 & \longrightarrow & \bar{E}_1' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{7}$$

Restrict every bundle over X_{μ_n} and split it into the direct sum of eigenbundles, then one can immediately repeat the argument given in [9, Theorem 1.20] (where the non-equivariant bundles were dealt with) to write down a proof of the fact that $\widetilde{\mathrm{ch}}_g(\bar{\varepsilon}) = \widetilde{\mathrm{ch}}_g(\bar{\varepsilon}_1) + \widetilde{\mathrm{ch}}_g(\bar{\varepsilon}_2)$ modulo $\mathrm{Im}\,\partial + \mathrm{Im}\,\bar{\partial}$. In the proof of [9, Theorem 1.20], the error terms were explicitly written down and were functorial (see [9, (1.71), (1.72), (1.78), (1.81) and (1.82)]). That means one can fix a functorial choice of differential forms $\Delta_\partial'(\bar{\varepsilon})$ and $\Delta_{\bar{\partial}}'(\bar{\varepsilon})$ such that

$$\widetilde{\mathrm{ch}}_g(\bar{\varepsilon}) - \left(\widetilde{\mathrm{ch}}_g(\bar{\varepsilon}_1) + \widetilde{\mathrm{ch}}_g(\bar{\varepsilon}_2)\right) = \partial\Delta_\partial'(\bar{\varepsilon}) + \bar{\partial}\Delta_{\bar{\partial}}'(\bar{\varepsilon}).$$

So, we may reduce our proof to the case, where $\bar{\varepsilon}$ is an emi-1-cube.

Now, we consider the following exact sequence on $X \times \mathbb{P}^1$

$$\Psi : \quad 0 \longrightarrow \bar{E}_{-1} \xrightarrow{\mathrm{Id}\otimes\sigma_\infty \oplus i\otimes\sigma_0} \bar{E}_{-1}(1) \oplus \bar{E}_0(1) \longrightarrow \mathrm{tr}_1(\bar{\varepsilon}) \longrightarrow 0,$$

we compute, using the fact that $\int_{\mathbb{P}^1} \mathrm{ch}_g(\bar{E}_{-1}) \log \mid z \mid = 0$, $\int_{\mathbb{P}^1} \mathrm{ch}_g(\bar{E}_{-1}(1) \oplus \bar{E}_0(1)) \log \mid z \mid = 0$ and the Stokes formula,

$$\begin{aligned}
2\Phi(\mathrm{ch}_g(\bar{\varepsilon})) &= \int_{\mathbb{P}^1} \Phi(\mathrm{ch}_g(\mathrm{tr}_1(\bar{\varepsilon}))) \log \mid z \mid^2 \\
&= -\int_{\mathbb{P}^1} \frac{\bar{\partial}\partial}{2\pi i} \widetilde{\mathrm{ch}}_g(\Psi) \log \mid z \mid^2 \\
&= \frac{1}{2\pi i} \int_{\mathbb{P}^1} (\partial_X + \partial_z)(\bar{\partial}_X + \bar{\partial}_z) \widetilde{\mathrm{ch}}_g(\Psi) \log \mid z \mid^2 \\
&= \partial_X\bar{\partial}_X \left(\frac{1}{2\pi i} \int_{\mathbb{P}^1} \widetilde{\mathrm{ch}}_g(\Psi) \log \mid z \mid^2\right) + \partial_X \left(\frac{1}{2\pi i} \int_{\mathbb{P}^1} \bar{\partial}_z \widetilde{\mathrm{ch}}_g(\Psi) \log \mid z \mid^2\right) \\
&\quad - \bar{\partial}_X \left(\frac{1}{2\pi i} \int_{\mathbb{P}^1} \partial_z \widetilde{\mathrm{ch}}_g(\Psi) \log \mid z \mid^2\right) + \frac{1}{2\pi i} \int_{\mathbb{P}^1} \partial_z\bar{\partial}_z \widetilde{\mathrm{ch}}_g(\Psi) \log \mid z \mid^2
\end{aligned}$$

$$= \frac{1}{2\pi i}\int_{\mathbb{P}^1} \widetilde{\mathrm{ch}}_g(\Psi)\partial_z\bar{\partial}_z \log \mid z \mid^2 + \partial_X\bar{\partial}_X\left(\frac{1}{2\pi i}\int_{\mathbb{P}^1} \widetilde{\mathrm{ch}}_g(\Psi)\log \mid z \mid^2\right)$$

$$+ \partial_X\left(\frac{1}{2\pi i}\int_{\mathbb{P}^1} \bar{\partial}_z\widetilde{\mathrm{ch}}_g(\Psi)\log \mid z \mid^2\right) - \bar{\partial}_X\left(\frac{1}{2\pi i}\int_{\mathbb{P}^1} \partial_z\widetilde{\mathrm{ch}}_g(\Psi)\log \mid z \mid^2\right).$$

Set

$$\Delta_\partial(\bar{\varepsilon}) = \bar{\partial}_X\left(\frac{1}{2\pi i}\int_{\mathbb{P}^1} \widetilde{\mathrm{ch}}_g(\Psi)\log \mid z \mid^2\right) + \frac{1}{2\pi i}\int_{\mathbb{P}^1} \bar{\partial}_z\widetilde{\mathrm{ch}}_g(\Psi)\log \mid z \mid^2$$

and

$$\Delta_{\bar{\partial}}(\bar{\varepsilon}) = -\frac{1}{2\pi i}\int_{\mathbb{P}^1} \partial_z\widetilde{\mathrm{ch}}_g(\Psi)\log \mid z \mid^2 .$$

We get

$$2\Phi(\mathrm{ch}_g(\bar{\varepsilon})) = \widetilde{\mathrm{ch}}_g(\Psi \mid_{X\times\{\infty\}}) - \widetilde{\mathrm{ch}}_g(\Psi \mid_{X\times\{0\}}) + \partial\Delta_\partial(\bar{\varepsilon}) + \bar{\partial}\Delta_{\bar{\partial}}(\bar{\varepsilon}).$$

By the construction, $\Psi \mid_{X\times\{0\}}$ is split and $\Psi \mid_{X\times\{\infty\}}$ is isometric to the direct sum of $\bar{\varepsilon}$ and a split exact sequence $0 \to 0 \to \bar{E}_{-1} \to \bar{E}_{-1} \to 0$, we finally have

$$2\Phi(\mathrm{ch}_g(\bar{\varepsilon})) - \widetilde{\mathrm{ch}}_g(\bar{\varepsilon}) = \partial\Delta_\partial(\bar{\varepsilon}) + \bar{\partial}\Delta_{\bar{\partial}}(\bar{\varepsilon}).$$

Since the 1-transgression bundle construction Ψ is functorial, the differential forms $\Delta_\partial(\bar{\varepsilon})$ and $\Delta_{\bar{\partial}}(\bar{\varepsilon})$ associated to $\widetilde{\mathrm{ch}}_g(\Psi)$ are also functorial, thus we complete the whole proof. □

Remark 2.7. (i) According to Theorem 2.6, we can make a functorial choice of differential forms $\Delta_\partial(\bar{\varepsilon})$ and $\Delta_{\bar{\partial}}(\bar{\varepsilon})$ for any hermitian 1-cube $\bar{\varepsilon}$ such that

$$2\Phi(\mathrm{ch}_g(\bar{\varepsilon})) - \widetilde{\mathrm{ch}}_g(\bar{\varepsilon}) = \partial\Delta_\partial(\bar{\varepsilon}) + \bar{\partial}\Delta_{\bar{\partial}}(\bar{\varepsilon}).$$

Set $\Delta(\bar{\varepsilon}) = -\Phi^{-1}(\frac{\Delta_\partial(\bar{\varepsilon})+\Delta_{\bar{\partial}}(\bar{\varepsilon})}{2})$, then $\Delta(\bar{\varepsilon})$ is functorial and by the definition of the Deligne complex $\mathfrak{D}^*(E_{\log}(X), p)$ we have

$$\mathrm{ch}_g(\bar{\varepsilon}) - \Phi^{-1}\left(\frac{\widetilde{\mathrm{ch}}_g(\bar{\varepsilon})}{2}\right) = -\pi\big((\partial+\bar{\partial})\Delta(\bar{\varepsilon})\big) = d_{\mathcal{D}}\Delta(\bar{\varepsilon}).$$

(ii) It is easily seen from the proof of Theorem 2.6 that if one uses another way to define the equivariant Bott–Chern form $\widetilde{\mathrm{ch}}_g$ which satisfies the axiomatic conditions in [**9**, Theorem 1.29] (at the level of differential forms) and which is additive for direct sum of short exact sequences, then one can also make a functorial choice of element $\Delta(\bar{\varepsilon})$ for any hermitian emi-1-cube $\bar{\varepsilon}$ such that

$$\mathrm{ch}_g(\bar{\varepsilon}) - \Phi^{-1}\left(\frac{\widetilde{\mathrm{ch}}_g(\bar{\varepsilon})}{2}\right) = d_{\mathcal{D}}\Delta(\bar{\varepsilon}).$$

If X is a regular μ_n-projective arithmetic scheme over an arithmetic ring (D, Σ, F_∞), we shall denote $X_{\mathbb{R}} := (X(\mathbb{C}), F_\infty)$ the real variety associated to X, where F_∞ is the antiholomorphic involution of $X(\mathbb{C})$ induced by the conjugate-linear involution F_∞ over (D, Σ, F_∞). For any sheaf of complex vector spaces V with a real structure over $X_{\mathbb{R}}$, we denote by σ the involution given by

$$\omega \mapsto \overline{F_\infty^*(\omega)}.$$

Write $D^*(X_{\mathbb{R}}, p) := D^*(X(\mathbb{C}), p)^\sigma$ for the subcomplex of $D^*(X(\mathbb{C}), p)$ consisting of the fixed elements under σ, we define the real Deligne–Beilinson cohomology of X as

$$H_D^*(X, \mathbb{R}(p)) := H^*(D^*(X_{\mathbb{R}}, p)).$$

Let us denote by $\widehat{\mathcal{P}}(X,\mu_n)$ the exact category of μ_n-equivariant hermitian vector bundles on X, and by $\widehat{S}(X,\mu_n)$ the simplicial set associated to the Waldhausen S-construction of $\widehat{\mathcal{P}}(X,\mu_n)$ (cf. [**34**, Section 2.3]). The forgetful functor (forget about the metrics) $\pi:\widehat{\mathcal{P}}(X,\mu_n)\to\mathcal{P}(X,\mu_n)$ induces an equivalence of categories, so we have homotopy equivalence

$$|\ \widehat{S}(X,\mu_n)\ |\simeq|\ S(X,\mu_n)\ |$$

and isomorphisms of abelian groups

$$K_m(X,\mu_n)\cong\pi_{m+1}(|\ \widehat{S}(X,\mu_n)\ |,0)$$

for any $m\geqslant 0$. To give the simplicial description of the equivariant regulator maps, we associate to each element in $S_k\widehat{\mathcal{P}}(X,\mu_n)$ a hermitian $(k-1)$-cube. First, note that an element A in $S_k\widehat{\mathcal{P}}(X,\mu_n)$ is a family of injections

$$A_{0,1}\rightarrowtail A_{0,2}\rightarrowtail\cdots\rightarrowtail A_{0,k}$$

of μ_n-equivariant hermitian vector bundles on X with quotients $A_{i,j}\simeq A_{0,j}/A_{0,i}$ for each $i<j$. For $k=1$, we write

$$\mathrm{Cub}(A_{0,1})=A_{0,1}.$$

Suppose that the map Cub is defined for all $l<k$, then Cub$A$ is the $(k-1)$-cube with

$$\partial_1^{-1}\mathrm{Cub}A=s^1_{k-2}\cdots s^1_1(A_{0,1}),$$

$$\partial_1^1\mathrm{Cub}A=\mathrm{Cub}(\partial_0 A).$$

Let $\mathbb{Z}\widehat{S}_*(X,\mu_n)$ be the simplicial abelian group generated by the simplicial set $\widehat{S}(X,\mu_n)$, and let $\mathcal{N}(\mathbb{Z}\widehat{S}_*(X,\mu_n))$ be the Moore complex associated to $\mathbb{Z}\widehat{S}_*(X,\mu_n)$ with differential $d=\sum_{i=0}^k(-1)^i\partial_i$, where ∂_i is the face map of $\widehat{S}(X,\mu_n)$. Then, according to [**17**, Corollary 4.8], the map Cub defined above extends by linearity to a morphism of homological complexes

$$\mathrm{Cub}:\quad\mathcal{N}\Big(\mathbb{Z}\widehat{S}_*(X,\mu_n)\Big)\to\widetilde{\mathbb{Z}}C_*(X,\mu_n)[-1],$$

and hence one gets a simplicial map

$$\mathrm{Cub}:\quad\mathbb{Z}\widehat{S}_*(X,\mu_n)\to\mathcal{K}\Big(\widetilde{\mathbb{Z}}C_*(X,\mu_n)[-1]\Big),$$

where $\mathcal{K}$ is the Dold–Puppe functor.

Definition 2.8. Let notations and assumptions be as above. We denote by $D^{2p-*}(X_{\mu_n},p)$ the homological complex associated to the complex $\tau_{\leqslant 0}(D^*(X_{\mu_n},p)[2p])$ which is the canonical truncation of $D^*(X_{\mu_n},p)[2p]$ at degree 0. We define a simplicial map

$$\begin{array}{ccc}\widetilde{\mathrm{ch}}_g:\widehat{S}(X,\mu_n)&\xrightarrow{\mathrm{Hu}}&\mathbb{Z}\widehat{S}_*(X,\mu_n)\\ &&\Big\downarrow{\scriptstyle\mathrm{Cub}}\\ &&\mathcal{K}\left(\widetilde{\mathbb{Z}}C_*(X,\mu_n)[-1]\right)\xrightarrow{\mathcal{K}(\mathrm{ch}_g)}\mathcal{K}\left(\bigoplus_{p\geq 0}D^{2p-*}(X_{\mu_n},p)[-1]_{R_n}\right),\end{array}$$

where Hu is the Hurewicz map.

Definition 2.9. Let X be a regular μ_n-projective scheme over an arithmetic ring (D,Σ,F_∞), and let X_{μ_n} be the fixed-point subscheme. The higher equivariant arithmetic K-groups of X are defined as

$$\widehat{K}_m(X,\mu_n):=\pi_{m+1}\Big(\text{homotopy fiber of }\ |\ \widetilde{\mathrm{ch}}_g\ |\Big)\quad\text{for}\quad m\geqslant 1,$$

and the equivariant regulator maps

$$\mathrm{ch}_g : \quad K_m(X,\mu_n) \to \bigoplus_{p\geqslant 0} H_D^{2p-m}(X_{\mu_n},\mathbb{R}(p))_{R_n}$$

are defined as the homomorphisms induced by $\widetilde{\mathrm{ch}}_g$ at the level of homotopy groups.

Remark 2.10. (i). We have the long exact sequence

$$\cdots \to \widehat{K}_m(X,\mu_n) \to K_m(X,\mu_n) \to \bigoplus_{p\geqslant 0} H_{\mathcal{D}}^{2p-m}(X_{\mu_n},\mathbb{R}(p))_{R_n} \to \widehat{K}_{m-1}(X,\mu_n) \to \cdots$$

ending with

$$\begin{array}{l} \cdots \to K_1(X,\mu_n) \longrightarrow \bigoplus_{p\geq 0} H_{\mathcal{D}}^{2p-1}(X_{\mu_n},\mathbb{R}(p))_{R_n} \\ \qquad\qquad\qquad\qquad\qquad\qquad \downarrow \\ \qquad\qquad \pi_1\left(\text{homotopy fiber of } \widetilde{\mathrm{ch}}_g\right) \longrightarrow K_0(X,\mu_n) \to \bigoplus_{p\geq 0} H_{\mathcal{D}}^{2p}(X_{\mu_n},\mathbb{R}(p))_{R_n}. \end{array}$$

(ii) When $n=1$, the equivariant higher Bott–Chern forms given in Definition 2.3 coincide with the higher Bott–Chern forms defined in [17] for non-equivariant proper varieties. So, in this case,

$$\mathrm{ch}_g : \quad K_m(X,\mu_1) \to \bigoplus_{p\geqslant 0} H_D^{2p-m}(X,\mathbb{R}(p))$$

is the Beilinson's regulator map.

(iii) The higher equivariant arithmetic K-groups $\widehat{K}_m(X,\mu_n)$ can be defined for non-proper X, for details, see [34, Section 2].

(iv) Let $s(\mathrm{ch}_g)$ denote the simple complex associated to the chain morphism

$$\mathrm{ch}_g : \qquad \widetilde{\mathbb{Z}}C_*(X,\mu_n) \xrightarrow{\mathrm{ch}_g} \bigoplus_{p\geq 0} D^{2p-*}(X_{\mu_n},p)_{R_n}.$$

Then, for any $m \geqslant 1$, there is an isomorphism

$$\widehat{K}_m(X,\mu_n)_{\mathbb{Q}} \cong H_m(s(\mathrm{ch}_g),\mathbb{Q}).$$

(v) A μ_n-equivariant hermitian sheaf on X is a μ_n-equivariant coherent sheaf on X which is locally free on $X(\mathbb{C})$ and is equipped with a μ_n-invariant hermitian metric. To a μ_n-equivariant hermitian sheaf, the higher equivariant Bott–Chern form can still be defined in the same way. Denote by $\widehat{\mathcal{P}}'(X,\mu_n)$ the category of μ_n-equivariant hermitian sheaves on X, then instead of $\widehat{\mathcal{P}}(X,\mu_n)$ one may define a new arithmetic K-theory $\widehat{K}'_*(X,\mu_n)$ which is called the equivariant arithmetic K′-theory. Since $\widehat{\mathcal{P}}'(X,\mu_n)$ and $\widehat{\mathcal{P}}(X,\mu_n)$ define the same algebraic K-theory when X is regular, it is easily seen from the Five-lemma that the natural inclusion $\widehat{\mathcal{P}}(X,\mu_n) \subset \widehat{\mathcal{P}}'(X,\mu_n)$ induces isomorphisms $\widehat{K}_m(X,\mu_n) \cong \widehat{K}'_m(X,\mu_n)$ for any $m \geqslant 1$.

2.2. *Equivariant analytic torsion for hermitian cubes*

In [12], Bismut and Köhler developed a theory of higher analytic torsion forms for holomorphic submersions of complex manifolds. The higher analytic torsion form solves a differential equation which gives a refinement of the Grothendieck–Riemann–Roch theorem at the level of characteristic forms. Later, in [27], Ma generalized Bismut and Köhler's results to the equivariant case. Considering the higher K-theory and the Deligne-Beilinson cohomology, to make a refinement of the Riemann–Roch theorem at the level of higher Bott–Chern forms representing the regulator maps, one needs an extension of higher analytic torsion

for hermitian cubes, this has been done in [30]. In this subsection, we do the equivariant case by using Ma's equivariant analytic torsion forms. Our construction is slightly different to Roessler's construction.

Let X, Y be two smooth μ_n-projective varieties over $\mathbb{C}$, and let $f : X \to Y$ be an equivariant and smooth morphism. A Kähler fibration structure on f is a real closed (1,1)-form ω on X which induces Kähler metrics on the fibers of f (cf. [12, Definition 1.1, Theorem 1.2]). For instance, we may fix a μ_n-invariant Kähler metric on X and choose corresponding Kähler form ω as a Kähler fibration structure on f. Let (E, h^E) be a μ_n-equivariant hermitian vector bundle on X such that E is f-acyclic, that is, the higher direct image $R^q f_* E$ vanishes for $q > 0$. The equivariant analytic torsion form $T_g(f, \omega, h^E)$ is an element of $\bigoplus_{p \geqslant 0} D^{2p-1}(Y_{\mu_n}, p)_{R_n}$, which depends on f, ω and (E, h^E) and satisfies the differential equation

$$d_{\mathcal{D}} T_g(f, \omega, h^E) = \mathrm{ch}_g(f_* E, f_* h^E) - \frac{1}{(2\pi i)^r} \int_{X_{\mu_n}/Y_{\mu_n}} \mathrm{Td}_g(Tf, h^{Tf}) \mathrm{ch}_g(E, h^E),$$

where h^{Tf} is the hermitian metric induced by ω on the holomorphic tangent bundle Tf, r is the rank of the bundle Tf_{μ_n}, and $f_* h^E$ is the L^2-metric on $f_* E$ (see the end of [30, Section 2.2] for a definition). By definition, for elements $u, v \in (f_* E)_y$ of the fiber of $f_* E$ over a point $y \in Y$, the L^2-hermitian product is given by

$$\langle u, v \rangle_{L^2} = \frac{1}{(2\pi)^b} \int_{f^{-1} y} \langle u, v \rangle_E \frac{\omega^b}{b!},$$

where b is the relative dimension of X over Y.

We would like to caution the reader that the equivariant analytic torsion form we use here coincides with Ma's definition only up to a rescaling. If we denote by $T'_g(f, \omega, h^E)$ Ma's equivariant torsion form, then the equality $2\Phi(T_g(f, \omega, h^E)) = T'_g(f, \omega, h^E)$ holds. From now on, we shall write $T_g(\omega, h^E)$ or $T_g(h^E)$ for $T_g(f, \omega, h^E)$, if there is no ambiguity about the underlying map or Kähler form. Now, let ω' be the form associated to another Kähler fibration structure on $f : X \to Y$ and let h'^{Tf} be the metric on Tf induced by this new fibration. Let $\widetilde{\mathrm{Td}}_g(Tf, h'^{Tf}, h^{Tf})$ be the equivariant secondary Todd form used in the Appendix (Section A.1 (A.1)), and set $\mathrm{Td}_g(Tf, h'^{Tf}, h^{Tf}) = \Phi^{-1}(\frac{\widetilde{\mathrm{Td}}_g(Tf, h'^{Tf}, h^{Tf})}{2})$. So,

$$d_{\mathcal{D}} \mathrm{Td}_g(Tf, h'^{Tf}, h^{Tf}) = \mathrm{Td}_g(Tf, h^{Tf}) - \mathrm{Td}_g(Tf, h'^{Tf}).$$

The following anomaly formula is useful for our later discussion.

THEOREM 2.11. *Let notations and assumptions be as above. The following identity holds in* $\bigoplus_{p \geqslant 0} (D^{2p-1}(Y_{\mu_n}, p)/\mathrm{Im}\, d_{\mathcal{D}})$*:*

$$T_g(\omega, h^E) - T_g(\omega', h^E) = \mathrm{ch}_g(f_* E, h'^{f_* E}, h^{f_* E}) - \frac{1}{(2\pi i)^r} \int_{X_{\mu_n}/Y_{\mu_n}} \mathrm{Td}_g(Tf, h'^{Tf}, h^{Tf}) \mathrm{ch}_g(E, h^E),$$

where $(f_* E, h'^{f_* E}, h^{f_* E})$ *stands for the emi-1-cubes of hermitian vector bundles*

$$0 \longrightarrow (f_* E, h'^{f_* E}) \xrightarrow{\mathrm{Id}} (f_* E, h^{f_* E}) \longrightarrow 0 \longrightarrow 0.$$

Proof. This is a translation of [27, Theorem 2.13], see also Theorem A.1. Considering the relation between the equivariant analytic torsion forms $T_g(\omega, h^E), T_g(\omega', h^E)$ and the ones used in Ma's paper, we only need to show

$$\mathrm{ch}_g(f_* E, h'^{f_* E}, h^{f_* E}) = \Phi^{-1}\left(\frac{\widetilde{\mathrm{ch}}_g(f_* E, h'^{f_* E}, h^{f_* E})}{2}\right) \in \bigoplus_{p \geqslant 0} \left(D^{2p-1}(Y_{\mu_n}, p)/\mathrm{Im}\, d_{\mathcal{D}}\right).$$

But this is the content of Remark 2.7. □

According to Remark 2.7 and Theorem A.2, there exists a functorial choice of the differential form which measures the difference

$$T_g(\omega,h^E)-T_g(\omega',h^E)-\mathrm{ch}_g(f_*E,h'^{f_*E},h^{f_*E})+\frac{1}{(2\pi i)^r}\int_{X_{\mu_n}/Y_{\mu_n}}\mathrm{Td}_g(Tf,h'^{Tf},h^{Tf})\mathrm{ch}_g(E,h^E)$$

in Theorem 2.11. With the same notations as in Remark 2.7 and Theorem A.2, we set

$$\Delta(f,\overline{E},\omega,\omega'):=-\Phi^{-1}\left(\frac{\Delta^0(f,\overline{E},\omega,\omega')+\Delta_0(f,\overline{E},\omega,\omega')}{2}\right)+\Delta(f_*E,h'^{f_*E},h^{f_*E}),$$

it satisfies the differential equation

$$\begin{aligned}d_{\mathcal{D}}\Delta(f,\overline{E},\omega,\omega')&=T_g(\omega,h^E)-T_g(\omega',h^E)-\mathrm{ch}_g(f_*E,h'^{f_*E},h^{f_*E})\\&\quad+\frac{1}{(2\pi i)^r}\int_{X_{\mu_n}/Y_{\mu_n}}\mathrm{Td}_g(Tf,h'^{Tf},h^{Tf})\mathrm{ch}_g(E,h^E).\end{aligned}$$

We consider the following setting. Let Z be a compact Kähler manifold and let Z_1 be a closed submanifold of Z. Choose a Kähler metric on Z and endow Z_1 with the restricted metric. Let $f_Z:X\times Z\to Y\times Z$ be the induced map and let ω,ω' be the Kähler forms of the product metrics on $X\times Z$ with respect to two Kähler fibrations on $f:X\to Y$. Similarly, let $f_{Z_1}:X\times Z_1\to Y\times Z_1$ be the induced map and let ω_1,ω_1' be the Kähler forms of the product metrics on $X\times Z_1$ with respect to the same two Kähler fibrations on $f:X\to Y$. We shall denote by j (respectively, i) the natural embedding $X\times Z_1\to X\times Z$ (respectively, $Y\times Z_1\to Y\times Z$). Then $j^*\omega=\omega_1$ and $j^*\omega'=\omega_1'$. Let $\overline{E}$ be an f_Z-acyclic hermitian bundle on $X\times Z$, we have the following result.

LEMMA 2.12. *The identity* $i_{\mu_n}^*\Delta(f_Z,\overline{E},\omega,\omega')=\Delta(f_{Z_1},j^*\overline{E},\omega_1,\omega_1')$ *holds.*

Proof. This is a consequence of Theorem A.2. □

DEFINITION 2.13. By a chain homotopy of a diagram of homological complexes

$$\begin{array}{ccc}A_*&\xrightarrow{i}&B_*\\\downarrow{\scriptstyle f}&&\downarrow{\scriptstyle l}\\C_*&\xrightarrow{j}&D_*,\end{array}$$

we understand a chain homotopy between the complex morphisms $j\circ f$ and $l\circ i$.

Roughly speaking, the equivariant analytic torsion for hermitian cubes is a chain homotopy of the following diagram:

$$\begin{array}{ccc}\widetilde{\mathbb{Z}}C_*^{f-\mathrm{ac}}(X,\mu_n)&\xrightarrow{\mathrm{ch}_g}&\bigoplus_{p\geq 0}D^{2p-*}(X_{\mu_n},p)_{R_n}\\\downarrow{\scriptstyle f_*}&&\downarrow{\scriptstyle f_{\mu_n *}\circ\mathrm{Td}_g(\overline{Tf})\bullet(\cdot)}\\\widetilde{\mathbb{Z}}C_*(Y,\mu_n)&\xrightarrow{\mathrm{ch}_g}&\bigoplus_{p\geq 0}D^{2p-*}(Y_{\mu_n},p)_{R_n}\end{array}\quad,\tag{8}$$

where $\widetilde{\mathbb{Z}}C_*^{f-\mathrm{ac}}(X,\mu_n)$ is the subcomplex of $\widetilde{\mathbb{Z}}C_*(X,\mu_n)$ made of f-acyclic bundles. Since the Waldhausen K-theory space of $\widehat{\mathcal{P}}(X,\mu_n)$ is homotopy equivalent to the Waldhausen K-theory space of the full subcategory of $\widehat{\mathcal{P}}(X,\mu_n)$ consisting of f-acyclic bundles, we shall always work with acyclic bundles.

Like the non-equivariant case treated in [**30**], the equivariant analytic torsion for hermitian cubes induces a commutative diagram at the level of homology groups and hence one gets an analytic proof of the equivariant version of Gillet's Riemann–Roch theorem for higher algebraic K-theory.

To construct a chain homotopy of (8), let us move in two steps. Note that the equivariant higher Bott–Chern form factors as

$$\widetilde{\mathbb{Z}}C_*(X,\mu_n) \xrightarrow{\lambda} \widetilde{\mathbb{Z}}C_*^{\mathrm{emi}}(X,\mu_n) \xrightarrow{\mathrm{ch}_g^0\circ \mathrm{tr}_*} \bigoplus_{p\geq 0} D^{2p-*}(X_{\mu_n},p)_{R_n},$$

we first clarify the difference between $f_*(\mathrm{tr}\circ\lambda(\cdot))$ and $\mathrm{tr}\circ\lambda(f_*(\cdot))$. Let $\overline{E}$ be a f-acyclic hermitian k-cube in $\widehat{\mathcal{P}}(X,\mu_n)$. The hermitian bundles $f_*(\mathrm{tr}_k\circ\lambda(\overline{E}))$ and $\mathrm{tr}_k\circ\lambda(f_*(\overline{E}))$ are canonically isomorphic as bundles, but carry in general different metrics. For instance, assume that $\overline{E}$ is a hermitian emi-1-cube, then $f_*(\mathrm{tr}_1(\overline{E}))$ and $\mathrm{tr}_1(f_*(\overline{E}))$ fit into the following two exact sequences:

$$0 \longrightarrow f_*(p_X^*\bar{E}_{-1}) \longrightarrow f_*(p_X^*\bar{E}_{-1})(1)\oplus f_*(p_X^*\bar{E}_0)(1) \longrightarrow f_*\left(\mathrm{tr}_1(\overline{E})\right) \longrightarrow 0,$$

and

$$0 \longrightarrow p_Y^*\left(f_*(\bar{E}_{-1})\right) \longrightarrow p_Y^*\left(f_*(\bar{E}_{-1})\right)(1)\oplus p_Y^*\left(f_*(\bar{E}_0)(1)\right) \longrightarrow \mathrm{tr}_1\left(f_*(\overline{E})\right) \longrightarrow 0.$$

Here, p_X (respectively, p_Y) stands for the obvious projection $X\times\mathbb{P}^1\to X$ (respectively, $Y\times\mathbb{P}^1\to Y$). By the definition of the L^2-metric, over the point (y,t) in $Y\times\mathbb{P}^1$, the hermitian product on $f_*(\mathrm{tr}_1(\overline{E}))_{(y,t)}$ relies on the integral of certain power of the Kähler form $\omega_{X\times\mathbb{P}^1}$ over the fiber f_t and hence relies on t. But the pullback hermitian products on $p_Y^*(f_*(\bar{E}_0)_{(y,t)}$ and on $p_Y^*(f_*(\bar{E}_{-1})_{(y,t)}$ equal the hermitian products on $f_*(\bar{E}_0)_y$ and on $f_*(\bar{E}_{-1})_y$ which do not rely on t, therefore the induced hermitian product on $\mathrm{tr}_1(f_*(\overline{E})_{(y,t)}$ does not rely on t neither. So, in general, $f_*(\mathrm{tr}_1(\overline{E}))$ and $\mathrm{tr}_1(f_*(\overline{E}))$ carry different metrics.

In the following, we shall write $H(\overline{E})$ for the short exact sequence

$$0 \longrightarrow f_*\left(\mathrm{tr}_k\circ\lambda(\overline{E})\right) \xrightarrow{\mathrm{Id}} \mathrm{tr}_k\circ\lambda\left(f_*(\overline{E})\right) \longrightarrow 0 \longrightarrow 0$$

which is an emi-1-cube of hermitian bundles on $Y\times(\mathbb{P}^1)^k$. The transgression bundle of $H(\overline{E})$ is a hermitian bundle on $Y\times(\mathbb{P}^1)^{k+1}=Y\times(\mathbb{P}^1)^k\times\mathbb{P}^1$. But here we change the order of the $\mathbb{P}^1$, let p_1 be the first projection from $Y\times\mathbb{P}^1\times(\mathbb{P}^1)^k$ to $Y\times(\mathbb{P}^1)^k$, we apply the transgression bundle construction to the short exact sequence $H(\overline{E})$ with respect to the projection p_1 to get a hermitian bundle on $Y\times(\mathbb{P}^1)^{k+1}$. With some abuse of notation, we still denote this hermitian bundle by $\mathrm{tr}_1(H(\overline{E}))$ and it satisfies the following relations:

$$\mathrm{tr}_1\big(H(\overline{E})\big)\mid_{Y\times\{0\}\times(\mathbb{P}^1)^k}=\mathrm{tr}_k\circ\lambda\big(f_*(\overline{E})\big),\quad \mathrm{tr}_1\big(H(\overline{E})\big)\mid_{Y\times\{\infty\}\times(\mathbb{P}^1)^k}=f_*\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big)$$

and

$$\mathrm{tr}_1\big(H(\overline{E})\big)\mid_{Y\times(\mathbb{P}^1)^i\times\{0\}\times(\mathbb{P}^1)^{k-i}}=\mathrm{tr}_1\big(H(\partial_i^0\overline{E})\big),$$

$$\mathrm{tr}_1\big(H(\overline{E})\big)\mid_{Y\times(\mathbb{P}^1)^i\times\{\infty\}\times(\mathbb{P}^1)^{k-i}}=\mathrm{tr}_1\big(H(\partial_i^{-1}\overline{E})\big)\oplus\mathrm{tr}_1\big(H(\partial_i^1\overline{E})\big)$$

for $i=1,\dots,k$. Now, we define

$$\Pi'_k(\overline{E}):=\frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}}\int_{(\mathbb{P}^1)^{k+1}}\mathrm{ch}_g^0\big(\mathrm{tr}_1\big(H(\overline{E})\big)\big)\wedge C_{k+1}(\log\mid z_1\mid^2,\dots,\log\mid z_{k+1}\mid^2).$$

The same reasoning as in [**30**, Lemma 3.3] proves that Π'_k vanishes on degenerate k-cubes, and hence we obtain a map $\Pi'_k:\widetilde{\mathbb{Z}}C_k^{f-\mathrm{ac}}(X,\mu_n)\to\bigoplus_{p\geqslant 0}D^{2p-k-1}(Y_{\mu_n},p)_{R_n}$ by linear extension.

Proposition 2.14. *The equality*

$$d_{\mathcal{D}}\circ\Pi'_k(\overline{E})+\Pi'_{k-1}\circ d(\overline{E})$$

$$=\mathrm{ch}_g(f_*\overline{E})-\frac{(-1)^k}{2k!(2\pi i)^k}\int_{(\mathbb{P}^1)^k}\mathrm{ch}_g^0\big(f_*\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big)\big)\wedge C_k(\log\mid z_1\mid^2,\dots,\log\mid z_k\mid^2)$$

holds.

Proof. We compute

$$
\begin{aligned}
&d_{\mathcal{D}}\circ\Pi'_k(\overline{E})\\
&=\frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}}\int_{(\mathbb{P}^1)^{k+1}}\mathrm{ch}_g^0\big(\mathrm{tr}_1\big(H(\overline{E})\big)\big)\wedge d_{\mathcal{D}}C_{k+1}(\log\mid z_1\mid^2,\dots,\log\mid z_{k+1}\mid^2)\\
&=\frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}}\int_{(\mathbb{P}^1)^{k+1}}\mathrm{ch}_g^0\big(\mathrm{tr}_1\big(H(\overline{E})\big)\big)\wedge\Bigg(\Big(-\frac{1}{2}\Big)(k+1)\sum_{j=1}^{k+1}(-1)^{j-1}(-4\pi i)\\
&\quad\times(\delta_{z_j=\infty}-\delta_{z_j=0})\wedge C_k(\log\mid z_1\mid^2,\dots,\widehat{\log\mid z_j\mid^2},\dots,\log\mid z_{k+1}\mid^2)\Bigg)\\
&=\frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}}\int_{(\mathbb{P}^1)^{k+1}}\mathrm{ch}_g^0\big(\mathrm{tr}_1\big(H(\overline{E})\big)\big)\wedge\Bigg(\Big(-\frac{1}{2}\Big)(k+1)\sum_{j=2}^{k+1}(-1)^{j-1}(-4\pi i)\\
&\quad\times(\delta_{z_j=\infty}-\delta_{z_j=0})\wedge C_k(\log\mid z_1\mid^2,\dots,\widehat{\log\mid z_j\mid^2},\dots,\log\mid z_{k+1}\mid^2)\Bigg)\\
&\quad+\frac{(-1)^k}{2k!(2\pi i)^k}\int_{(\mathbb{P}^1)^k}\mathrm{ch}_g^0\big(\mathrm{tr}_k\circ\lambda\big(f_*(\overline{E})\big)\big)\wedge C_k(\log\mid z_1\mid^2,\dots,\log\mid z_k\mid^2)\\
&\quad-\frac{(-1)^k}{2k!(2\pi i)^k}\int_{(\mathbb{P}^1)^k}\mathrm{ch}_g^0\big(f_*\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big)\big)\wedge C_k(\log\mid z_1\mid^2,\dots,\log\mid z_k\mid^2)\\
&=\frac{(-1)^{k+1}}{2k!(2\pi i)^k}\int_{(\mathbb{P}^1)^k}\Bigg(\Big(\sum_{j=2}^{k+1}(-1)^{j-1}\mathrm{ch}_g^0\Big(\mathrm{tr}_1\big(H(\partial_j^{-1}\overline{E}\oplus\partial_j^1\overline{E})\big)\Big)-\mathrm{ch}_g^0\Big(\mathrm{tr}_1\big(H(\partial_i^0\overline{E})\big)\Big)\Big)\\
&\quad\wedge C_k(\log\mid z_1\mid^2,\dots,\log\mid z_k\mid^2)\Bigg)+\mathrm{ch}_g(f_*\overline{E})\\
&\quad-\frac{(-1)^k}{2k!(2\pi i)^k}\int_{(\mathbb{P}^1)^k}\mathrm{ch}_g^0\big(f_*\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big)\big)\wedge C_k(\log\mid z_1\mid^2,\dots,\log\mid z_k\mid^2)\\
&=\frac{(-1)^k}{2k!(2\pi i)^k}\int_{(\mathbb{P}^1)^k}\mathrm{ch}_g^0\big(\mathrm{tr}_1\big(H(-d\overline{E})\big)\big)\wedge C_k(\log\mid z_1\mid^2,\dots,\log\mid z_k\mid^2)+\mathrm{ch}_g(f_*\overline{E})\\
&\quad-\frac{(-1)^k}{2k!(2\pi i)^k}\int_{(\mathbb{P}^1)^k}\mathrm{ch}_g^0\big(f_*\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big)\big)\wedge C_k(\log\mid z_1\mid^2,\dots,\log\mid z_k\mid^2)\\
&=-\Pi'_{k-1}\circ d(\overline{E})+\mathrm{ch}_g(f_*\overline{E})-\frac{(-1)^k}{2k!(2\pi i)^k}\int_{(\mathbb{P}^1)^k}\mathrm{ch}_g^0\big(f_*\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big)\big)\\
&\quad\wedge C_k(\log\mid z_1\mid^2,\dots,\log\mid z_k\mid^2).
\end{aligned}
$$

So, we are done. □

On the other hand, we equip $X\times(\mathbb{P}^1)^k$ with the product metric and we define

$$\Pi''_k(\overline{E})=\frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k}\int_{(\mathbb{P}^1)^k}C_{k+1}\big(T_g\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big),\log\mid z_1\mid^2,\dots,\log\mid z_k\mid^2\big),$$

where $T_g(\mathrm{tr}_k \circ \lambda(\overline{E}))$ is the equivariant higher analytic torsion of the hermitian bundle $\mathrm{tr}_k \circ \lambda(\overline{E})$ with respect to the fibration $f : X \times (\mathbb{P}^1)^k \to Y \times (\mathbb{P}^1)^k$. By [30, Lemma 3.5], the map Π''_k vanishes on degenerate k-cubes and hence we obtain a map $\Pi''_k : \widetilde{\mathbb{Z}}C_k^{f-\mathrm{ac}}(X, \mu_n) \to \bigoplus_{p \geqslant 0} D^{2p-k-1}(Y_{\mu_n}, p)_{R_n}$ by linear extension.

THEOREM 2.15. *Set $\Pi_k = \Pi'_k + \Pi''_k$, then Π_k defines a chain homotopy of the diagram (8). This map $\Pi_k : \widetilde{\mathbb{Z}}C_k^{f-\mathrm{ac}}(X, \mu_n) \to \bigoplus_{p \geqslant 0} D^{2p-k-1}(Y_{\mu_n}, p)_{R_n}$ is called the equivariant higher analytic torsion for hermitian cubes.*

Proof. Let $\overline{E}$ be a hermitian k-cube in $\widetilde{\mathbb{Z}}C_k^{f-\mathrm{ac}}(X, \mu_n)$, we compute

$$
\begin{aligned}
&d_{\mathcal{D}} \circ \Pi_k(\overline{E}) + \Pi_{k-1} \circ d(\overline{E}) \\
&\quad = d_{\mathcal{D}} \circ \Pi'_k(\overline{E}) + \Pi'_{k-1} \circ d(\overline{E}) + d_{\mathcal{D}} \circ \Pi''_k(\overline{E}) + \Pi''_{k-1} \circ d(\overline{E}) \\
&\quad = \mathrm{ch}_g(f_*\overline{E}) - \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \mathrm{ch}_g^0\big(f_*\big(\mathrm{tr}_k \circ \lambda(\overline{E})\big)\big) \wedge C_k(\log \mid z_1 \mid^2, \ldots, \log \mid z_k \mid^2) \\
&\qquad + d_{\mathcal{D}} \circ \Pi''_k(\overline{E}) + \Pi''_{k-1} \circ d(\overline{E}).
\end{aligned}
$$

and

$$
\begin{aligned}
&d_{\mathcal{D}} \circ \Pi''_k(\overline{E}) \\
&\quad = \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} d_{\mathcal{D}} C_{k+1}\big(T_g\big(\mathrm{tr}_k \circ \lambda(\overline{E})\big), \log \mid z_1 \mid^2, \ldots, \log \mid z_k \mid^2\big) \\
&\quad = \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \left(-\frac{1}{2}\right)(k+1) \Bigg(d_{\mathcal{D}} T_g\big(\mathrm{tr}_k \circ \lambda(\overline{E})\big) \bullet C_k(\log \mid z_1 \mid^2, \ldots, \log \mid z_k \mid^2) \\
&\qquad + \sum_{j=1}^{k} (-1)^j (-4\pi i)(\delta_{z_j=\infty} - \delta_{z_j=0}) \\
&\qquad \wedge C_k\Big(T_g\big(\mathrm{tr}_k \circ \lambda(\overline{E})\big), \log \mid z_1 \mid^2, \ldots, \widehat{\log \mid z_j \mid^2}, \ldots, \log \mid z_k \mid^2\Big)\Bigg) \\
&\quad = \frac{(-1)^k}{k!(2\pi i)^{k-1}} \int_{(\mathbb{P}^1)^{k-1}} \sum_{j=1}^{k} (-1)^j \Big(C_k\big(T_g\big(\mathrm{tr}_{k-1} \circ \lambda(\partial_j^0 \overline{E})\big), \log \mid z_1 \mid^2, \ldots, \log \mid z_{k-1} \mid^2\big) \\
&\qquad - C_k\big(T_g\big(\mathrm{tr}_{k-1} \circ \lambda(\partial_j^{-1} \overline{E}) \oplus \mathrm{tr}_{k-1} \circ \lambda(\partial_j^1 \overline{E})\big), \log \mid z_1 \mid^2, \ldots, \log \mid z_{k-1} \mid^2\big)\Big) \\
&\qquad + \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \Bigg(\Bigg(\mathrm{ch}_g^0(f_*(\mathrm{tr}_k \circ \lambda(\overline{E}))) \\
&\qquad - \frac{1}{(2\pi i)^r} \int_{X_{\mu_n} \times (\mathbb{P}^1)^k / Y_{\mu_n} \times (\mathbb{P}^1)^k} \mathrm{Td}_g(\overline{Tf}) \mathrm{ch}_g^0\big(\mathrm{tr}_k \circ \lambda(\overline{E})\big) \Bigg) \\
&\qquad \bullet C_k(\log \mid z_1 \mid^2, \ldots, \log \mid z_k \mid^2) \Bigg)
\end{aligned}
$$

$$= -\Pi''_{k-1}\circ d(\overline{E}) + \frac{(-1)^k}{2k!(2\pi i)^k}\int_{(\mathbb{P}^1)^k} \mathrm{ch}_g^0\big(f_*\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big)\big)\wedge C_k(\log\mid z_1\mid^2,\ldots,\log\mid z_k\mid^2)$$

$$-\frac{1}{(2\pi i)^r}\int_{X_{\mu_n}/Y_{\mu_n}} \mathrm{Td}_g(\overline{Tf})\bullet \mathrm{ch}_g(\overline{E}).$$

Combining these two computations, we finally get

$$d_{\mathcal{D}}\circ\Pi_k(\overline{E}) + \Pi_{k-1}\circ d(\overline{E}) = \mathrm{ch}_g(f_*\overline{E}) - \frac{1}{(2\pi i)^r}\int_{X_{\mu_n}/Y_{\mu_n}} \mathrm{Td}_g(\overline{Tf})\bullet \mathrm{ch}_g(\overline{E}).$$

So, we are done. □

If we are given another fibration structure ω', then for any f-acyclic hermitian k-cube $\overline{E}$ in $\widehat{\mathcal{P}}(X,\mu_n)$, the short exact sequence

$$0\longrightarrow (f_*E, h'^{f_*E}) \xrightarrow{\mathrm{Id}} (f_*E, h^{f_*E})\longrightarrow 0\longrightarrow 0$$

forms a hermitian $(k+1)$-cube $H_f(\overline{E})$ on Y such that the transgression bundle $\mathrm{tr}_{k+1}(\lambda(H_f(\overline{E})))$ satisfies the relations

$$\mathrm{tr}_{k+1}\big(\lambda\big(H_f(\overline{E})\big)\big)\mid_{Y\times\{0\}\times(\mathbb{P}^1)^k} = \mathrm{tr}_k\big(\lambda(f_*E, h^{f_*E})\big),$$

$$\mathrm{tr}_{k+1}\big(\lambda\big(H_f(\overline{E})\big)\big)\mid_{Y\times\{\infty\}\times(\mathbb{P}^1)^k} = \mathrm{tr}_k\big(\lambda(f_*E, h'^{f_*E})\big)$$

and

$$\mathrm{tr}_{k+1}\big(\lambda\big(H_f(\overline{E})\big)\big)\mid_{Y\times(\mathbb{P}^1)^i\times\{0\}\times(\mathbb{P}^1)^{k-i}} = \mathrm{tr}_k\big(\lambda\big(H_f(\partial_i^0\overline{E})\big)\big),$$

$$\mathrm{tr}_{k+1}\big(\lambda\big(H_f(\overline{E})\big)\big)\mid_{Y\times(\mathbb{P}^1)^i\times\{\infty\}\times(\mathbb{P}^1)^{k-i}} = \mathrm{tr}_k\big(\lambda\big(H_f(\partial_i^{-1}\overline{E})\big)\big)\oplus \mathrm{tr}_k\big(\lambda\big(H_f(\partial_i^1\overline{E})\big)\big)$$

for $i=1,\ldots,k$. Therefore, the following map

$$\Pi_k^{(1)}(\overline{E}) = \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}}\int_{(\mathbb{P}^1)^{k+1}} \mathrm{ch}_g^0\big(\mathrm{tr}_{k+1}\circ\lambda\big(H_f(\overline{E})\big)\big)\wedge C_{k+1}(\log\mid z_1\mid^2,\ldots,\log\mid z_{k+1}\mid^2)$$

which vanishes on degenerate cubes provides a chain homotopy of homological complexes between the maps $\mathrm{ch}_g\circ f_*$ and $\mathrm{ch}_g\circ f'_*$, where $f'_*(\overline{E}) := (f_*E, h'^{f_*E})$ is the pushforward with respect to the new fibration ω'. Similarly, by projection formula, the map

$$\Pi_k^{(3)}(\overline{E}) := \frac{(-1)^k}{2k!(2\pi i)^k}\int_{(\mathbb{P}^1)^k}\Bigg(\Bigg(\frac{1}{(2\pi i)^r}\int_{X_{\mu_n}\times(\mathbb{P}^1)^k/Y_{\mu_n}\times(\mathbb{P}^1)^k} \mathrm{Td}_g(Tf, h'^{Tf}, h^{Tf})\mathrm{ch}_g^0\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big)\Bigg)$$

$$\bullet C_k(\log\mid z_1\mid^2,\ldots,\log\mid z_k\mid^2)\Bigg)$$

gives a chain homotopy of homological complexes between the maps $f_{\mu_n *}\circ(\mathrm{Td}_g(Tf,h^{Tf})\bullet \mathrm{ch}_g)$ and $f_{\mu_n *}\circ(\mathrm{Td}_g(Tf,h'^{Tf})\bullet \mathrm{ch}_g)$. Finally we write $\Pi_k^{(2)} = \Pi_k'^{(2)} + \Pi_k''^{(2)}$ for the chain homotopy defined in Theorem 2.15 between the maps $\mathrm{ch}_g\circ f'_*$ and $f_{\mu_n *}\circ(\mathrm{Td}_g(Tf,h'^{Tf})\bullet \mathrm{ch}_g)$ with respect to the new fibration ω'. Then $\Pi_k^{(1)}+\Pi_k^{(2)}-\Pi_k^{(3)}$ defines a chain homotopy between $\mathrm{ch}_g\circ f_*$ and $f_{\mu_n *}\circ(\mathrm{Td}_g(Tf,h^{Tf})\bullet \mathrm{ch}_g)$. At the end of this subsection, we compare this homotopy $\Pi_k^{(1)}+\Pi_k^{(2)}-\Pi_k^{(3)}$ with Π_k constructed in Theorem 2.15.

Definition 2.16. Let f,l be two morphisms of homological complexes $A_*\to B_*$, and let h_1,h_2 be two chain homotopies between f and l. We say that h_1 is homotopic to h_2 if there exists a map $H: A_*\to B_{*+2}$ satisfying the condition that $Hd-dH = h_1-h_2$.

Now, we denote by $H_f^{f'}(\overline{E})$ the following emi-2-cube of hermitian bundles on $Y \times (\mathbb{P}^1)^k$

$$\begin{array}{ccccc}
f'_*\left(\mathrm{tr}_k \circ \lambda(\overline{E})\right) & \xrightarrow{\mathrm{Id}} & \mathrm{tr}_k \circ \lambda\left(f'_*(\overline{E})\right) & \longrightarrow & 0 \\
\downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow \\
f_*\left(\mathrm{tr}_k \circ \lambda(\overline{E})\right) & \xrightarrow{\mathrm{Id}} & \mathrm{tr}_k \circ \lambda\left(f_*(\overline{E})\right) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & 0 & \longrightarrow & 0.
\end{array}$$

Changing the order of the $\mathbb{P}^1 \times \mathbb{P}^1$ in $(\mathbb{P}^1)^{k+2} = (\mathbb{P}^1)^k \times \mathbb{P}^1 \times \mathbb{P}^1$, so that $(\mathbb{P}^1)^{k+2} = \mathbb{P}^1 \times \mathbb{P}^1 \times (\mathbb{P}^1)^k$, we construct a hermitian bundle $\mathrm{tr}_2(H_f^{f'}(\overline{E}))$ on $Y \times (\mathbb{P}^1)^{k+2}$ as the second transgression bundle of $H_f^{f'}(\overline{E})$ such that it satisfies the following relations:

$$\mathrm{tr}_2\Big(H_f^{f'}(\overline{E})\Big) \mid_{Y\times\{0\}\times(\mathbb{P}^1)^{k+1}} = \mathrm{tr}_{k+1}\big(\lambda\big(H_f(\overline{E})\big)\big),$$

$$\mathrm{tr}_2\Big(H_f^{f'}(\overline{E})\Big) \mid_{Y\times\{\infty\}\times(\mathbb{P}^1)^{k+1}} = \mathrm{tr}_1\big(H_f\big(\mathrm{tr}_k \circ \lambda(\overline{E})\big)\big),$$

$$\mathrm{tr}_2\Big(H_f^{f'}(\overline{E})\Big) \mid_{Y\times\mathbb{P}^1\times\{0\}\times(\mathbb{P}^1)^k} = \mathrm{tr}_1\big(H(\overline{E})\big), \quad \mathrm{tr}_2\Big(H_f^{f'}(\overline{E})\Big) \mid_{Y\times\mathbb{P}^1\times\{\infty\}\times(\mathbb{P}^1)^k} = \mathrm{tr}_1\big(H'(\overline{E})\big)$$

and

$$\mathrm{tr}_2\Big(H_f^{f'}(\overline{E})\Big) \mid_{Y\times(\mathbb{P}^1)^{i+1}\times\{0\}\times(\mathbb{P}^1)^{k-i}} = \mathrm{tr}_2\Big(H_f^{f'}(\partial_i^0\overline{E})\Big),$$

$$\mathrm{tr}_2\Big(H_f^{f'}(\overline{E})\Big) \mid_{Y\times(\mathbb{P}^1)^{i+1}\times\{\infty\}\times(\mathbb{P}^1)^{k-i}} = \mathrm{tr}_2\Big(H_f^{f'}(\partial_i^{-1}\overline{E})\Big) \oplus \mathrm{tr}_2\Big(H_f^{f'}(\partial_i^1\overline{E})\Big)$$

for $i = 1, \ldots, k$. We set

$$\Pi_{f,k}^{f'}(\overline{E}) := \frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}} \int_{(\mathbb{P}^1)^{k+2}} \mathrm{ch}_g^0\Big(\mathrm{tr}_2\Big(H_f^{f'}(\overline{E})\Big)\Big) \wedge C_{k+2}(\log \mid z_1 \mid^2, \ldots, \log \mid z_{k+2} \mid^2).$$

Then $\Pi_{f,k}^{f'}$ vanishes on degenerate k-cubes, and we obtain a map

$$\Pi_{f,k}^{f'} : \widetilde{\mathbb{Z}}C_k^{f-\mathrm{ac}}(X, \mu_n) \to \bigoplus_{p\geqslant 0} D^{2p-k-2}(Y_{\mu_n}, p)_{R_n}$$

by linear extension.

Proposition 2.17. *Let notations and assumptions be as above. Then the chain homotopy* Π_k *is homotopic to the chain homotopy* $\Pi_k^{(1)} + \Pi_k^{(2)} - \Pi_k^{(3)}$.

Proof. First, we set

$$\Pi_k^{(3')}(\overline{E}) := \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}$$

$$\left(\frac{1}{(2\pi i)^r} \int_{X_{\mu_n}\times(\mathbb{P}^1)^k / Y_{\mu_n}\times(\mathbb{P}^1)^k} \mathrm{Td}_g(Tf, h'^{Tf}, h^{Tf}) \mathrm{ch}_g^0\big(\mathrm{tr}_k \circ \lambda(\overline{E})\big), \right.$$

$$\left. \log \mid z_1 \mid^2, \ldots, \log \mid z_k \mid^2 \right).$$

It also defines a chain homotopy between the maps $f_{\mu_n *}\circ(\mathrm{Td}_g(Tf,h^{Tf})\bullet \mathrm{ch}_g)$ and $f_{\mu_n *}\circ(\mathrm{Td}_g(Tf,h'^{Tf})\bullet \mathrm{ch}_g)$. Since the product $\bullet$ on Deligne complex is graded commutative and is associative up to homotopy, we claim that $\Pi_k^{(3')}(\overline{E})$ is homotopic to $\Pi_k^{(3)}(\overline{E})$, so that we are left to show that Π_k is homotopic to $\Pi_k^{(1)}+\Pi_k^{(2)}-\Pi_k^{(3')}$. Actually, our claim follows from the fact that $d_{\mathcal{D}}\Pi_k^{(3)}(\overline{E})-d_{\mathcal{D}}\Pi_k^{(3')}(\overline{E})=\Pi_{k-1}^{(3)}(-d\overline{E})-\Pi_{k-1}^{(3')}(-d\overline{E})$ and [**34**, Remark 2.4, Lemma 2.5].

Now, let $\overline{E}$ be a hermitian k-cube in $\widehat{\mathcal{P}}(X,\mu_n)$ which is f-acyclic. We compute

$$
\begin{aligned}
&d_{\mathcal{D}}\circ\Pi_{f,k}^{f'}(\overline{E})\\
&\quad=\frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}}\int_{(\mathbb{P}^1)^{k+2}}\mathrm{ch}_g^0\Big(\mathrm{tr}_2\Big(H_f^{f'}(\overline{E})\Big)\Big)\wedge d_{\mathcal{D}}C_{k+2}(\log\mid z_1\mid^2,\dots,\log\mid z_{k+2}\mid^2)\\
&\quad=\frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}}\int_{(\mathbb{P}^1)^{k+2}}\mathrm{ch}_g^0\Big(\mathrm{tr}_2\big(H_f^{f'}(\overline{E})\big)\Big)\wedge\Bigg(\Big(-\frac{1}{2}\Big)(k+2)\sum_{j=1}^{k+2}(-1)^{j-1}(-4\pi i)\\
&\qquad\times(\delta_{z_j=\infty}-\delta_{z_j=0})\wedge C_{k+1}(\log\mid z_1\mid^2,\dots,\widehat{\log\mid z_j\mid^2},\dots,\log\mid z_{k+2}\mid^2)\Bigg)\\
&\quad=\Pi_{f,k-1}^{f'}\circ d(\overline{E})-\frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}}\int_{(\mathbb{P}^1)^{k+1}}\big[\big(\mathrm{ch}_g^0\big(\mathrm{tr}_1\big(H(\overline{E})\big)\big)-\mathrm{ch}_g^0\big(\mathrm{tr}_1\big(H'(\overline{E})\big)\big)\big)\\
&\qquad\wedge C_{k+1}(\log\mid z_1\mid^2,\dots,\log\mid z_{k+1}\mid^2)\big]\\
&\qquad+\frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}}\int_{(\mathbb{P}^1)^{k+1}}\big[\big(\mathrm{ch}_g^0\big(\mathrm{tr}_{k+1}\big(\lambda\big(H_f(\overline{E})\big)\big)\big)-\mathrm{ch}_g^0\big(\mathrm{tr}_1\big(H_f\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big)\big)\big)\big)\\
&\qquad\wedge C_{k+1}(\log\mid z_1\mid^2,\dots,\log\mid z_{k+1}\mid^2)\big]\\
&\quad=\Pi_{f,k-1}^{f'}\circ d(\overline{E})-\Pi_k'(\overline{E})+\Pi_k'^{(2)}(\overline{E})+\Pi_k^{(1)}(\overline{E})-\frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}}\\
&\qquad\times\int_{(\mathbb{P}^1)^{k+1}}\mathrm{ch}_g^0\big(\mathrm{tr}_1\big(H_f\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big)\big)\big)\wedge C_{k+1}(\log\mid z_1\mid^2,\dots,\log\mid z_{k+1}\mid^2).
\end{aligned}
$$

On the other hand, according to the anomaly formula Theorem 2.11, we have

$$
\begin{aligned}
\Pi_k''(\overline{E})-\Pi_k''^{(2)}(\overline{E})&=\frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k}\int_{(\mathbb{P}^1)^k}C_{k+1}\big(T_g\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big),\log\mid z_1\mid^2,\dots,\log\mid z_k\mid^2\big)\\
&\quad-\frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k}\int_{(\mathbb{P}^1)^k}C_{k+1}\big(T_g'\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big),\log\mid z_1\mid^2,\dots,\log\mid z_k\mid^2\big)\\
&=\frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k}\int_{(\mathbb{P}^1)^k}C_{k+1}\Bigg(\frac{1}{4\pi i}\int_{Y_{\mu_n}\times(\mathbb{P}^1)^{k+1}/Y_{\mu_n}\times(\mathbb{P}^1)^k}\mathrm{ch}_g^0\big(\mathrm{tr}_1\big(H_f\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big)\big)\big)\log\mid z_0\mid^2,\\
&\qquad\log\mid z_1\mid^2,\dots,\log\mid z_k\mid^2\Bigg)\\
&\quad-\frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k}\int_{(\mathbb{P}^1)^k}C_{k+1}\Bigg(\frac{1}{(2\pi i)^r}\int_{X_{\mu_n}\times(\mathbb{P}^1)^k/Y_{\mu_n}\times(\mathbb{P}^1)^k}\mathrm{Td}_g(Tf,h'^{Tf},h^{Tf})\mathrm{ch}_g^0\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big),
\end{aligned}
$$

$$\log \mid z_1 \mid^2, \ldots, \log \mid z_k \mid^2 \Big)$$

$$+\frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k}\int_{(\mathbb{P}^1)^k} C_{k+1}\big(d_{\mathcal{D}}\Delta\big(f, \mathrm{tr}_k\circ\lambda(\overline{E}), \omega, \omega'\big), \log \mid z_1 \mid^2, \ldots, \log \mid z_k \mid^2\big)$$

$$= \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}}\int_{(\mathbb{P}^1)^{k+1}} \mathrm{ch}_g^0\big(\mathrm{tr}_1\big(H_f\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big)\big)\big)\wedge C_{k+1}(\log \mid z_1 \mid^2, \ldots, \log \mid z_{k+1} \mid^2)$$

$$+\frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k}\int_{(\mathbb{P}^1)^k} C_{k+1}\big(d_{\mathcal{D}}\Delta\big(f, \mathrm{tr}_k\circ\lambda(\overline{E}), \omega, \omega'\big), \log \mid z_1 \mid^2, \ldots, \log \mid z_k \mid^2\big) - \Pi_k^{(3')}(\overline{E}).$$

We formally define a product $C_{k+1}(\Delta(f, \mathrm{tr}_k\circ\lambda(\overline{E}), \omega, \omega'), \log \mid z_1 \mid^2, \ldots, \log \mid z_k \mid^2)$ in a similar way to $C_{k+1}(\cdot, \ldots, \cdot)$ like follows.

$$C_{k+1}\big(\Delta\big(f, \mathrm{tr}_k\circ\lambda(\overline{E}), \omega, \omega'\big), \log \mid z_1 \mid^2, \ldots, \log \mid z_k \mid^2\big)$$

$$= -\Big(-\frac{1}{2}\Big)^k \sum_{\sigma\in\mathfrak{S}_k} (-1)^\sigma \Delta \bullet (\log \mid z_{\sigma(1)} \mid^2 \bullet(\log \mid z_{\sigma(2)} \mid^2 \bullet(\cdots \log \mid z_{\sigma(k)} \mid^2)\cdots)$$

$$- \Big(-\frac{1}{2}\Big)^k \sum_{\sigma\in\mathfrak{S}_k} (-1)^\sigma \log \mid z_{\sigma(1)} \mid^2 \bullet(\Delta \bullet (\log \mid z_{\sigma(2)} \mid^2 \bullet(\cdots \log \mid z_{\sigma(k)} \mid^2)\cdots)$$

$$\cdots$$

$$- \Big(-\frac{1}{2}\Big)^k \sum_{\sigma\in\mathfrak{S}_k} (-1)^\sigma \log \mid z_{\sigma(1)} \mid^2 \bullet(\log \mid z_{\sigma(2)} \mid^2 \bullet(\cdots \log \mid z_{\sigma(k)} \mid^2 \bullet\Delta)\cdots) \tag{9}$$

Then we set

$$\Delta_k(\overline{E}) = \frac{(-1)^k}{(k+1)!(2\pi i)^k}\int_{(\mathbb{P}^1)^k} C_{k+1}\big(\Delta\big(f, \mathrm{tr}_k\circ\lambda(\overline{E}), \omega, \omega'\big), \log \mid z_1 \mid^2, \ldots, \log \mid z_k \mid^2\big),$$

and it is readily checked by Lemma 2.12 that

$$\Delta_{k-1}(d\overline{E}) - d_{\mathcal{D}}\Delta_k(\overline{E}) = \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k}$$

$$\times\int_{(\mathbb{P}^1)^k} C_{k+1}\big(d_{\mathcal{D}}\Delta\big(f, \mathrm{tr}_k\circ\lambda(\overline{E}), \omega, \omega'\big), \log \mid z_1 \mid^2, \ldots, \log \mid z_k \mid^2\big).$$

Combing all the above computations, we finally get

$$(\Pi_{f,k-1}^{f'} + \Delta_{k-1})\circ d(\overline{E}) - d_{\mathcal{D}}\circ(\Pi_{f,k}^{f'} + \Delta_k)(\overline{E})$$

$$= -\Pi_k'^{(2)}(\overline{E}) + \Pi_k'(\overline{E}) - \Pi_k^{(1)}(\overline{E}) + \Delta_{k-1}(d\overline{E}) - d_{\mathcal{D}}\Delta_k(\overline{E})$$

$$+\frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}}\int_{(\mathbb{P}^1)^{k+1}} \mathrm{ch}_g^0\big(\mathrm{tr}_1\big(H_f\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big)\big)\big)\wedge C_{k+1}(\log \mid z_1 \mid^2, \ldots, \log \mid z_{k+1} \mid^2)$$

$$= -\Pi_k'^{(2)}(\overline{E}) + \Pi_k'(\overline{E}) - \Pi_k^{(1)}(\overline{E}) - \Pi_k''^{(2)}(\overline{E}) + \Pi_k''(\overline{E}) + \Pi_k^{(3')}(\overline{E})$$

$$= \Pi_k(\overline{E}) - \Big(\Pi_k^{(1)}(\overline{E}) + \Pi_k^{(2)}(\overline{E}) - \Pi_k^{(3')}(\overline{E})\Big).$$

So, we are done. □

2.3. *Direct image map between arithmetic K-groups*

In this subsection, we define the direct image map between arithmetic K-groups of regular μ_n-projective arithmetic schemes by means of the equivariant higher analytic torsion for hermitian cubes constructed in last subsection.

Let now X and Y be two regular μ_n-projective schemes over an arithmetic ring (D, Σ, F_∞). Assume that $f : X \to Y$ is an equivariant and flat morphism from X to Y such that f is smooth over the generic fiber. Note that the chain homotopy

$$\Pi_* : \widetilde{\mathbb{Z}}C_*^{f-\mathrm{ac}}(X(\mathbb{C}), \mu_n) \to \bigoplus_{p \geqslant 0} D^{2p-*-1}(Y(\mathbb{C})_{\mu_n}, p)_{R_n}$$

is σ-invariant and the following diagrams:

$$\begin{array}{ccccc}
\widehat{S}^{f-\mathrm{ac}}(X, \mu_n) & \xrightarrow{\mathrm{Hu}} & \mathbb{Z}\widehat{S}_*^{f-\mathrm{ac}}(X, \mu_n) & \xrightarrow{\mathrm{Cub}} & \mathcal{K}\left(\widetilde{\mathbb{Z}}C_*^{f-\mathrm{ac}}(X, \mu_n)[-1]\right) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_*} \\
\widehat{S}(Y, \mu_n) & \xrightarrow{\mathrm{Hu}} & \mathbb{Z}\widehat{S}_*(Y, \mu_n) & \xrightarrow{\mathrm{Cub}} & \mathcal{K}\left(\widetilde{\mathbb{Z}}C_*(Y, \mu_n)[-1]\right)
\end{array}$$

are commutative, the chain homotopy Π_* induces a simplicial homotopy between the maps $\widetilde{\mathrm{ch}}_g \circ f_*$ and $f_{\mu_n *} \circ \mathrm{Td}_g(\overline{Tf}) \bullet (\cdot) \circ \widetilde{\mathrm{ch}}_g$ in the following square

$$\begin{array}{ccc}
\widehat{S}^{f-\mathrm{ac}}(X, \mu_n) & \xrightarrow{\widetilde{\mathrm{ch}}_g} & \mathcal{K}\left(\bigoplus_{p \geq 0} D^{2p-*}(X_{\mu_n}, p)[-1]_{R_n}\right) \\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_{\mu_n *} \circ \mathrm{Td}_g(\overline{Tf}) \bullet (\cdot)} \\
\widehat{S}(Y, \mu_n) & \xrightarrow{\widetilde{\mathrm{ch}}_g} & \mathcal{K}\left(\bigoplus_{p \geq 0} D^{2p-*}(Y_{\mu_n}, p)[-1]_{R_n}\right).
\end{array}$$

To see the construction of this simplicial homotopy and general theory on homotopies in the category of simplical abelian groups, the reader is referred to [22, Sections 2.1, 2.3 and 3.2], especially [22, p. 160, p. 162, Proposition 2.18, p. 72, Proposition 1.8, Corollary 1.9].

We remark that, according to the construction given in [22], the resulting simplicial homotopy is unique up to a homotopy in a strong sense: let h_1, h_2 be two simplicial homotopies arising from Π_*, then there exists a homotopy

$$\widetilde{H} : \quad \widehat{S}^{f-\mathrm{ac}}(X, \mu_n) \times \Delta^1 \times \Delta^1 \longrightarrow \mathcal{K}\left(\bigoplus_{p \geq 0} D^{2p-*}(Y_{\mu_n}, p)[-1]_{R_n}\right)$$

such that $\widetilde{H}(\cdot, \cdot, 0) = h_1$, $\widetilde{H}(\cdot, \cdot, 1) = h_2$, $\widetilde{H}(\cdot, 0, \cdot)$ is the constant homotopy on $\widetilde{\mathrm{ch}}_g \circ f_*$ and $\widetilde{H}(\cdot, 1, \cdot)$ is the constant homotopy on $f_{\mu_n *} \circ \mathrm{Td}_g(\overline{Tf}) \bullet (\cdot) \circ \widetilde{\mathrm{ch}}_g$ (cf. [22, Proposition 3.8]). Thus, applying the geometric realization construction to the above simplicial square, we get a continuous map between homotopy fibers

$$|\, f \,| \colon \text{ homotopy fiber of } |\, \widetilde{\mathrm{ch}}_g^{\,X} \,| \longrightarrow \text{homotopy fiber of } |\, \widetilde{\mathrm{ch}}_g^{\,Y} \,|$$

which is unique up to a homotopy. So, we may have a well-defined direct image map between arithmetic K-groups as follows.

Definition 2.18. For $m \geqslant 1$, the direct image map $f_* : \widehat{K}_m(X, \mu_n) \to \widehat{K}_m(Y, \mu_n)$ is defined as the homomorphism of abelian groups induced by the map $|\, f \,|$ at the level of homotopy groups.

REMARK 2.19. The condition 'flatness' of the map f is only used to guarantee that the direct image of a f-acyclic bundle is locally free. By introducing the arithmetic K′-theory and using the isomorphisms $\widehat{K}_m(X,\mu_n)\cong\widehat{K}'_m(X,\mu_n)$ which hold for regular schemes, the condition 'flatness' can certainly be removed.

To study the direct image map up to torsion, we need the following lemma.

LEMMA 2.20. *Consider the following diagram of homological complexes*

$$\begin{array}{ccc} A_* & \xrightarrow{\quad i\quad} & B_* \\ {\scriptstyle f_1}\big(\big.\big)\,{\scriptstyle f_2} & & {\scriptstyle l_1}\big(\big.\big)\,{\scriptstyle l_2} \\ C_* & \xrightarrow{\quad j\quad} & D_*. \end{array}$$

Assume that $j\circ f_1$ (respectively, $j\circ f_2$) is homotopic to $l_1\circ i$ (respectively, $l_2\circ i$) via the chain homotopy h_1 (respectively, h_2), and that f_1 (respectively, l_1) is homotopic to f_2 (respectively, l_2) via the chain homotopy π_f (respectively, π_l). Suppose that the chain homotopy $j\circ\pi_f+h_2-\pi_l\circ i$ is homotopic to the chain homotopy h_1, then the morphism on simple complexes

$$(f_1,l_1,h_1):s_*(i:A_*\to B_*)\to s_*(j:C_*\to D_*)$$

is chain homotopic to (f_2,l_2,h_2).

Proof. Let $(a,b)\in A_k\bigoplus B_{k+1}$, the morphism (f_1,l_1,h_1) (respectively, (f_2,l_2,h_2)) sends (a,b) to $(f_1(a),l_1(b)+h_1(a))$ (respectively, $(f_2(a),l_2(b)+h_2(a))$). Let $H:A_*\to D_{*+2}$ be the homotopy such that

$$Hd-dH=h_1-(j\circ\pi_f+h_2-\pi_l\circ i),$$

and we define $\widetilde{H}(a,b)=(\pi_f(a),-\pi_l(b)+H(a))$. Then we compute

$$\begin{aligned} d\widetilde{H}(a,b)&=d(\pi_f(a),-\pi_l(b)+H(a))\\ &=(d\pi_f(a),j\circ\pi_f(a)+d\pi_l(b)-dH(a))\\ &=(f_1(a)-f_2(a)-\pi_f(da),l_1(b)-l_2(b)-\pi_l(db)-Hd(a)+h_1(a)-h_2(a)+\pi_l\circ i(a))\\ &=(f_1(a),l_1(b)+h_1(a))-(f_2(a),l_2(b)+h_2(a))-(\pi_f d(a),\pi_l(db)-\pi_l\circ i(a)+Hd(a))\\ &=(f_1(a),l_1(b)+h_1(a))-(f_2(a),l_2(b)+h_2(a))-\widetilde{H}(da,i(a)-db)\\ &=(f_1(a),l_1(b)+h_1(a))-(f_2(a),l_2(b)+h_2(a))-\widetilde{H}d(a,b). \end{aligned}$$

So, we are done. □

COROLLARY 2.21. *Let notations and assumptions be as above, then the direct image map $f_*:\widehat{K}_m(X,\mu_n)_{\mathbb{Q}}\to\widehat{K}_m(Y,\mu_n)_{\mathbb{Q}}$ without torsion is independent of the choice of the Kähler fibration structure.*

Proof. This follows from Remark 2.10(iv), Theorem 2.17 and Lemma 2.20. □

3. *Transitivity of the direct image maps*

Let $f:X\to Y$, $h:Y\to Z$ and $l:X\to Z$ be three equivariant morphisms between regular μ_n-projective schemes, which are all smooth over the generic fibers. Assume $l=h\circ f$, in this

section, we shall compare the direct image map l_* with the composition $h_* \circ f_*$. To this aim, we shall first discuss the functoriality of the equivariant analytic torsion forms with respect to a composition of submersions.

3.1. *Analytic torsion forms and families of submersions*

Let W, V and S be three smooth μ_n-equivariant algebraic varieties over $\mathbb{C}$ with $S = S_{\mu_n}$. Suppose that $f : W \to V$ and $h : V \to S$ are two proper smooth morphisms, then passing to their analytifications the maps $f : W(\mathbb{C}) \to V(\mathbb{C})$ and $h : V(\mathbb{C}) \to S(\mathbb{C})$ are holomorphic submersions with compact fibers. Set $l = h \circ f$, it is also a proper smooth morphism and $l : W(\mathbb{C}) \to S(\mathbb{C})$ is a holomorphic submersion with compact fiber as well.

Let ω^W and ω^V be two μ_n-invariant Kähler forms on W and on V. As before, ω^W and ω^V imply Kähler fibration structures on the morphisms f, h and l and they induce μ_n-invariant hermitian metrics on relative tangent bundles Tf, Th and Tl. Consider the following short exact sequence of hermitian vector bundles

$$\overline{T}(f,h,h\circ f): \quad 0 \to \overline{Tf} \to \overline{Tl} \to f^*\overline{Th} \to 0,$$

denote by $\mathrm{Td}_g(\overline{T}(f,h,h\circ f)) = \Phi^{-1}(\frac{\widetilde{\mathrm{Td}}_g(\overline{T}(f,h,h\circ f))}{2})$ (see Section A.2 in the Appendix) the equivariant secondary Todd form such that

$$d_{\mathcal{D}}\mathrm{Td}_g\big(\overline{T}(f,h,h\circ f)\big) = \mathrm{Td}_g(\overline{Tl}) - f_{\mu_n}^*\,\mathrm{Td}_g(\overline{Th})\mathrm{Td}_g(\overline{Tf}).$$

Now, let $\overline{E}$ be a hermitian vector bundle on W, we shall assume that E is f-acyclic and l-acyclic. Then the Leray spectral sequence $E_2^{i,j} = R^i h_*(R^j f_* E)$ degenerates at E_2, so that $f_*E = R^0 f_*(E)$ is h-acyclic and $l_*E \cong h_* f_* E$. Clearly, l_*E and $h_* f_* E$ carry in general different L^2-metrics (see Section A.2 in the Appendix). Consider the following short exact sequence of hermitian vector bundles

$$\overline{E}(f,h,h\circ f): \quad 0 \to h_* f_* \overline{E} \to l_*\overline{E} \to 0 \to 0,$$

it can be regarded as an emi-1-cube of hermitian bundles on S. Then the equivariant higher Bott–Chern form $\mathrm{ch}_g(\overline{E}(f,h,h\circ f))$ satisfies the differential equation

$$d_{\mathcal{D}}\mathrm{ch}_g\big(\overline{E}(f,h,h\circ f)\big) = \mathrm{ch}_g(l_*\overline{E}) - \mathrm{ch}_g(h_* f_* \overline{E}).$$

The main result in this subsection is the following.

THEOREM 3.1. *Let notations and assumptions be as above. Then the following identity holds in* $\bigoplus_{p\geqslant 0}(D^{2p-1}(S,p)/\mathrm{Im}\, d_{\mathcal{D}})$*:*

$$\begin{aligned} &T_g(l,\omega^W,h^E) - T_g(h,\omega^V,h^{f_*E}) - \frac{1}{(2\pi i)^{r_h}}\int_{V_{\mu_n}/S}\mathrm{Td}_g(\overline{Th})T_g(f,\omega^W,h^E) \\ &\quad = \mathrm{ch}_g\big(\overline{E}(f,h,h\circ f)\big) - \frac{1}{(2\pi i)^{r_l}}\int_{W_{\mu_n}/S}\mathrm{Td}_g\big(\overline{T}(f,h,h\circ f)\big)\mathrm{ch}_g(\overline{E}), \end{aligned}$$

where r_h *and* r_l *are the relative dimensions of* V_{μ_n}/S *and* W_{μ_n}/S*, respectively.*

Proof. This is a translation of Theorem A.3. □

LEMMA 3.2. *With the same notations as in Remark 2.7 and Theorem A.4 , we set*

$$\Delta(f,h,\omega^W,\omega^V,\overline{E}) := -\Phi^{-1}\left(\frac{\Delta^0(f,h,\omega^W,\omega^V,\overline{E}) + \Delta_0(f,h,\omega^W,\omega^V,\overline{E})}{2}\right) + \Delta\big(\overline{E}(f,h,h\circ f)\big).$$

Then $d_{\mathcal{D}}\Delta(f,h,\omega^W,\omega^V,\overline{E})$ *measures the difference*

$$T_g(l,\omega^W,h^E)-T_g(h,\omega^V,h^{f_*E})-\frac{1}{(2\pi i)^{r_h}}\int_{V_{\mu_n}/S}\mathrm{Td}_g(\overline{Th})T_g(f,\omega^W,h^E)$$

$$-\mathrm{ch}_g\big(\overline{E}(f,h,h\circ f)\big)+\frac{1}{(2\pi i)^{r_l}}\int_{W_{\mu_n}/S}\mathrm{Td}_g\big(\overline{T}(f,h,h\circ f)\big)\mathrm{ch}_g(\overline{E})$$

in Theorem 3.1. Assume that we are in the same situation described before Lemma 2.12. Call $l: S\times Z_1\to S\times Z$ *the natural inclusion, then similar to Lemma 2.12, we have*

$$l^*\Delta(f_Z,h_Z,\omega^W,\omega^V,\overline{E})=\Delta(f_{Z_1},h_{Z_1},\omega_1^W,\omega_1^V,j^*\overline{E}).$$

Proof. This is a consequence of Theorem A.4. □

3.2. *The transitivity property*

In this subsection, we present certain transitivity property of direct image maps between equivariant higher arithmetic K-groups. To do this, we first write down the following diagram of homological complexes:

$$\begin{array}{ccc}
\widetilde{\mathbb{Z}}C_*^{(f,l)-\mathrm{ac}}(X,\mu_n) & \xrightarrow{\mathrm{ch}_g} & \bigoplus_{p\geq 0}D^{2p-*}(X_{\mu_n},p)_{R_n}\\
\downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_{\mu_n *}\circ\mathrm{Td}_g(\overline{Tf})\bullet(\cdot)}\\
\widetilde{\mathbb{Z}}C_*^{f-\mathrm{ac}}(Y,\mu_n) & \xrightarrow{\mathrm{ch}_g} & \bigoplus_{p\geq 0}D^{2p-*}(Y_{\mu_n},p)_{R_n}\\
\downarrow{\scriptstyle h_*} & & \downarrow{\scriptstyle h_{\mu_n *}\circ\mathrm{Td}_g(\overline{Th})\bullet(\cdot)}\\
\widetilde{\mathbb{Z}}C_*(Z,\mu_n) & \xrightarrow{\mathrm{ch}_g} & \bigoplus_{p\geq 0}D^{2p-*}(Z_{\mu_n},p)_{R_n},
\end{array}\tag{10}$$

where l is $h\circ f$ and $\widetilde{\mathbb{Z}}C_*^{(f,l)-\mathrm{ac}}(X,\mu_n)$ is the subcomplex of $\widetilde{\mathbb{Z}}C_*(X,\mu_n)$ made of those bundles which are f-acyclic and l-acyclic simultaneously.

Let $\overline{E}$ be a hermitian k-cube in $\widehat{\mathcal{P}}(X,\mu_n)$ which is f-acyclic and l-acyclic, the short exact sequence

$$0\longrightarrow h_*f_*\overline{E}\xrightarrow{\mathrm{Id}}l_*\overline{E}\longrightarrow 0\longrightarrow 0$$

can be regarded as a hermitian $(k+1)$-cube $H_{h\circ f}(\overline{E})$ on Z such that the transgression bundle $\mathrm{tr}_{k+1}(\lambda(H_{h\circ f}(\overline{E})))$ satisfies the relations

$$\mathrm{tr}_{k+1}\big(\lambda\big(H_{h\circ f}(\overline{E})\big)\big)\mid_{Z\times\{0\}\times(\mathbb{P}^1)^k}=\mathrm{tr}_k\big(\lambda(l_*\overline{E})\big),$$

$$\mathrm{tr}_{k+1}\big(\lambda\big(H_{h\circ f}(\overline{E})\big)\big)\mid_{Z\times\{\infty\}\times(\mathbb{P}^1)^k}=\mathrm{tr}_k\big(\lambda(h_*f_*\overline{E})\big)$$

and

$$\mathrm{tr}_{k+1}\big(\lambda\big(H_{h\circ f}(\overline{E})\big)\big)\mid_{Z\times(\mathbb{P}^1)^i\times\{0\}\times(\mathbb{P}^1)^{k-i}}=\mathrm{tr}_k\big(\lambda\big(H_{h\circ f}(\partial_i^0\overline{E})\big)\big),$$

$$\mathrm{tr}_{k+1}\big(\lambda\big(H_{h\circ f}(\overline{E})\big)\big)\mid_{Z\times(\mathbb{P}^1)^i\times\{\infty\}\times(\mathbb{P}^1)^{k-i}}=\mathrm{tr}_k\big(\lambda\big(H_{h\circ f}(\partial_i^{-1}\overline{E})\big)\big)\oplus\mathrm{tr}_k\big(\lambda\big(H_{h\circ f}(\partial_i^1\overline{E})\big)\big)$$

for $i=1,\ldots,k$.

PROPOSITION 3.3. *The following map*

$$\Pi_k^{(1)}(\overline{E}) = \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \mathrm{ch}_g^0\big(\mathrm{tr}_{k+1} \circ \lambda\big(H_{h\circ f}(\overline{E})\big)\big) \wedge C_{k+1}(\log \mid z_1 \mid^2, \ldots, \log \mid z_{k+1} \mid^2)$$

which vanishes on degenerate cubes provides a chain homotopy of homological complexes between the maps $\mathrm{ch}_g \circ l_*$ *and* $\mathrm{ch}_g \circ (h_* \circ f_*)$.

Proof. Using the above relations that the transgression bundle $\mathrm{tr}_{k+1}(\lambda(H_{h\circ f}(\overline{E})))$ satisfies and the expression of $d_{\mathcal{D}} C_{k+1}$, the proof is straightforward. This can be also seen from the fact that $H_{h\circ f}(\overline{E})$ provides a chain homotopy between l_* and $h_* \circ f_*$. □

PROPOSITION 3.4. *The composition* $h_{\mu_n *} \circ \mathrm{Td}_g(\overline{Th}) \bullet (f_{\mu_n *} \circ \mathrm{Td}_g(\overline{Tf}) \bullet (\cdot))$ *is equal to* $l_{\mu_n *} \circ f_{\mu_n}^* \mathrm{Td}_g(\overline{Th}) \mathrm{Td}_g(\overline{Tf}) \bullet (\cdot)$. *The following maps*

$$\Pi_k^{(3)}(\overline{E}) := \frac{(-1)^k}{2k!(2\pi i)^k} \times \int_{(\mathbb{P}^1)^k} \left(\left(\frac{1}{(2\pi i)^{r_l}} \int_{X_{\mu_n} \times (\mathbb{P}^1)^k / Z_{\mu_n} \times (\mathbb{P}^1)^k} \mathrm{Td}_g\big(\overline{T}(f, h, h\circ f)\big) \mathrm{ch}_g^0\big(\mathrm{tr}_k \circ \lambda(\overline{E})\big) \right) \bullet C_k(\log \mid z_1 \mid^2, \ldots, \log \mid z_k \mid^2) \right)$$

and

$$\Pi_k^{(3')}(\overline{E}) := \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \times \int_{(\mathbb{P}^1)^k} C_{k+1} \left(\frac{1}{(2\pi i)^{r_l}} \int_{X_{\mu_n} \times (\mathbb{P}^1)^k / Z_{\mu_n} \times (\mathbb{P}^1)^k} \mathrm{Td}_g\big(\overline{T}(f, h, h\circ f)\big) \mathrm{ch}_g^0\big(\mathrm{tr}_k \circ \lambda(\overline{E})\big), \log \mid z_1 \mid^2, \ldots, \log \mid z_k \mid^2 \right)$$

give two chain homotopies of homological complexes between the maps $l_{\mu_n *} \circ \mathrm{Td}_g(\overline{Tg}) \bullet (\mathrm{ch}_g(\cdot))$ *and* $l_{\mu_n *} \circ f_{\mu_n}^* \mathrm{Td}_g(\overline{Th}) \mathrm{Td}_g(\overline{Tf}) \bullet (\mathrm{ch}_g(\cdot))$. *Moreover,* $\Pi_k^{(3)}(\overline{E})$ *and* $\Pi_k^{(3')}(\overline{E})$ *are homotopic to each other.*

Proof. The first statement follows from the projection formula, the second statement follows from a straightforward computation and the third follows from [**34**, Remark 2.4, Lemma. 2.5]. □

Now, we write $\Pi_k^f = \Pi_k'^f + \Pi_k''^f$ for the chain homotopy of the upper square in (10) and $\Pi_k^h = \Pi_k'^h + \Pi_k''^h$ for the chain homotopy of the lower square in (10). Then $\Pi_k^{(1)} + h_{\mu_n *} \circ (\mathrm{Td}_g(\overline{Th}) \bullet \Pi_k^f) + \Pi_k^h \circ f_* - \Pi_k^{(3)}$ defines a chain homotopy between maps $\mathrm{ch}_g \circ l_*$ and $l_{\mu_n *} \circ \mathrm{Td}_g(\overline{Tg}) \bullet (\mathrm{ch}_g(\cdot))$. Suppose that the μ_n-action on Z is trivial, it is the main result of this subsection that the chain homotopy $\Pi_k^{(1)} + h_{\mu_n *} \circ (\mathrm{Td}_g(\overline{Th}) \bullet \Pi_k^f) + \Pi_k^h \circ f_* - \Pi_k^{(3)}$ is homotopic to the chain homotopy $\Pi_k^l = \Pi_k'^l + \Pi_k''^l$ for the whole square in (10). According to Proposition 3.4, it is equivalent to show that $\Pi_k^{(1)} + h_{\mu_n *} \circ (\mathrm{Td}_g(\overline{Th}) \bullet \Pi_k^f) + \Pi_k^h \circ f_* - \Pi_k^{(3')}$ is homotopic to Π_k^l.

To see this, we first denote by $H^l_{h\circ f}(\overline{E})$ the following emi-2-cube of hermitian bundles on $Z\times(\mathbb{P}^1)^k$

$$\begin{array}{ccccc}
h_*f_*\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big) & \xrightarrow{\mathrm{Id}} & \mathrm{tr}_k\circ\lambda\big(h_*f_*(\overline{E})\big) & \longrightarrow & 0\\
\downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow\\
l_*\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big) & \xrightarrow{\mathrm{Id}} & \mathrm{tr}_k\circ\lambda\big(l_*(\overline{E})\big) & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow\\
0 & \longrightarrow & 0 & \longrightarrow & 0.
\end{array}$$

Then, like before, we construct a hermitian bundle $\mathrm{tr}_2(H^l_{h\circ f}(\overline{E}))$ on $(\mathbb{P}^1)^{k+2}$ as the second transgression bundle of $H^l_{h\circ f}(\overline{E})$ such that it satisfies the following relations:

$$\begin{aligned}
&\mathrm{tr}_2\big(H^l_{h\circ f}(\overline{E})\big)\mid_{Z\times\{0\}\times(\mathbb{P}^1)^{k+1}}=\mathrm{tr}_{k+1}\big(\lambda\big(H_{h\circ f}(\overline{E})\big)\big),\\
&\mathrm{tr}_2\big(H^l_{h\circ f}(\overline{E})\big)\mid_{Z\times\{\infty\}\times(\mathbb{P}^1)^{k+1}}=\mathrm{tr}_1\big(H_{h\circ f}\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big)\big),\\
&\mathrm{tr}_2\big(H^l_{h\circ f}(\overline{E})\big)\mid_{Z\times\mathbb{P}^1\times\{0\}\times(\mathbb{P}^1)^k}=\mathrm{tr}_1\big(H(\overline{E},l_*)\big),\\
&\mathrm{tr}_2\big(H^l_{h\circ f}(\overline{E})\big)\mid_{Z\times\mathbb{P}^1\times\{\infty\}\times(\mathbb{P}^1)^k}=\mathrm{tr}_1\big(H(\overline{E},h_*f_*)\big)
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{tr}_2\big(H^l_{h\circ f}(\overline{E})\big)\mid_{Z\times(\mathbb{P}^1)^{i+1}\times\{0\}\times(\mathbb{P}^1)^{k-i}}=\mathrm{tr}_2\big(H^l_{h\circ f}(\partial_i^0\overline{E})\big),\\
&\mathrm{tr}_2\big(H^l_{h\circ f}(\overline{E})\big)\mid_{Z\times(\mathbb{P}^1)^{i+1}\times\{\infty\}\times(\mathbb{P}^1)^{k-i}}=\mathrm{tr}_2\big(H^l_{h\circ f}(\partial_i^{-1}\overline{E})\big)\oplus\mathrm{tr}_2\big(H^l_{h\circ f}(\partial_i^1\overline{E})\big)
\end{aligned}$$

for $i=1,\dots,k$. We set

$$\mathbf{H}_{1,k}(\overline{E}):=\frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}}\int_{(\mathbb{P}^1)^{k+2}}\mathrm{ch}_g^0\big(\mathrm{tr}_2\big(H^l_{h\circ f}(\overline{E})\big)\big)\wedge C_{k+2}(\log\mid z_1\mid^2,\dots,\log\mid z_{k+2}\mid^2).$$

Then $\mathbf{H}_{1,k}$ vanishes on degenerate k-cubes, and we obtain a map

$$\mathbf{H}_{1,k}:\widetilde{\mathbb{Z}}C_k^{(f,l)-\mathrm{ac}}(X,\mu_n)\to\bigoplus_{p\geqslant 0}D^{2p-k-2}(Z,p)_{R_n}$$

by linear extension. This map satisfies the following differential equation:

$$\begin{aligned}
&d_{\mathcal{D}}\circ\mathbf{H}_{1,k}(\overline{E})\\
&=\frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}}\int_{(\mathbb{P}^1)^{k+2}}\mathrm{ch}_g^0\Big(\mathrm{tr}_2\Big(H^l_{h\circ f}(\overline{E})\Big)\Big)\wedge d_{\mathcal{D}}C_{k+2}(\log\mid z_1\mid^2,\dots,\log\mid z_{k+2}\mid^2)\\
&=\frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}}\int_{(\mathbb{P}^1)^{k+2}}\mathrm{ch}_g^0\Big(\mathrm{tr}_2\Big(H^l_{h\circ f}(\overline{E})\Big)\Big)\wedge\Bigg(\Big(-\frac{1}{2}\Big)(k+2)\sum_{j=1}^{k+2}(-1)^{j-1}(-4\pi i)\\
&\quad\times(\delta_{z_j=\infty}-\delta_{z_j=0})\wedge C_{k+1}(\log\mid z_1\mid^2,\dots,\widehat{\log\mid z_j\mid^2},\dots,\log\mid z_{k+2}\mid^2)\Bigg)\\
&=\mathbf{H}_{1,k-1}\circ d(\overline{E})-\frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}}\int_{(\mathbb{P}^1)^{k+1}}\Big[\Big(\mathrm{ch}_g^0\big(\mathrm{tr}_1\big(H(\overline{E},l_*)\big)\big)-\mathrm{ch}_g^0\big(\mathrm{tr}_1\big(H(\overline{E},h_*f_*)\big)\big)\Big)\\
&\quad\wedge C_{k+1}(\log\mid z_1\mid^2,\dots,\log\mid z_{k+1}\mid^2)\Big]
\end{aligned}$$

$$+\frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}}\int_{(\mathbb{P}^1)^{k+1}}\Big[\Big(\mathrm{ch}_g^0\big(\mathrm{tr}_{k+1}\big(\lambda\big(H_{h\circ f}(\overline{E})\big)\big)\big)-\mathrm{ch}_g^0\big(\mathrm{tr}_1\big(H_{h\circ f}\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big)\big)\big)\Big)$$

$$\wedge C_{k+1}(\log\mid z_1\mid^2,\dots,\log\mid z_{k+1}\mid^2)\Big]$$

$$=\mathbf{H}_{1,k-1}\circ d(\overline{E})-\Pi_k'^{l}(\overline{E})+\Pi_k^{(1)}(\overline{E})$$

$$+\frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}}\int_{(\mathbb{P}^1)^{k+1}}\mathrm{ch}_g^0\big(\mathrm{tr}_1\big(H(\overline{E},h_*f_*)\big)\big)\wedge C_{k+1}(\log\mid z_1\mid^2,\dots,\log\mid z_{k+1}\mid^2)$$

$$-\frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}}\int_{(\mathbb{P}^1)^{k+1}}\mathrm{ch}_g^0\big(\mathrm{tr}_1\big(H_{h\circ f}\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big)\big)\big)\wedge C_{k+1}(\log\mid z_1\mid^2,\dots,\log\mid z_{k+1}\mid^2).$$

Second, we denote by $H_{h\circ f}'^{l}(\overline{E})$ the following emi-2-cube of hermitian bundles on $Z\times(\mathbb{P}^1)^k$

$$\begin{array}{ccccc}
h_*f_*\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big) & \xrightarrow{\mathrm{Id}} & h_*\mathrm{tr}_k\circ\lambda\big(f_*(\overline{E})\big) & \longrightarrow & 0\\
\downarrow{\scriptstyle\mathrm{Id}} & & \downarrow{\scriptstyle\mathrm{Id}} & & \downarrow\\
h_*f_*\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big) & \xrightarrow{\mathrm{Id}} & \mathrm{tr}_k\circ\lambda\big(h_*f_*(\overline{E})\big) & \longrightarrow & 0\\
\downarrow & & \downarrow & & \downarrow\\
0 & \longrightarrow & 0 & \longrightarrow & 0.
\end{array}$$

Again, we construct a hermitian bundle $\mathrm{tr}_2(H_{h\circ f}'^{l}(\overline{E}))$ on $Z\times(\mathbb{P}^1)^{k+2}$ as the second transgression bundle of $H_{h\circ f}'^{l}(\overline{E})$ such that it satisfies the following relations:

$$\mathrm{tr}_2\big(H_{h\circ f}'^{l}(\overline{E})\big)\mid_{Z\times\{0\}\times(\mathbb{P}^1)^{k+1}}=\mathrm{tr}_1\big(H(f_*\overline{E},h_*)\big),$$

$$\mathrm{tr}_2\big(H_{h\circ f}'^{l}(\overline{E})\big)\mid_{Z\times\{\infty\}\times(\mathbb{P}^1)^{k+1}}=\mathrm{tr}_1\big(h_*f_*\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big)\to h_*f_*\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big)\big),$$

$$\mathrm{tr}_2\big(H_{h\circ f}'^{l}(\overline{E})\big)\mid_{Z\times\mathbb{P}^1\times\{0\}\times(\mathbb{P}^1)^{k}}=\mathrm{tr}_1\big(H(\overline{E},h_*f_*)\big),$$

$$\mathrm{tr}_2\big(H_{h\circ f}'^{l}(\overline{E})\big)\mid_{Z\times\mathbb{P}^1\times\{\infty\}\times(\mathbb{P}^1)^{k}}=\mathrm{tr}_1\big(h_*H(\overline{E},f_*)\big)$$

and

$$\mathrm{tr}_2\big(H_{h\circ f}'^{l}(\overline{E})\big)\mid_{Z\times(\mathbb{P}^1)^{i+1}\times\{0\}\times(\mathbb{P}^1)^{k-i}}=\mathrm{tr}_2\big(H_{h\circ f}'^{l}(\partial_i^0\overline{E})\big),$$

$$\mathrm{tr}_2\big(H_{h\circ f}'^{l}(\overline{E})\big)\mid_{Z\times(\mathbb{P}^1)^{i+1}\times\{\infty\}\times(\mathbb{P}^1)^{k-i}}=\mathrm{tr}_2\big(H_{h\circ f}'^{l}(\partial_i^{-1}\overline{E})\big)\oplus\mathrm{tr}_2\big(H_{h\circ f}'^{l}(\partial_i^1\overline{E})\big)$$

for $i=1,\dots,k$. We set

$$\mathbf{H}_{2,k}(\overline{E}):=\frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}}\int_{(\mathbb{P}^1)^{k+2}}\mathrm{ch}_g^0\big(\mathrm{tr}_2\big(H_{h\circ f}'^{l}(\overline{E})\big)\big)\wedge C_{k+2}(\log\mid z_1\mid^2,\dots,\log\mid z_{k+2}\mid^2).$$

Then $\mathbf{H}_{2,k}$ defines a map

$$\mathbf{H}_{2,k}:\widetilde{\mathbb{Z}}C_k^{(f,l)-\mathrm{ac}}(X,\mu_n)\to\bigoplus_{p\geqslant 0}D^{2p-k-2}(Z,p)_{R_n}$$

which satisfies the following differential equation:

$$d_{\mathcal{D}}\circ\mathbf{H}_{2,k}(\overline{E})$$

$$=\frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}}\int_{(\mathbb{P}^1)^{k+2}}\mathrm{ch}_g^0\Big(\mathrm{tr}_2\Big(H_{h\circ f}'^{l}(\overline{E})\Big)\Big)\wedge d_{\mathcal{D}}C_{k+2}(\log\mid z_1\mid^2,\dots,\log\mid z_{k+2}\mid^2)$$

$$= \frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}} \int_{(\mathbb{P}^1)^{k+2}} \mathrm{ch}_g^0\Big(\mathrm{tr}_2\Big(H'^{l}_{h\circ f}(\overline{E})\Big)\Big) \wedge \Bigg(\Big(-\frac{1}{2}\Big)(k+2) \sum_{j=1}^{k+2} (-1)^{j-1}(-4\pi i)$$

$$\times(\delta_{z_j=\infty} - \delta_{z_j=0}) \wedge C_{k+1}(\log \mid z_1 \mid^2, \ldots, \widehat{\log \mid z_j \mid^2}, \ldots, \log \mid z_{k+2} \mid^2)\Bigg)$$

$$= \mathbf{H}_{2,k-1} \circ d(\overline{E}) - \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \Big[\Big(\mathrm{ch}_g^0\big(\mathrm{tr}_1\big(H(\overline{E}, h_* f_*)\big)\big) - \mathrm{ch}_g^0\big(\mathrm{tr}_1\big(h_* H(\overline{E}, f_*)\big)\big)\Big)$$

$$\wedge C_{k+1}(\log \mid z_1 \mid^2, \ldots, \log \mid z_{k+1} \mid^2)\Big]$$

$$+ \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \mathrm{ch}_g^0\big(\mathrm{tr}_1\big(H(f_*\overline{E}, h_*)\big)\big) \wedge C_{k+1}(\log \mid z_1 \mid^2, \ldots, \log \mid z_{k+1} \mid^2)$$

$$= \mathbf{H}_{2,k-1} \circ d(\overline{E}) + \Pi_k'^{h}(f_*\overline{E})$$

$$- \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \mathrm{ch}_g^0\big(\mathrm{tr}_1\big(H(\overline{E}, h_* f_*)\big)\big) \wedge C_{k+1}(\log \mid z_1 \mid^2, \ldots, \log \mid z_{k+1} \mid^2)$$

$$+ \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \mathrm{ch}_g^0\big(\mathrm{tr}_1\big(h_* H(\overline{E}, f_*)\big)\big) \wedge C_{k+1}(\log \mid z_1 \mid^2, \ldots, \log \mid z_{k+1} \mid^2).$$

Third, note that the short exact sequence

$$0 \longrightarrow h_* \mathrm{tr}_1\big(H(\overline{E}, f_*)\big) \xrightarrow{\mathrm{Id}} \mathrm{tr}_1\big(h_* H(\overline{E}, f_*)\big) \longrightarrow 0 \longrightarrow 0$$

forms an emi-1-cube of hermitian bundles on $Z \times \mathbb{P}^1 \times (\mathbb{P}^1)^k$, we denote it by $\widetilde{H}_{h\circ f}(\overline{E})$. Using the same construction as before, we construct a transgression bundle $\mathrm{tr}_1(\widetilde{H}_{h\circ f}(\overline{E}))$ on $Z \times \mathbb{P}^1 \times \mathbb{P}^1 \times (\mathbb{P}^1)^k$ satisfying

$$\mathrm{tr}_1\Big(\widetilde{H}_{h\circ f}(\overline{E})\Big) \mid_{Z\times\{0\}\times(\mathbb{P}^1)^{k+1}} = \mathrm{tr}_1\big(h_* H(\overline{E}, f_*)\big),$$

$$\mathrm{tr}_1\Big(\widetilde{H}_{h\circ f}(\overline{E})\Big) \mid_{Z\times\{\infty\}\times(\mathbb{P}^1)^{k+1}} = h_* \mathrm{tr}_1\big(H(\overline{E}, f_*)\big),$$

$$\mathrm{tr}_1\Big(\widetilde{H}_{h\circ f}(\overline{E})\Big) \mid_{Z\times\mathbb{P}^1\times\{0\}\times(\mathbb{P}^1)^{k}} = \mathrm{tr}_1\big(h_* \mathrm{tr}_k \circ \lambda(f_*\overline{E}) \to h_* \mathrm{tr}_k \circ \lambda(f_*\overline{E}) \to 0\big),$$

$$\mathrm{tr}_1\Big(\widetilde{H}_{h\circ f}(\overline{E})\Big) \mid_{Z\times\mathbb{P}^1\times\{\infty\}\times(\mathbb{P}^1)^{k}} = \mathrm{tr}_1\big(h_* f_* \mathrm{tr}_k \circ \lambda(\overline{E}) \to h_* f_* \mathrm{tr}_k \circ \lambda(\overline{E}) \to 0\big)$$

and

$$\mathrm{tr}_1\Big(\widetilde{H}_{h\circ f}(\overline{E})\Big) \mid_{Z\times(\mathbb{P}^1)^{i+1}\times\{0\}\times(\mathbb{P}^1)^{k-i}} = \mathrm{tr}_1\Big(\widetilde{H}_{h\circ f}(\partial_i^0\overline{E})\Big),$$

$$\mathrm{tr}_1\Big(\widetilde{H}_{h\circ f}(\overline{E})\Big) \mid_{Z\times(\mathbb{P}^1)^{i+1}\times\{\infty\}\times(\mathbb{P}^1)^{k-i}} = \mathrm{tr}_1\Big(\widetilde{H}_{h\circ f}(\partial_i^{-1}\overline{E})\Big) \oplus \mathrm{tr}_1\Big(\widetilde{H}_{h\circ f}(\partial_i^1\overline{E})\Big)$$

for $i = 1, \ldots, k$. So, if we set

$$\mathbf{H}_{3,k}(\overline{E}) := \frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}} \int_{(\mathbb{P}^1)^{k+2}} \mathrm{ch}_g^0\Big(\mathrm{tr}_1\Big(\widetilde{H}_{h\circ f}(\overline{E})\Big)\Big) \wedge C_{k+2}(\log \mid z_1 \mid^2, \ldots, \log \mid z_{k+2} \mid^2),$$

it satisfies the differential equation

$$\begin{aligned}
&d_{\mathcal{D}}\circ\mathbf{H}_{3,k}(\overline{E})\\
&=\frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}}\int_{(\mathbb{P}^1)^{k+2}}\mathrm{ch}_g^0\Big(\mathrm{tr}_1(\widetilde{H}_{h\circ f})\Big)\wedge d_{\mathcal{D}}C_{k+2}(\log\mid z_1\mid^2,\dots,\log\mid z_{k+2}\mid^2)\\
&=\mathbf{H}_{3,k-1}\circ d(\overline{E})+\frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}}\int_{(\mathbb{P}^1)^{k+1}}\mathrm{ch}_g^0\big(\mathrm{tr}_1\big(h_*H(\overline{E},f_*)\big)\big)\\
&\quad\wedge C_{k+1}(\log\mid z_1\mid^2,\dots,\log\mid z_{k+1}\mid^2)\\
&\quad-\frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}}\int_{(\mathbb{P}^1)^{k+1}}\mathrm{ch}_g^0(h_*\mathrm{tr}_1\big(H(\overline{E},f_*)\big)\wedge C_{k+1}(\log\mid z_1\mid^2,\dots,\log\mid z_{k+1}\mid^2).
\end{aligned}$$

Finally, we set

$$\mathbf{H}_{4,k}(\overline{E}):=\frac{(-1)^{k+2}}{(k+2)!(2\pi i)^{k+1}}\int_{(\mathbb{P}^1)^{k+1}}C_{k+2}\Big(T_g\Big(h,h^{\mathrm{tr}_1\left(H(\overline{E},f_*)\right)}\Big),\log\mid z_1\mid^2,\dots,\log\mid z_{k+1}\mid^2\Big),$$

then it satisfies

$$\begin{aligned}
&d_{\mathcal{D}}\circ\mathbf{H}_{4,k}(\overline{E})\\
&=\frac{(-1)^{k+2}}{(k+2)!(2\pi i)^{k+1}}\int_{(\mathbb{P}^1)^{k+1}}C_{k+2}\Big(T_g\Big(h,h^{\mathrm{tr}_1\left(H(\overline{E},f_*)\right)}\Big),\log\mid z_1\mid^2,\dots,\log\mid z_{k+1}\mid^2\Big)\\
&=\mathbf{H}_{4,k-1}\circ d(\overline{E})+\frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}}\int_{(\mathbb{P}^1)^{k+1}}\mathrm{ch}_g^0\big(h_*\mathrm{tr}_1\big(H(\overline{E},f_*)\big)\big)\\
&\quad\wedge C_{k+1}(\log\mid z_1\mid^2,\dots,\log\mid z_{k+1}\mid^2)\\
&\quad-\frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}}\int_{(\mathbb{P}^1)^{k+1}}\left(\frac{1}{(2\pi i)^{r_h}}\int_{Y_{\mu_n}}\mathrm{Td}_g(\overline{Th})\mathrm{ch}_g^0\big(\mathrm{tr}_1\big(H(\overline{E},f_*)\big)\big)\right)\\
&\quad\wedge C_{k+1}(\log\mid z_1\mid^2,\dots,\log\mid z_{k+1}\mid^2)\\
&\quad-\frac{(-1)^{k+1}}{(k+1)!(2\pi i)^{k}}\int_{(\mathbb{P}^1)^{k}}C_{k+1}\Big(T_g\Big(h,h^{\mathrm{tr}_k\circ\lambda(f_*\overline{E})}\Big),\log\mid z_1\mid^2,\dots,\log\mid z_{k}\mid^2\Big)\\
&\quad+\frac{(-1)^{k+1}}{(k+1)!(2\pi i)^{k}}\int_{(\mathbb{P}^1)^{k}}C_{k+1}\Big(T_g\Big(h,h^{f_*\mathrm{tr}_k\circ\lambda(\overline{E})}\Big),\log\mid z_1\mid^2,\dots,\log\mid z_{k}\mid^2\Big)\\
&=\mathbf{H}_{4,k-1}\circ d(\overline{E})-h_{\mu_n*}\circ\Big(\mathrm{Td}_g(\overline{Th})\bullet\Pi_k'^f(\overline{E})\Big)\\
&\quad+\frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}}\int_{(\mathbb{P}^1)^{k+1}}\mathrm{ch}_g^0\big(h_*\mathrm{tr}_1\big(H(\overline{E},f_*)\big)\big)\wedge C_{k+1}(\log\mid z_1\mid^2,\dots,\log\mid z_{k+1}\mid^2)\\
&\quad-\Pi_k''^h(f_*\overline{E})+\frac{(-1)^{k+1}}{(k+1)!(2\pi i)^{k}}\int_{(\mathbb{P}^1)^{k}}C_{k+1}\Big(T_g\Big(h,h^{f_*\mathrm{tr}_k\circ\lambda(\overline{E})}\Big),\log\mid z_1\mid^2,\dots,\log\mid z_{k}\mid^2\Big).
\end{aligned}$$

Proposition 3.5. *Let notations and assumptions be as above, then the chain homotopy* $\Pi_k^l=\Pi_k'^l+\Pi_k''^l$ *is homotopic to* $\Pi_k^{(1)}+h_{\mu_n*}\circ(\mathrm{Td}_g(\overline{Th})\bullet\Pi_k^f)+\Pi_k^h\circ f_*-\Pi_k^{(3')}$.

Proof. Let $\overline{E}$ be a hermitian k-cube in $\widehat{\mathcal{P}}(X,\mu_n)$ which is f-acyclic and l-acyclic. Using the above differential equations concerning $\mathbf{H}_{i,k}$, we obtain that

$$\begin{aligned}
&(\mathbf{H}_{1,k-1}+\mathbf{H}_{2,k-1}-\mathbf{H}_{3,k-1}-\mathbf{H}_{4,k-1})\circ d(\overline{E})-d_{\mathcal{D}}\circ(\mathbf{H}_{1,k}+\mathbf{H}_{2,k}-\mathbf{H}_{3,k}-\mathbf{H}_{4,k})(\overline{E})\\
&=\Pi_k'^{l}(\overline{E})-\Pi_k^{(1)}(\overline{E})-\Pi_k'^{h}(f_*\overline{E})-\Pi_k''^{h}(f_*\overline{E})-h_{\mu_n*}\circ\Big(\mathrm{Td}_g(\overline{Th})\bullet\Pi_k'^{f}(\overline{E})\Big)\\
&\quad+\frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}}\int_{(\mathbb{P}^1)^{k+1}}\mathrm{ch}_g^0\big(\mathrm{tr}_1\big(H_{h\circ f}\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big)\big)\big)\\
&\quad\wedge C_{k+1}(\log\mid z_1\mid^2,\ldots,\log\mid z_{k+1}\mid^2)\\
&\quad+\frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k}\int_{(\mathbb{P}^1)^k}C_{k+1}\Big(T_g\Big(h,h^{f_*\mathrm{tr}_k\circ\lambda(\overline{E})}\Big),\log\mid z_1\mid^2,\ldots,\log\mid z_k\mid^2\Big).
\end{aligned}$$

On the other hand, according to Theorem 3.1, we have

$$\begin{aligned}
&\frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k}\int_{(\mathbb{P}^1)^k}C_{k+1}\Big(T_g\Big(g,h^{\mathrm{tr}_k\circ\lambda(\overline{E})}\Big),\log\mid z_1\mid^2,\ldots,\log\mid z_k\mid^2\Big)\\
&-\frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k}\int_{(\mathbb{P}^1)^k}C_{k+1}\Big(T_g\Big(h,h^{f_*\mathrm{tr}_k\circ\lambda(\overline{E})}\Big),\log\mid z_1\mid^2,\ldots,\log\mid z_k\mid^2\Big)\\
&-h_{\mu_n*}\circ\Big(\mathrm{Td}_g(\overline{Th})\bullet\Pi_k''^{f}(\overline{E})\Big)=\frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}}\int_{(\mathbb{P}^1)^{k+1}}\mathrm{ch}_g^0\big(\mathrm{tr}_1\big(H_{h\circ f}\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big)\big)\big)\\
&\quad\wedge C_{k+1}(\log\mid z_1\mid^2,\ldots,\log\mid z_{k+1}\mid^2)-\Pi_k^{(3')}(\overline{E})\\
&\quad+\frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k}\int_{(\mathbb{P}^1)^k}C_{k+1}\big(d_{\mathcal{D}}\Delta\big(f,h,\omega^X,\omega^Y,\mathrm{tr}_k\circ\lambda(\overline{E})\big),\log\mid z_1\mid^2,\ldots,\log\mid z_k\mid^2\big).
\end{aligned}$$

We then formally define a product

$$C_{k+1}\big(\Delta\big(f,h,\omega^X,\omega^Y,\mathrm{tr}_k\circ\lambda(\overline{E})\big),\log\mid z_1\mid^2,\ldots,\log\mid z_k\mid^2\big)$$

in the same way as (9), and we set

$$\Delta_k(\overline{E})=\frac{(-1)^k}{(k+1)!(2\pi i)^k}\int_{(\mathbb{P}^1)^k}C_{k+1}\big(\Delta\big(f,h,\omega^X,\omega^Y,\mathrm{tr}_k\circ\lambda(\overline{E})\big),\log\mid z_1\mid^2,\ldots,\log\mid z_k\mid^2\big).$$

It is readily checked by Lemma 3.2 that

$$\begin{aligned}
&\Delta_{k-1}(d\overline{E})-d_{\mathcal{D}}\Delta_k(\overline{E})\\
&=\frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k}\int_{(\mathbb{P}^1)^k}C_{k+1}\big(d_{\mathcal{D}}\Delta\big(f,h,\omega^X,\omega^Y,\mathrm{tr}_k\circ\lambda(\overline{E})\big),\log\mid z_1\mid^2,\ldots,\log\mid z_k\mid^2\big).
\end{aligned}$$

Combing all the above computations, we finally get

$$\begin{aligned}
&(\mathbf{H}_{1,k-1}+\mathbf{H}_{2,k-1}-\mathbf{H}_{3,k-1}-\mathbf{H}_{4,k-1}+\Delta_{k-1})(d\overline{E})\\
&\quad-d_{\mathcal{D}}(\mathbf{H}_{1,k}+\mathbf{H}_{2,k}-\mathbf{H}_{3,k}-\mathbf{H}_{4,k}+\Delta_k)(\overline{E})
\end{aligned}$$

$$
\begin{aligned}
&= \Pi_k'^{l}(\overline{E}) - \Pi_k^{(1)}(\overline{E}) - \Pi_k'^{h}(f_*\overline{E}) - \Pi_k''^{h}(f_*\overline{E}) + \Pi_k^{(3')}(\overline{E}) + \Pi_k''^{l}(\overline{E}) \\
&\quad -h_{\mu_n *} \circ \Big(\mathrm{Td}_g(\overline{Th}) \bullet \Pi_k'^{f}(\overline{E})\Big) - h_{\mu_n *} \circ \Big(\mathrm{Td}_g(\overline{Th}) \bullet \Pi_k''^{f}(\overline{E})\Big) \\
&= \Pi_k^{l}(\overline{E}) - \Big(\Pi_k^{(1)}(\overline{E}) + h_{\mu_n *} \circ \Big(\mathrm{Td}_g(\overline{Th}) \bullet \Pi_k^{f}(\overline{E})\Big) + \Pi_k^{h}(f_*\overline{E}) - \Pi_k^{(3')}(\overline{E})\Big).
\end{aligned}
$$

So, we are done. □

Corollary 3.6. *Let $f : X \to Y$, $h : Y \to Z$ and $l : X \to Z$ be three equivariant morphisms between regular μ_n-projective schemes, which are all smooth over the generic fibers. Assume $l = h \circ f$ and that the μ_n-action on Z is trivial. Then the direct image map l_* is equal to the composition $h_* \circ f_*$ from $\widehat{K}_m(X, \mu_n)_{\mathbb{Q}}$ to $\widehat{K}_m(Z, \mu_n)_{\mathbb{Q}}$ for any $m \geqslant 1$.*

4. *The Lefschetz–Riemann–Roch theorem*

4.1. *The statement*

In order to formulate the Lefschetz–Riemann–Roch theorem for higher equivariant arithmetic K-groups, we need to introduce the equivariant R-genus due to Bismut. Let X be a μ_n-equivariant smooth algebraic variety over $\mathbb{C}$, and let $\overline{E}$ be a μ_n-equivariant hermitian vector bundle on X. For $\zeta \in \mu_n(\mathbb{C})$ and $s > 1$, we consider the following Lerch zeta function

$$L(\zeta, s) = \sum_{k=1}^{\infty} \frac{\zeta^k}{k^s}$$

and its meromorphic continuation to the whole complex plane. Define a formal power series in the variable x as

$$\widetilde{R}(\zeta, x) := \sum_{n=0}^{\infty} \left(\frac{\partial L}{\partial s}(\zeta, -n) + L(\zeta, -n) \sum_{j=1}^{n} \frac{1}{2j} \right) \frac{x^n}{n!}.$$

Definition 4.1. The Bismut's equivariant R-genus of an equivariant hermitian vector bundle $\overline{E}$ with $\overline{E}\,|_{X_{\mu_n}} = \sum_{\zeta \in \mu_n(\mathbb{C})} \overline{E}_\zeta$ is defined as

$$R_g(\overline{E}) := \sum_{\zeta \in \mu_n(\mathbb{C})} \Big(\mathrm{Tr}\widetilde{R}(\zeta, -\Omega^{\overline{E}_\zeta}) - \mathrm{Tr}\widetilde{R}(1/\zeta, \Omega^{\overline{E}_\zeta}) \Big),$$

where $\Omega^{\overline{E}_\zeta}$ is the curvature form associated to $\overline{E}_\zeta$.

Now, let X be a regular μ_n-projective arithmetic scheme over an arithmetic ring (D, Σ, F_∞) and we construct a naive commutative diagram of homological complexes

$$
\begin{array}{ccc}
\widetilde{\mathbb{Z}}C_*(X, \mu_n) & \xrightarrow{\mathrm{ch}_g} & \bigoplus_{p \geq 0} D^{2p-*}(X_{\mu_n}, p)_{R_n} \\
\big\downarrow 0 & & \big\downarrow 0 \\
\widetilde{\mathbb{Z}}C_*(X, \mu_n) & \xrightarrow{\mathrm{ch}_g} & \bigoplus_{p \geq 0} D^{2p-*}(X_{\mu_n}, p)_{R_n}
\end{array}, \tag{11}
$$

where 0 stands for the zero map. Let $\overline{N}$ be a μ_n-equivariant hermitian vector bundle on X, we shall formally regard the R-genus $R_g(\overline{N})$ as an element in $\bigoplus_{p \geqslant 0} D^{2p-1}(X, p)$. It is a d-

closed form. Denote by p_0 the projection from $X \times (\mathbb{P}^1)^{\cdot}$ to X. For any hermitian k-cube $\overline{E}$ in $\widehat{\mathcal{P}}(X, \mu_n)$, we set

$$\Pi_R(\overline{E}) = \frac{(-1)^k}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}\big(R_g(p_0^*\overline{N})\mathrm{ch}_g^0\big(\mathrm{tr}_k \circ \lambda(\overline{E})\big), \log \mid z_1 \mid^2, \ldots, \log \mid z_k \mid^2\big).$$

It is clear that $\Pi_R(\overline{E})$ extends to be a map $\Pi_R : \widetilde{\mathbb{Z}}C_k(X, \mu_n) \to \bigoplus_{p \geqslant 0} D^{2p-k-1}(X_{\mu_n}, p)_{R_n}$ which provides a chain homotopy of the square (11). Therefore, we get an endomorphism of $\widehat{K}_m(X, \mu_n)$ for any $m \geqslant 1$. This endomorphism will be denoted by $\otimes R_g(\overline{N})$.

Again, by [34, Remark 2.4, Lemma 2.5], the chain homotopy Π_R is homotopic to the chain homotopy Π'_R defined by

$$\Pi'_R(\overline{E}) = \frac{(-1)^{k+1}}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} R_g(p_0^*\overline{N}) \bullet \mathrm{ch}_g^0\big(\mathrm{tr}_k \circ \lambda(\overline{E})\big) \bullet C_k(\log \mid z_1 \mid^2, \ldots, \log \mid z_k \mid^2),$$

and hence is homotopic to $-R_g(\overline{N}) \bullet \mathrm{ch}_g(\overline{E})$ by the projection formula. Let (x, α) be an element in $\widehat{K}_m(X, \mu_n)_{\mathbb{Q}}$, then $dx = 0$ and $\mathrm{ch}_g(x)$ is a $d_{\mathcal{D}}$-closed form. Let $(0, \alpha)$ and $(0, \alpha')$ be two elements in $\widehat{K}_m(X, \mu_n)_{\mathbb{Q}}$, then $(0, \alpha) = (0, \alpha')$ if α and α' have the same cohomology class in $\bigoplus_{p \geqslant 0} H^*_{\mathcal{D}}(X_{\mu_n}, \mathbb{R}(p))_{R_n}$. Note that the product $\bullet$ on the Deligne-Beilinson complex induces the product on the real Deligne-Beilinson cohomology. Then, modulo torsion, the endomorphism $\otimes R_g(\overline{N})$ is independent of the choice of the metric on N and it can be written as $\otimes R_g(N)$.

Assume that ρ is any prime ideal in $R(\mu_n) := K_0(\mathrm{Spec}\mathbb{Z}, \mu_n) \cong \mathbb{Z}[T]/(1 - T^n)$ which does not contain the elements $1 - T^k$ for $k = 1, \ldots, n-1$. For instance, ρ can be chosen to be the kernel of the natural morphism $\mathbb{Z}[T]/(1 - T^n) \to \mathbb{Z}[T]/(\Phi_n)$, where Φ_n stands for the nth cyclotomic polynomial. Let X_{μ_n} be the fixed-point subscheme of X, and let $\overline{N}_{X/X_{\mu_n}}$ be the normal bundle of X_{μ_n} in X with some μ_n-invariant hermitian metric. We set

$$\Lambda_R := \big(\mathrm{Id} - \otimes R_g(N_{X/X_{\mu_n}})\big) \circ \otimes \lambda_{-1}^{-1}(\overline{N}^{\vee}_{X/X_{\mu_n}}),$$

it is a well-defined endomorphism of $\widehat{K}_m(X_{\mu_n}, \mu_n)_\rho \otimes \mathbb{Q}$. Then the arithmetic Lefschetz–Riemann–Roch theorem for higher equivariant arithmetic K-groups can be formulated as follows.

THEOREM 4.2 (arithmetic Lefschetz–Riemann–Roch). *Let $f : X \to Y$ be an equivariant morphism between two regular μ_n-projective arithmetic schemes, which is smooth over the generic fiber. Suppose that the μ_n-action on the base Y is trivial. Then, for any $m \geqslant 1$, the following diagram*

$$\begin{array}{ccc} \widehat{K}_m(X, \mu_n) & \xrightarrow{\Lambda_R \circ \tau} & \widehat{K}_m(X_{\mu_n}, \mu_n)_\rho \otimes \mathbb{Q} \\ \downarrow{\scriptstyle f_*} & & \downarrow{\scriptstyle f_{\mu_n *}} \\ \widehat{K}_m(Y, \mu_n) & \xrightarrow{\tau} & \widehat{K}_m(Y, \mu_n)_\rho \otimes \mathbb{Q} \end{array}$$

where τ is the restriction map, is commutative.

The proof of Theorem 4.2 will be given in next two subsections.

4.2. *Arithmetic K-theoretic form of Bismut–Ma immersion formula*

Let $Y \hookrightarrow X$ be a μ_n-equivariant closed immersion of regular μ_n-projective arithmetic schemes over (D, Σ, F_∞). In [34, Section 4], we have proved an arithmetic purity theorem

$$\widehat{K}_m(Y, \mu_n) \cong \widehat{K}_{Y,m}(X, \mu_n)$$

for any integer $m \geqslant 1$. As a byproduct, we get an embedding morphism $\widehat{K}_m(Y,\mu_n) \to \widehat{K}_m(X,\mu_n)$. This embedding morphism is realized by constructing an explicit chain homotopy of the square

$$\begin{array}{ccc} \widetilde{\mathbb{Z}}C_*(Y,\mu_n) & \xrightarrow{\mathrm{ch}_g} & \bigoplus_{p\geq 0} {}'D^{2p-*}(Y_{\mu_n},p)_{R_n} \\ \downarrow{\scriptstyle i_*} & & \downarrow{\scriptstyle i_{\mu_n!}\circ \mathrm{Td}_g^{-1}(\overline{N}_{X/Y})\bullet(\cdot)} \\ \widetilde{\mathbb{Z}}C_*(P,\mu_n) & \xrightarrow{\mathrm{ch}_g} & \bigoplus_{p\geq 0} {}'D^{2p-*}(P_{\mu_n},p)_{R_n}, \end{array} \tag{12}$$

where ${}'D^{2p-*}(\cdot,p)$ stands for the Deligne complex of currents computing the Deligne homology groups, $(i_{\mu_n!}T)(\eta) = T(i_{\mu_n}^*\eta)$ for a current T and a test form η, $i : Y \hookrightarrow P := \mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y)$ is the associated zero section embedding with projection $\pi : P \to Y$ and

$$i_* : \widetilde{\mathbb{Z}}C_*(Y,\mu_n) \to \widetilde{\mathbb{Z}}C_*(P,\mu_n)$$

is the complex morphism defined by sending a hermitian cube $\overline{E}$ to $\sum_{j=0}^n (-1)^j \overline{Q}^\vee \otimes \pi^*\overline{E}$ provided the Koszul resolution

$$K(\overline{E},\overline{N}_{X/Y}) : \quad 0 \to \wedge^n \overline{Q}^\vee \otimes \pi^*\overline{E} \to \cdots \to \wedge \overline{Q}^\vee \otimes \pi^*\overline{E} \to \pi^*\overline{E} \to i_*\overline{E} \to 0.$$

For any hermitian k-cube $\overline{E}$, one chain homotopy $\mathbf{H}_k(\overline{E})$ of (12) is given by the formula

$$\mathbf{H}_k(\overline{E}) = T_g\big(K(\overline{\mathcal{O}}_Y,\overline{N}_{X/Y})\big) \bullet \mathrm{ch}_g(\pi^*\overline{E}),$$

where $T_g(K(\overline{\mathcal{O}}_Y,\overline{N}_{X/Y}))$ is the equivariant Bott–Chern singular current associated to the Koszul resolution which satisfies

$$d_{\mathcal{D}} T_g\big(K(\overline{\mathcal{O}}_Y,\overline{N}_{X/Y})\big) = \sum_{j=1}^n (-1)^j \mathrm{ch}_g(\wedge^n \overline{Q}^\vee) - i_{\mu_n!}\big(\mathrm{ch}_g(\overline{O}_Y)\mathrm{Td}_g^{-1}(\overline{N}_{X/Y})\big).$$

For more details the reader is referred to [**34**, Section 4.2].

It is clear that if we choose another resolution

$$0 \to \overline{F}_n \to \cdots \to \overline{F}_1 \to \overline{F}_0 \to i_*\overline{O}_Y \to 0$$

with respect to the zero section embedding $i : Y \hookrightarrow \mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y)$ such that the metrics on $F.$ satisfy the Bismut's assumption (A), we may construct a different homotopy of (12) and we shall get a different embedding morphism $i_* : \widehat{K}_m(Y,\mu_n) \to \widehat{K}_m(P,\mu_n)$. Our first result in this subsection is the following.

Proposition 4.3. *The embedding morphism over rational arithmetic K-groups*

$$i_* : \widehat{K}_m(Y,\mu_n)_{\mathbb{Q}} \to \widehat{K}_m(P,\mu_n)_{\mathbb{Q}}$$

is independent of the choice of the resolution of $i_*\overline{\mathcal{O}}_Y$ on $\mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y)$ which satisfies the Bismut's assumption (A).

Proof. Since any two resolutions of $i_*\overline{\mathcal{O}}_Y$ on $\mathbb{P}(N_{X/Y}\oplus\mathcal{O}_Y)$ are dominated by a third one, we may assume that $\overline{F}.$ and $\wedge\cdot\overline{Q}^\vee$ fit into the following diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{A}_n & \longrightarrow & \overline{F}_n & \longrightarrow & \wedge^n\overline{Q}^\vee & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{A}_1 & \longrightarrow & \overline{F}_1 & \longrightarrow & \wedge\overline{Q}^\vee & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \overline{A}_0 & \longrightarrow & \overline{F}_0 & \longrightarrow & \overline{\mathcal{O}}_P & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & i_*\overline{\mathcal{O}}_Y & \longrightarrow & i_*\overline{\mathcal{O}}_Y & &
\end{array}$$

where $\overline{A}.$ is an exact sequence of hermitian vector bundles on P. We endow $A.$ with the metrics coming form $\overline{F}.$ via the natural inclusion. We split $\overline{A}.$ into a family of short exact sequence of hermitian bundles from $j=1$ to $n-1$

$$\chi_j: \quad 0\longrightarrow \operatorname{Ker} d_j \longrightarrow \overline{A}_j \xrightarrow{d_j} \operatorname{Ker} d_{j-1}\longrightarrow 0\ .$$

Moreover, we denote by ε_j the short exact sequence

$$0\longrightarrow \overline{A}_j\longrightarrow \overline{F}_j\longrightarrow \wedge^j\overline{Q}^\vee\longrightarrow 0$$

from $j=0$ to n. Write i_* (respectively, i'_*) for the morphism $\widetilde{\mathbb{Z}}C_*(Y,\mu_n)\to\widetilde{\mathbb{Z}}C_*(P,\mu_n)$ with respect to the Koszul resolution $K(\overline{\mathcal{O}}_Y,\overline{N}_{X/Y})$ (respectively, the resolution $\overline{F}.$). Then, for any hermitian k-cube $\overline{E}$ on Y, the assignment

$$H_i(\overline{E}):=\sum_{j=0}^{n}(-1)^j\varepsilon_j\otimes\pi^*\overline{E}+\sum_{j=1}^{n-1}(-1)^j\chi_j\otimes\pi^*\overline{E}\in\widetilde{\mathbb{Z}}C_{k+1}(P,\mu_n)$$

provides a chain homotopy between i'_* and i_*. Consequently, the formula

$$\mathbf{H}_k^{(1)}(\overline{E})=\left(\sum_{j=0}^{n}(-1)^j\mathrm{ch}_g(\varepsilon_j)+\sum_{j=1}^{n-1}(-1)^j\mathrm{ch}_g(\chi_j)\right)\mathrm{ch}_g(\pi^*\overline{E})$$

defines a chain homotopy between $\mathrm{ch}_g\circ i'_*$ and $\mathrm{ch}_g\circ i_*$. We claim that there exists a homotopy of chain homotopies between $\mathbf{H}'_k(\overline{E})$ and $\mathbf{H}_k^{(1)}(\overline{E})+\mathbf{H}_k(\overline{E})$.

In fact, according to [24, Theorem 3.14, Corollary 3.10], we have

$$-\sum_{j=1}^{n-1}(-1)^j\mathrm{ch}_g(\chi_j)+T_g(\overline{F}.)-T_g\big(K(\overline{\mathcal{O}}_Y,\overline{N}_{X/Y})\big)=\sum_{j=0}^{n}(-1)^j\mathrm{ch}_g(\varepsilon_j)$$

up to $\operatorname{Im} d_{\mathcal{D}}$. We fix an element Δ such that

$$d_{\mathcal{D}}\Delta = \sum_{j=0}^{n}(-1)^j \mathrm{ch}_g(\varepsilon_j) + \sum_{j=1}^{n-1}(-1)^j \mathrm{ch}_g(\chi_j) - T_g(\overline{F}.) + T_g\big(K(\overline{\mathcal{O}}_Y, \overline{N}_{X/Y})\big)$$

and set

$$\widetilde{\mathbf{H}}_k(\overline{E}) := \Delta \bullet \mathrm{ch}_g(\pi^*\overline{E}).$$

Then

$$d_{\mathcal{D}} \circ \widetilde{\mathbf{H}}_k(\overline{E}) = \mathbf{H}_k^{(1)}(\overline{E}) + \mathbf{H}_k(\overline{E}) - \mathbf{H}'_k(\overline{E}) + \widetilde{\mathbf{H}}_{k-1} \circ d(\overline{E}).$$

So, we are done. □

Note that the product $P \times (\mathbb{P}^1)^{\cdot}$ can be identified with the projective space bundle over $Y \times (\mathbb{P}^1)^{\cdot}$ with respect to the vector bundle $p_0^* N_{X/Y}$, and

$$0 \to p_0^* \wedge^n \overline{Q}^{\vee} \to \cdots \to p_0^* \wedge \overline{Q}^{\vee} \to \overline{\mathcal{O}}_{P\times(\mathbb{P}^1)^{\cdot}} \to i_* \overline{\mathcal{O}}_{Y\times(\mathbb{P}^1)^{\cdot}} \to 0$$

is the Koszul resolution, so that the corresponding Bott–Chern singular current is the pullback $p_0^* T_g(K(\overline{\mathcal{O}}_Y, \overline{N}_{X/Y}))$. We shall still write it as $T_g(K(\overline{\mathcal{O}}_Y, \overline{N}_{X/Y})$ for the sake of simplicity. Then, like before, by the projection formula and [34, Remark 2.4, Lemma 2.5], $\mathbf{H}_k(\overline{E})$ is homotopic to the following chain homotopy

$$\frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}\big(T_g\big(K(\overline{\mathcal{O}}_Y, \overline{N}_{X/Y})\big) \bullet \mathrm{ch}_g^0\big(\mathrm{tr}_k \circ \lambda(\overline{E})\big), \log \mid z_1 \mid^2, \ldots, \log \mid z_k \mid^2\big),$$

which will be still denoted by $\mathbf{H}_k(\overline{E})$.

Now, let us recall the Bismut–Ma immersion formula which relates analytic torsion forms and the Bott–Chern singular current. Let X be a smooth μ_n-equivariant algebraic variety over $\mathbb{C}$ and let $i : Y \hookrightarrow X$ be an equivariant closed smooth subvariety. Let S be a smooth algebraic variety with trivial μ_n-action, and let $f : Y \to S$, $l : X \to S$ be two equivariant proper smooth morphisms such that $f = l \circ i$. Assume that $\overline{\eta}$ is an equivariant hermitian bundle on Y and $\overline{\xi}.$ is a complex of equivariant hermitian bundles on X which provides a resolution of $i_*\overline{\eta}$ such that the metrics on $\xi.$ satisfy the Bismut's assumption (A). Let ω^Y, ω^X be two Kähler fibrations on f and on l, respectively. We shall assume that ω^Y is the pullback of ω^X, so that the Kähler metric on Y is induced by the Kähler metric on X. Consider the following exact sequence

$$\overline{\mathcal{N}} : \quad 0 \to \overline{Tf} \to \overline{Tl}\mid_Y \to \overline{N}_{X/Y} \to 0,$$

where $N_{X/Y}$ is endowed with the quotient metric. Denote by $\mathrm{Td}_g(\overline{\mathcal{N}}) = \Phi^{-1}(\frac{\widetilde{\mathrm{Td}}_g(\overline{\mathcal{N}})}{2})$ (see Section A.3 in the Appendix) the equivariant secondary Todd form of $\overline{\mathcal{N}}$ which satisfies the identity

$$d_{\mathcal{D}}\mathrm{Td}_g(\overline{\mathcal{N}}) = \mathrm{Td}_g(Tl\mid_Y, h^{Tl}) - \mathrm{Td}_g(Tf, h^{Tf})\mathrm{Td}_g(\overline{N}_{X/Y}).$$

We suppose that in the resolution $\xi.$, ξ_j are all $l-$acyclic and moreover η is $f-$acyclic. Denote by $h^{H(\xi.)}$ the hermitian metric on $f_*\eta$ corresponding to the L^2-metric on the hypercohomology of $\xi.$ over the fiber of $l : X \to S$ (see Section A.3 in the Appendix). By an easy argument of long exact sequence, we have the following exact sequence of hermitian vector bundles on S

$$\overline{\Xi} : \quad 0 \to l_*(\overline{\xi}_m) \to l_*(\overline{\xi}_{m-1}) \to \ldots \to l_*(\overline{\xi}_0) \to \Big(f_*\eta, h^{H(\xi.)}\Big) \to 0.$$

We may split $\Xi.$ into a family of short exact sequence of hermitian bundles from $j = 1$ to m

$$\chi_j : \quad 0 \longrightarrow \mathrm{Ker}\, d_j \longrightarrow \overline{\Xi}_j \xrightarrow{d_j} \mathrm{Ker}\, d_{j-1} \longrightarrow 0$$

such that the kernel of every map d_{j-1} for $j = 2, \dots, m$ carries the metric induced by $\overline{\Xi}_j$ and $\operatorname{Ker} d_0 = \overline{\Xi}_0 = (f_*\eta, h^{H(\xi.)})$, $\operatorname{Ker} d_m = \overline{\Xi}_{m+1} = l_*(\overline{\xi}_m)$. We regard χ_j as a hermitian 1-cube on S and we set $\operatorname{ch}_g(\overline{\Xi}.) = \sum_{j=1}^{m}(-1)^j \operatorname{ch}_g(\chi_j)$. Then it satisfies the differential equation

$$d_{\mathcal{D}}\operatorname{ch}_g(\overline{\Xi}.) = \operatorname{ch}_g\Big(f_*\eta, h^{H(\xi.)}\Big) - \sum_{j=0}^{m} \operatorname{ch}_g\big(l_*(\overline{\xi}_j)\big).$$

Set $\operatorname{ch}_g(\overline{\Xi}., f_*\overline{\eta}) := \operatorname{ch}_g(\overline{\Xi}.) + \operatorname{ch}_g(f_*\eta, h^{H(\xi.)}, f_*h^{\eta})$, it satisfies the differential equation

$$d_{\mathcal{D}}\operatorname{ch}_g(\overline{\Xi}., f_*\overline{\eta}) = \operatorname{ch}_g(f_*\overline{\eta}) - \sum_{j=0}^{m} \operatorname{ch}_g\big(l_*(\overline{\xi}_j)\big).$$

With some abuse of notations, we still use $\overline{\Xi}$ to denote the long exact sequence

$$0 \to l_*(\overline{\xi}_m) \to l_*(\overline{\xi}_{m-1}) \to \dots \to l_*(\overline{\xi}_0) \to f_*\overline{\eta} \to 0$$

and identify $\operatorname{ch}_g(\overline{\Xi}.)$ with $\operatorname{ch}_g(\overline{\Xi}., f_*\overline{\eta})$.

THEOREM 4.4 (Immersion formula). *Let notations and assumptions be as above. Then the following identity holds in* $\bigoplus_{p\geqslant 0}(D^{2p-1}(S,p)/\operatorname{Im} d_{\mathcal{D}})$.

$$\begin{aligned}
&\sum_{i=0}^{m}(-1)^i T_g(\omega^X, h^{\xi_i}) - T_g(\omega^Y, h^{\eta}) + \operatorname{ch}_g(\overline{\Xi}.) \\
&\quad = -\frac{1}{(2\pi i)^{r_l}}\int_{X_{\mu_n}/S} \operatorname{Td}_g(\overline{Tl})T_g(\overline{\xi}.) - \frac{1}{(2\pi i)^{r_f}}\int_{Y_{\mu_n}/S} \operatorname{Td}_g(\overline{\mathcal{N}})\operatorname{Td}_g^{-1}(\overline{N}_{X/Y})\operatorname{ch}_g(\overline{\eta}) \\
&\qquad + \frac{1}{(2\pi i)^{r_f}}\int_{Y_{\mu_n}/S} \operatorname{Td}_g(\overline{Tf})R_g(\overline{N}_{X/Y})\operatorname{ch}_g(\overline{\eta}),
\end{aligned}$$

where r_f and r_l are the relative dimensions of Y_{μ_n}/S and of X_{μ_n}/S, respectively. Note that the equivariant Bott–Chern singular current we use here differs from the one used in the appendix by a minus sign.

Proof. This is a translation of [**14**, Theorem 0.1 and 0.2] (see also Theorem A.5). □

With the same notations as in Remark 2.7 and Theorem A.6, we set

$$\Delta(f, l, i_*\overline{\eta}, \overline{\xi}.) := -\Phi^{-1}\left(\frac{\Delta^0(f, l, i_*\overline{\eta}, \overline{\xi}.) + \Delta_0(f, l, i_*\overline{\eta}, \overline{\xi}.)}{2}\right) - \Delta(\overline{\Xi}.).$$

Then $d_{\mathcal{D}}\Delta(f, l, i_*\overline{\eta}, \overline{\xi}.)$ measures the difference

$$\begin{aligned}
&\sum_{i=0}^{m}(-1)^i T_g(\omega^X, h^{\xi_i}) - T_g(\omega^Y, h^{\eta}) + \operatorname{ch}_g(\overline{\Xi}.) + \frac{1}{(2\pi i)^{r_l}}\int_{X_{\mu_n}/S} \operatorname{Td}_g(\overline{Tl})T_g(\overline{\xi}.) \\
&\quad + \frac{1}{(2\pi i)^{r_f}}\int_{Y_{\mu_n}/S} \operatorname{Td}_g(\overline{\mathcal{N}})\operatorname{Td}_g^{-1}(\overline{N}_{X/Y})\operatorname{ch}_g(\overline{\eta}) - \frac{1}{(2\pi i)^{r_f}}\int_{Y_{\mu_n}/S} \operatorname{Td}_g(\overline{Tf})R_g(\overline{N}_{X/Y})\operatorname{ch}_g(\overline{\eta})
\end{aligned}$$

in Theorem 4.4. Let us go back to the same situation described before Lemma 2.12 and assume that the following diagrams

$$\begin{array}{ccc} Y \times Z & \xrightarrow{i_Z} & X \times Z \\ & {\scriptstyle f_Z}\searrow \quad \swarrow{\scriptstyle l_Z} & \\ & S \times Z & \end{array} \quad \text{and} \quad \begin{array}{ccc} Y \times Z_1 & \xrightarrow{i_{Z_1}} & X \times Z_1 \\ & {\scriptstyle f_{Z_1}}\searrow \quad \swarrow{\scriptstyle l_{Z_1}} & \\ & S \times Z_1 & \end{array}$$

are obtained by smooth base changes. Then $Y \times Z$ and $X \times Z_1$ intersect transversely along $Y \times Z_1$ and the singular currents can be pulled back.

LEMMA 4.5. *The restriction of* $\Delta(f_Z, l_Z, i_{Z*}\overline{\eta}, \overline{\xi}.)$ *over* $S \times Z_1$ *is equal to the differential form* $\Delta(f_{Z_1}, l_{Z_1}, i_{Z_1*}\overline{\eta}\,|_{Y\times Z_1}, \overline{\xi}.\,|_{X\times Z_1})$.

Proof. This is a consequence of Theorem A.6. □

PROPOSITION 4.6. *Let* Y *be a regular* μ_n*-projective arithmetic scheme over* (D, Σ, F_∞) *and let* $\overline{N}$ *be a* μ_n*-equivariant hermitian vector bundle on* Y. *Suppose that the* μ_n*-action on* Y *is trivial and consider the zero section embedding*

$$i : Y \hookrightarrow P := \mathbb{P}(N \oplus \mathcal{O}_Y)$$

with hermitian normal bundle $\overline{N}$ *and the natural projection* $\pi : P \to Y$. *Then for any element* $x \in \widehat{K}_m(Y, \mu_n)_{\mathbb{Q}}$ *with integer* $m \geqslant 1$, *the following identity*

$$x - R_g(N) \cdot x = \pi_* i_*(x)$$

holds in $\widehat{K}_m(Y, \mu_n)_{\mathbb{Q}}$.

Proof. By the definition of the action of $R_g(N)$ on $\widehat{K}_m(Y, \mu_n)_{\mathbb{Q}}$, the map $x \mapsto x - R_g(N) \cdot x$ is defined via the chain homotopy

$$\Pi_k^{(0)}(\overline{E}) = \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}\big(R_g(\overline{N}) \bullet \mathrm{ch}_g^0\big(\mathrm{tr}_k \circ \lambda(\overline{E})\big), \log \mid z_1 \mid^2, \ldots, \log \mid z_k \mid^2\big)$$

of the square

$$\begin{array}{ccc} \widetilde{\mathbb{Z}} C_*(Y, \mu_n) & \xrightarrow{\mathrm{ch}_g} & \bigoplus_{p\geq 0} D^{2p-*}(Y_{\mu_n}, p)_{R_n} \\ \downarrow \mathrm{Id} & & \downarrow \mathrm{Id} \\ \widetilde{\mathbb{Z}} C_*(Y, \mu_n) & \xrightarrow{\mathrm{ch}_g} & \bigoplus_{p\geq 0} D^{2p-*}(Y_{\mu_n}, p)_{R_n}. \end{array}$$

According to Proposition 4.3, to define the morphism $i_* : \widehat{K}_m(Y, \mu_n)_{\mathbb{Q}} \to \widehat{K}_m(P, \mu_n)_{\mathbb{Q}}$, we may choose a resolution $F.$ of $i_*\mathcal{O}_Y$ on P such that every F_j is π-acyclic. We shall endow $F.$ with the metrics satisfying the Bismut's assumption (A). Then we have an exact sequence of hermitian bundles on Y

$$\overline{\Xi} : \quad 0 \to \pi_*(\overline{F}_m) \to \pi_*(\overline{F}_{m-1}) \to \ldots \to \pi_*(\overline{F}_0) \to \overline{\mathcal{O}}_Y \to 0.$$

Like before, splitting $\overline{\Xi}$ into a family of short exact sequence of hermitian bundles from $j = 1$ to m

$$\chi_j : \quad 0 \longrightarrow \mathrm{Ker}\, d_j \longrightarrow \overline{\Xi}_j \xrightarrow{d_j} \mathrm{Ker}\, d_{j-1} \longrightarrow 0,$$

we may construct a chain homotopy

$$H_{\pi\circ i}(\overline{E}) := \sum_{j=1}^{m} (-1)^j \chi_j \otimes \overline{E} \in \widetilde{\mathbb{Z}} C_{k+1}(Y, \mu_n)$$

between the maps Id and $\pi_* \circ i_* : \widetilde{\mathbb{Z}} C_*(Y, \mu_n) \to \widetilde{\mathbb{Z}} C_*(Y, \mu_n)$. Consequently, the formula

$$\mathbf{H}_k^{(1)}(\overline{E}) = \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \mathrm{ch}_g^0\big(\mathrm{tr}_{k+1} \circ \lambda\big(H_{\pi\circ i}(\overline{E})\big)\big) \wedge C_{k+1}(\log \mid z_1 \mid^2, \ldots, \log \mid z_{k+1} \mid^2)$$

defines a chain homotopy between $\mathrm{ch}_g \circ \mathrm{Id}$ and $\mathrm{ch}_g \circ \pi_* \circ i_*$. Then $\mathbf{H}_k^{(1)} + \Pi_k^{\pi} \circ i_* + \pi_{\mu_n *} \circ (\mathrm{Td}_g(\overline{T\pi}) \bullet \mathbf{H}_k)$ also defines a chain homotopy between $\mathrm{ch}_g \circ \mathrm{Id}$ and $\mathrm{Id} \circ \mathrm{ch}_g$. We compare it with $\Pi_k^{(0)}$.

First, denote by Pr_P (respectively, Pr_Y) the projection from $P \times (\mathbb{P}^1)^k$ (respectively, $Y \times (\mathbb{P}^1)^k$) to P (respectively, Y). Then, according to the functoriality of projective space bundle construction we have used before, $\mathrm{Pr}_P^* \overline{F}.$ provides a resolution of $i_* \overline{\mathcal{O}}_{Y \times (\mathbb{P}^1)^k}$ on $P \times (\mathbb{P}^1)^k$. Hence, we have an exact sequence

$$\overline{\Xi}' : \quad 0 \to \pi_*(\mathrm{Pr}_P^* \overline{F}_m) \to \pi_*(\mathrm{Pr}_P^* \overline{F}_{m-1}) \to \ldots \to \pi_*(\mathrm{Pr}_P^* \overline{F}_0) \to \overline{\mathcal{O}}_{Y \times (\mathbb{P}^1)^k} \to 0$$

which can be split into a family of short exact sequence of hermitian bundles from $j = 1$ to m

$$\chi_j' : \quad 0 \longrightarrow \mathrm{Ker}\, d_j \longrightarrow \overline{\Xi}_j' \xrightarrow{d_j} \mathrm{Ker}\, d_{j-1} \longrightarrow 0.$$

Furthermore, the short exact sequence of hermitian 1-cube

$$H^{(j)}(\overline{E}) : \ 0 \longrightarrow \chi_j' \otimes \mathrm{tr}_k \circ \lambda(\overline{E}) \xrightarrow{\mathrm{Id}} \mathrm{Pr}_Y^* \chi_j \otimes \mathrm{tr}_k \circ \lambda(\overline{E}) \longrightarrow 0 \longrightarrow 0$$

forms a hermitian 2-cube on $Y \times (\mathbb{P}^1)^k$. We set

$$\widetilde{\mathbf{H}}_k(\overline{E}) := \frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}} \int_{(\mathbb{P}^1)^{k+2}} \mathrm{ch}_g^0 \left(\sum_{j=1}^{m} (-1)^j \mathrm{tr}_2 \circ \lambda\Big(H^{(j)}(\overline{E})\Big) \right) \wedge C_{k+2}(\log \mid z_1 \mid^2, \ldots, \log \mid z_{k+2} \mid^2),$$

it satisfies the differential equation

$$\begin{aligned}
d_{\mathcal{D}} \circ \widetilde{\mathbf{H}}_k(\overline{E}) &= \widetilde{\mathbf{H}}_{k-1} \circ d\overline{E} + \sum_{j=0}^{m} (-1)^j \Pi_k'^{\pi}(\overline{F}_j \otimes \pi^* \overline{E}) \\
&+ \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \mathrm{ch}_g^0\big(\mathrm{tr}_{k+1} \circ \lambda\big(H_{\pi \circ i}(\overline{E})\big)\big) \wedge C_{k+1}(\log \mid z_1 \mid^2, \ldots, \log \mid z_{k+1} \mid^2) \\
&- \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \mathrm{ch}_g^0 \left(\sum_{j=1}^{m} (-1)^j \mathrm{tr}_1 \circ \lambda(\chi_j') \boxtimes \mathrm{tr}_k \circ \lambda(\overline{E}) \right) \\
&\wedge C_{k+1}(\log \mid z_1 \mid^2, \ldots, \log \mid z_{k+1} \mid^2) \\
&= \widetilde{\mathbf{H}}_{k-1} \circ d\overline{E} + \mathbf{H}_k^{(1)}(\overline{E}) + \Pi_k'^{\pi} \circ i_*(\overline{E}) \\
&- \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}\Big(\mathrm{ch}_g\Big(\overline{\Xi}' \otimes \mathrm{tr}_k \circ \lambda(\overline{E})\Big), \log \mid z_1 \mid^2, \ldots, \log \mid z_k \mid^2\Big).
\end{aligned}$$

On the other hand, we apply the immersion formula to the resolution $\mathrm{Pr}_P^* \overline{F}. \otimes \mathrm{tr}_k \circ \lambda(\overline{E})$. We then have

$$\begin{aligned}
&\Pi_k''^{\pi} \circ i_*(\overline{E}) \\
&= -\frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}\Big(\mathrm{ch}_g\Big(\overline{\Xi}' \otimes \mathrm{tr}_k \circ \lambda(\overline{E})\Big), \log \mid z_1 \mid^2, \ldots, \log \mid z_k \mid^2\Big) \\
&\quad - \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} C_{k+1}\left(\frac{1}{(2\pi i)^{r_\pi}} \int_{P_{\mu_n}/Y} \mathrm{Td}_g(\overline{T\pi}) T_g\big(\mathrm{Pr}_P^* \overline{F}. \otimes \mathrm{tr}_k \circ \lambda(\overline{E})\big), \right.
\end{aligned}$$

$$\log \mid z_1 \mid^2, \dots, \log \mid z_k \mid^2 \Big)$$

$$+\frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k}\int_{(\mathbb{P}^1)^k} C_{k+1}\big(R_g(\overline{N})\bullet \mathrm{ch}_g^0\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big), \log \mid z_1 \mid^2, \dots, \log \mid z_k \mid^2\big)$$

$$+\frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k}\int_{(\mathbb{P}^1)^k} C_{k+1}\big(d_{\mathcal{D}}\Delta\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big), \log \mid z_1 \mid^2, \dots, \log \mid z_k \mid^2\big)$$

$$= -\frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k}\int_{(\mathbb{P}^1)^k} C_{k+1}\Big(\mathrm{ch}_g\Big(\overline{\Xi}'\otimes \mathrm{tr}_k\circ\lambda(\overline{E})\Big), \log \mid z_1 \mid^2, \dots, \log \mid z_k \mid^2\Big)$$

$$-\pi_{\mu_n *}\circ\big(\mathrm{Td}_g(\overline{T\pi})\bullet \mathbf{H}_k(\overline{E})\big)$$

$$+\Pi_k^{(0)}(\overline{E}) + \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k}\int_{(\mathbb{P}^1)^k} C_{k+1}\big(d_{\mathcal{D}}\Delta\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big), \log \mid z_1 \mid^2, \dots, \log \mid z_k \mid^2\big).$$

We then formally define a product $C_{k+1}(\Delta(\mathrm{tr}_k\circ\lambda(\overline{E})), \log \mid z_1 \mid^2, \dots, \log \mid z_k \mid^2)$ in the same way as (9), and we set

$$\Delta_k(\overline{E}) = \frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k}\int_{(\mathbb{P}^1)^k} C_{k+1}\big(\Delta\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big), \log \mid z_1 \mid^2, \dots, \log \mid z_k \mid^2\big).$$

Again, it is readily checked by Lemma 4.5 that

$$\Delta_{k-1}(d\overline{E}) - d_{\mathcal{D}}\Delta_k(\overline{E}) = \frac{(-1)^k}{(k+1)!(2\pi i)^k}\int_{(\mathbb{P}^1)^k} C_{k+1}\Big(d_{\mathcal{D}}\Delta\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big), \log \mid z_1 \mid^2, \dots, \log \mid z_k \mid^2\Big).$$

Getting together all the above discussions, we see that $\widetilde{\mathbf{H}}_k + \Delta_k$ provides a homotopy between $\Pi_k^{(0)}$ and $\mathbf{H}_k^{(1)} + \Pi_k^{\pi}\circ i_* + \pi_{\mu_n *}\circ(\mathrm{Td}_g(\overline{T\pi})\bullet \mathbf{H}_k)$ which implies that $x - R_g(N)\cdot x = \pi_* i_*(x)$ for any element $x\in \widehat{K}_m(Y,\mu_n)_{\mathbb{Q}}$ with integer $m\geqslant 1$. □

Corollary 4.7. *Let S be another regular μ_n-projective arithmetic scheme with the trivial μ_n-action. Let $f: Y\to S$ and $l = f\circ\pi : P\to S$ be two equivariant morphisms which are smooth over the generic fibers. Then the identity*

$$f_*(x) - f_*(R_g(N)\cdot x) = l_*\circ i_*(x)$$

holds in $\widehat{K}_m(S,\mu_n)_{\mathbb{Q}}$ for any element $x\in\widehat{K}_m(Y,\mu_n)$.

Proof. This is an immediate consequence of Proposition 4.6 and Corollary 3.6. □

Now, we consider general situation. Let X, S be two regular μ_n-projective arithmetic schemes over (D,Σ,F_∞), and let Y be a regular μ_n-equivariant arithmetic closed subscheme of X with immersion $i: Y\to X$. Let $l: X\to S$ and $f = l\circ i : Y\to S$ be two equivariant morphisms which are smooth over the generic fibers. We shall suppose that the μ_n-actions on Y and on S are trivial (for example, $Y = X_{\mu_n}, S = \mathrm{Spec}D$). Then the main result in this subsection is the following.

Theorem 4.8. *For any element $x\in\widehat{K}_m(Y,\mu_n)$ with integer $m\geqslant 1$, the identity*

$$f_*(x) - f_*(R_g(N_{X/Y})\cdot x) = l_*\circ i_*(x)$$

holds in $\widehat{K}_m(S,\mu_n)_{\mathbb{Q}}$.

To prove Theorem 4.8, we use the deformation to the normal cone construction. Denote by W the blowing up of $X \times \mathbb{P}^1$ along $Y \times \{0\}$, and denote by $q_W : W \to \mathbb{P}^1$ the composition of the blow-down map $W \to X \times \mathbb{P}^1$ with the projection $X \times \mathbb{P}^1 \to \mathbb{P}^1$. For any point $t \in \mathbb{A}^1 \subset \mathbb{P}^1$, t is called a $\mathbb{Z}$-point if it corresponds to a prime ideal $(x - a)$ in $D[x]$ with $a \in \mathbb{Z}$. Then for any $\mathbb{Z}$-point $t \in \mathbb{P}^1$ we have

$$q_W^{-1}(t) \cong \begin{cases} X \times \{t\}, & \text{if } t \neq 0, \\ P \cup \widetilde{X}, & \text{if } t = 0, \end{cases}$$

where $\widetilde{X}$ is isomorphic to the blowing up of X along Y and P is the projective space bundle $\mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y)$. Let $j : Y \times \mathbb{P}^1 \to W$ be the closed immersion induced by $i \times \mathrm{Id}$, then the component $\widetilde{X}$ does not meet $j(Y \times \mathbb{P}^1)$ and the intersection of $j(Y \times \mathbb{P}^1)$ with P is exactly the image of Y under the zero section embedding. Moreover, denote by s_t the obvious section $Y \cong Y \times \{t\} \hookrightarrow Y \times \mathbb{P}^1$ for every $\mathbb{Z}$-point t and denote by u_t the natural inclusion $q_W^{-1}(t) \hookrightarrow W$. We have two Tor-independent squares

$$\begin{array}{ccc} Y \times \mathbb{P}^1 & \xrightarrow{j} & W \\ {\scriptstyle s_t}\uparrow & & \uparrow{\scriptstyle u_t} \\ Y & \xrightarrow{i} & X \end{array}$$

with $t \neq 0$ and

$$\begin{array}{ccc} Y \times \mathbb{P}^1 & \xrightarrow{j} & W \\ {\scriptstyle s_0}\uparrow & & \uparrow{\scriptstyle u_0} \\ Y & \xrightarrow{i_0} & \mathbb{P}(N_{X/Y} \oplus \mathcal{O}_Y). \end{array}$$

Note that the complement $X \setminus Y$ is contained in $W \setminus Y \times \mathbb{P}^1$, we have pullback morphism $u_t^* : \widehat{K}_{Y \times \mathbb{P}^1, m}(W, \mu_n) \to \widehat{K}_{Y,m}(X, \mu_n)$.

LEMMA 4.9. *For any $\mathbb{Z}$-point $t \neq 0$, the diagram*

$$\begin{array}{ccc} \widehat{K}_m(Y \times \mathbb{P}^1, \mu_n) & \xrightarrow[j_*]{\cong} & \widehat{K}_{Y \times \mathbb{P}^1, m}(W, \mu_n) \\ \downarrow{\scriptstyle s_t^*} & & \downarrow{\scriptstyle u_t^*} \\ \widehat{K}_m(Y, \mu_n) & \xrightarrow[i_*]{\cong} & \widehat{K}_{Y,m}(X, \mu_n) \end{array}$$

is commutative.

Proof. The commutativity of the algebraic prototype of this diagram follows from the Tor-independence of the deformation diagrams, but for arithmetic K-theory it is more complicated because the morphisms j_* and i_* are defined via another deformation to the normal cone construction according to the $\mathbb{A}^1$-homotopy invariance of the K-theory and the Deligne–Beilinson cohomology.

Write $c_t^* : \widehat{K}_m(Y \times \mathbb{P}^1, \mu_n) \to \widehat{K}_m(Y, \mu_n)$ for the composition $i_*^{-1} \circ u_t^* \circ j_*$. We need to show that $c_t^* = s_t^*$. The morphism s_t^* is induced by the commutativity between s_t^* and $\widetilde{\mathrm{ch}}_g$, while the morphism c_t^* is induced by the homotopy defining j_* and the homotopy defining i_*. Again, using the $\mathbb{A}^1$-homotopy invariance of the K-theory and the Deligne–Beilinson cohomology, we may consider the pullbacks of s_t^* and c_t^* to $\widehat{K}_m(Y \times \mathbb{P}^1 \times \mathbb{A}^1, \mu_n) \to \widehat{K}_m(Y \times \mathbb{A}^1, \mu_n)$ and restrict

them to $\{0\} \hookrightarrow \mathbb{A}^1$, then the statement in this lemma will follows from the commutativity of the diagram

$$\begin{array}{ccc} \widehat{K}_m(Y \times \mathbb{P}^1, \mu_n) & \xrightarrow[j_{0*}]{\cong} & \widehat{K}_{Y\times\mathbb{P}^1,m}(P', \mu_n) \\ \downarrow{\scriptstyle s_t^*} & & \downarrow{\scriptstyle u_t^*} \\ \widehat{K}_m(Y, \mu_n) & \xrightarrow[i_{0*}]{\cong} & \widehat{K}_{Y,m}(P, \mu_n) \end{array} \tag{13}$$

where $P' = \mathbb{P}((N_{X/Y} \boxtimes \mathcal{O}(-1)) \oplus \mathcal{O}_{Y\times\mathbb{P}^1})$ is the projective completion of $N_{W/Y\times\mathbb{P}^1}$ over $Y \times \mathbb{P}^1$. It is equivalent to show that the following diagram

$$\begin{array}{ccc} \widehat{K}_m(Y \times \mathbb{P}^1, \mu_n) & \xrightarrow{j_{0*}} & \widehat{K}_m(P', \mu_n) \\ \downarrow{\scriptstyle s_t^*} & & \downarrow{\scriptstyle u_t^*} \\ \widehat{K}_m(Y, \mu_n) & \xrightarrow{i_{0*}} & \widehat{K}_m(P, \mu_n) \end{array} \tag{14}$$

is commutative because the morphism $i_{0*} : \widehat{K}_m(Y, \mu_n) \to \widehat{K}_m(P, \mu_n)$ is injective. We endow $N_{X/Y} \boxtimes \mathcal{O}(-1)$ with the product metric coming from the metric on $N_{X/Y}$ and the Fubini–Study metric on $\mathcal{O}(-1)$, then the pullback of $\overline{N}_{W/Y\times\mathbb{P}^1}$ along s_t is isometric to $\overline{N}_{X/Y}$, so that the pullback along s_t of the Koszul resolution and of the corresponding Bott–Chern singular current with respect to j_0 is exactly the Koszul resolution and the corresponding Bott–Chern singular current with respect to i_0. According to the construction of the homotopies defining j_{0*} and i_{0*}, we get the commutativity of the diagram (14) and hence of (13). So, we are done. □

COROLLARY 4.10. *For any $\mathbb{Z}$-point $t \neq 0$, the diagram*

$$\begin{array}{ccc} \widehat{K}_m(Y \times \mathbb{P}^1, \mu_n) & \xrightarrow{j_*} & \widehat{K}_m(W, \mu_n) \\ \downarrow{\scriptstyle s_t^*} & & \downarrow{\scriptstyle u_t^*} \\ \widehat{K}_m(Y, \mu_n) & \xrightarrow{i_*} & \widehat{K}_m(X, \mu_n) \end{array}$$

is commutative.

REMARK 4.11. Using the same argument as in Lemma 4.9, we know that the diagram

$$\begin{array}{ccc} \widehat{K}_m(Y \times \mathbb{P}^1, \mu_n) & \xrightarrow{j_*} & \widehat{K}_m(W, \mu_n) \\ \downarrow{\scriptstyle s_0^*} & & \downarrow{\scriptstyle u_0^*} \\ \widehat{K}_m(Y, \mu_n) & \xrightarrow{i_{0*}} & \widehat{K}_m(P, \mu_n) \end{array}$$

is also commutative.

Next, we consider the commutative diagram

$$\begin{array}{ccc} W & & \\ \uparrow{\scriptstyle u_t} & \searrow^{l} & \\ X & \xrightarrow{f} & S \end{array}$$

with $\mathbb{Z}$-point $t \neq 0$ and we compare the map $f_* \circ u_t^*$ with the map l_* from $\widehat{K}_m(W, \mu_n)_{\mathbb{Q}}$ to $\widehat{K}_m(S, \mu_n)_{\mathbb{Q}}$.

First, for any μ_n-invariant Kähler metric ω^X on X which induces an invariant Kähler metric ω^Y on Y, there exists a μ_n-invariant Kähler metric ω^W on W such that the restrictions of ω^W over $X \cong X \times \{t\}$ with $t \neq 0$ and to $Y \cong Y \times \{0\}$ are exactly ω^X and ω^Y. This fact follows from [**32**, Lemma 3.5]. Actually, such a metric is constructed via the Grassmannian graph construction. In this construction, we have an embedding $W \to X \times \mathbb{P}^r \times \mathbb{P}^1$ and the metric ω^W is the μ_n-average of the restriction of a product metric on $X \times \mathbb{P}^r \times \mathbb{P}^1$. We fix such an invariant Kähler metric ω^W on W and endow all submanifolds of W with the induced metrics. Moreover, all normal bundles appearing in the construction of the deformation to the normal cone will be endowed with the quotient metrics.

Second, to the three divisors $u_t(X)$, $u_0(P)$ and $u_0(\widetilde{X})$ in W, we have the following result.

LEMMA 4.12. *Over W, there are μ_n-invariant hermitian metrics on $\mathcal{O}(X)$, $\mathcal{O}(P)$ and $\mathcal{O}(\widetilde{X})$ such that the isometry $\overline{\mathcal{O}}(X) \cong \overline{\mathcal{O}}(P) \otimes \overline{\mathcal{O}}(\widetilde{X})$ holds and such that the restriction of $\overline{\mathcal{O}}(X)$ over X yields the metric of $N_{W/X}$, the restriction of $\overline{\mathcal{O}}(\widetilde{X})$ over $\widetilde{X}$ yields the metric of $N_{W/\widetilde{X}}$ and the restriction of $\overline{\mathcal{O}}(P)$ over P yields the metric of $N_{W/P}$.*

Proof. choose metric on $\mathcal{O}(P)$ in a small neighborhood of P such that the restriction of $\overline{\mathcal{O}}(P)$ over P yields the metric of the normal bundle. Do the same for $\mathcal{O}(\widetilde{X})$. Since X is closed and disjoint from $\widetilde{X}$ and P, we can extend these metrics via a partition of unity to metrics defined on W, so that the restriction of the metric that $\mathcal{O}(X)$ inherits from the isomorphism $\mathcal{O}(X) \cong \mathcal{O}(P) \otimes \mathcal{O}(\widetilde{X})$ yields the metric of the normal bundle $N_{W/X}$. We then take the μ_n-averages of these metrics to make them μ_n-invariant. Since the metrics on $N_{W/X}$, $N_{W/P}$ and $N_{W/\widetilde{X}}$ are already μ_n-invariant, the μ_n-invariant metrics on $\mathcal{O}(X)$, $\mathcal{O}(P)$ and $\mathcal{O}(\widetilde{X})$ obtained as above have the properties that we require. □

Now, consider the Koszul resolution

$$0 \to \overline{\mathcal{O}}(-X) \to \overline{\mathcal{O}}_W \to u_{t*}\overline{\mathcal{O}}_X \to 0.$$

The associated equivariant singular Bott–Chern current $T_g(W/X)$ satisfies the identity

$$d_{\mathcal{D}} T_g(W/X) = \mathrm{ch}_g^0(\overline{\mathcal{O}}_W) - \mathrm{ch}_g^0\big(\overline{\mathcal{O}}(-X)\big) - u_{t*}[\mathrm{ch}_g^0(\overline{\mathcal{O}}_X)\mathrm{Td}_g^{-1}(\overline{N}_{W/X})].$$

We claim the following result.

LEMMA 4.13. *For any element $x \in \widehat{K}_m(W, \mu_n)_{\mathbb{Q}}$ with integer $m \geqslant 1$, the identity*

$$f_* \circ u_t^*(x) - f_*(R_g(N_{W/X}) \cdot u_t^* x) = l_*(x) - l_*(\overline{O}(-X) \otimes x)$$

hold in $\widehat{K}_m(S, \mu_n)_{\mathbb{Q}}$.

Proof. Let $\overline{E}$ be a l-acyclic hermitian k-cube in $\widehat{\mathcal{P}}(W, \mu_n)$. Since W admits a very ample invertible μ_n-sheaf which is relative to the morphism $l : W \to S$ (cf. [**32**, Lemma 3.9]), we may assume that $\overline{O}(-X) \otimes \overline{E}$ is also l-acyclic and $u_t^* \overline{E}$ is f-acyclic. Then we have a short exact sequence of hermitian k-cubes in $\widehat{\mathcal{P}}(S, \mu_n)$

$$\chi(\overline{E}) : \quad 0 \to l_*(\overline{\mathcal{O}}(-X) \otimes \overline{E}) \to l_*(\overline{E}) \to f_*({u_t}^* \overline{E}) \to 0,$$

which will be regarded as a hermitian $(k+1)$-cube and as a chain homotopy between the maps $l_* - l_*(\overline{\mathcal{O}}(-X)\otimes)$ and $f_* \circ u_t^*$. Consequently, the formula

$$\mathbf{H}_k^{(1)}(\overline{E}) = \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \mathrm{ch}_g^0\big(\mathrm{tr}_{k+1} \circ \lambda\big(\chi(\overline{E})\big)\big) \wedge C_{k+1}(\log \mid z_1 \mid^2, \ldots, \log \mid z_{k+1} \mid^2)$$

defines a chain homotopy between $\mathrm{ch}_g \circ l_* - \mathrm{ch}_g \circ l_*(\overline{\mathcal{O}}(-X)\otimes)$ and $\mathrm{ch}_g \circ f_* \circ u_t^*$.

On the other hand, for any element $\alpha \in \bigoplus_{p\geqslant 0} D^{2p-*}(W_{\mu_n}, p)_{R_n}$, the formula

$$\frac{1}{(2\pi i)^{r_l}} \int_{W_{\mu_n}/S} T_g(W/X) \bullet \mathrm{Td}_g(\overline{Tg}) \bullet \alpha + \frac{1}{(2\pi i)^{r_f}} \int_{X_{\mu_n}/S} \mathrm{Td}_g(\overline{\mathcal{N}}) \bullet \mathrm{Td}_g^{-1}(\overline{N}_{W/X}) \bullet \alpha$$

gives a chain homotopy between the maps $l_{\mu_n !} \circ (\mathrm{Td}_g(\overline{Tg})\bullet) - l_{\mu_n !} \circ (\mathrm{Td}_g(\overline{Tg})\mathrm{ch}_g^0(\overline{\mathcal{O}}(-X))\bullet)$ and $f_{\mu_n !} \circ (\mathrm{Td}_g(\overline{Tf}) \bullet u_t^*)$. Hence, it defines a chain homotopy between $l_{\mu_n !} \circ (\mathrm{Td}_g(\overline{Tg}) \bullet \mathrm{ch}_g) - l_{\mu_n !} \circ (\mathrm{Td}_g(\overline{Tg})\mathrm{ch}_g^0(\overline{\mathcal{O}}(-X)) \bullet \mathrm{ch}_g)$ and $f_{\mu_n !} \circ (\mathrm{Td}_g(\overline{Tf}) \bullet u_t^* \circ \mathrm{ch}_g)$. Like before, using the projection formula and the fact that the deformation to the normal cone construction is base-change invariant along smooth morphisms, we write the induced homotopy as

$$\begin{aligned}
\mathbf{H}_k^{(2)}(\overline{E}) = {} & \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \Bigg(\Bigg(\frac{1}{(2\pi i)^{r_l}} \int_{W_{\mu_n}\times(\mathbb{P}^1)^k/S\times(\mathbb{P}^1)^k} T_g(W/X) \bullet \mathrm{Td}_g(\overline{Tg}) \mathrm{ch}_g^0\big(\mathrm{tr}_k \circ \lambda(\overline{E})\big) \Bigg) \\
& \wedge C_k(\log \mid z_1 \mid^2, \ldots, \log \mid z_k \mid^2) \Bigg) \\
& + \frac{(-1)^k}{2k!(2\pi i)^k} \int_{(\mathbb{P}^1)^k} \Bigg(\Bigg(\frac{1}{(2\pi i)^{r_f}} \int_{X_{\mu_n}\times(\mathbb{P}^1)^k/S\times(\mathbb{P}^1)^k} \mathrm{Td}_g(\overline{\mathcal{N}}) \bullet \mathrm{Td}_g^{-1}(\overline{N}_{W/X}) \mathrm{ch}_g^0\big(\mathrm{tr}_k \circ \lambda(u_t^*\overline{E})\big) \Bigg) \\
& \wedge C_k(\log \mid z_1 \mid^2, \ldots, \log \mid z_k \mid^2) \Bigg).
\end{aligned}$$

Now, we denote by $H_\chi(\overline{E})$ the following 2-cube of hermitian bundles on $S \times (\mathbb{P}^1)^k$

$$\begin{array}{ccccc}
l_*\left(\mathrm{tr}_k \circ \lambda(\overline{\mathcal{O}}(-X) \otimes \overline{E})\right) & \xrightarrow{\mathrm{Id}} & \mathrm{tr}_k \circ \lambda\left(l_*(\overline{\mathcal{O}}(-X) \otimes \overline{E})\right) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
l_*\left(\mathrm{tr}_k \circ \lambda(\overline{E})\right) & \xrightarrow{\mathrm{Id}} & \mathrm{tr}_k \circ \lambda\left(l_*(\overline{E})\right) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
f_*\left(\mathrm{tr}_k \circ \lambda(u_t^*\overline{E})\right) & \xrightarrow{\mathrm{Id}} & \mathrm{tr}_k \circ \lambda\left(f_*(u_t^*\overline{E})\right) & \longrightarrow & 0
\end{array}$$

and we set

$$\widetilde{\mathbf{H}}_k(\overline{E}) := \frac{(-1)^{k+2}}{2(k+2)!(2\pi i)^{k+2}} \int_{(\mathbb{P}^1)^{k+2}} \mathrm{ch}_g^0\big(\mathrm{tr}_2 \circ \lambda\big(H_\chi(\overline{E})\big)\big) \wedge C_{k+2}(\log \mid z_1 \mid^2, \ldots, \log \mid z_{k+2} \mid^2),$$

it satisfies the differential equation

$$\begin{aligned}
& d_{\mathcal{D}} \circ \widetilde{\mathbf{H}}_k(\overline{E}) \\
& = \widetilde{\mathbf{H}}_{k-1} \circ d\overline{E} + \frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}} \int_{(\mathbb{P}^1)^{k+1}} \mathrm{ch}_g^0\big(\mathrm{tr}_{k+1} \circ \lambda\big(\chi(\overline{E})\big)\big) \\
& \quad \wedge C_{k+1}(\log \mid z_1 \mid^2, \ldots, \log \mid z_{k+1} \mid^2)
\end{aligned}$$

$$-\frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}}\int_{(\mathbb{P}^1)^{k+1}}\mathrm{ch}_g^0\big(\mathrm{tr}_1\circ\lambda\big(\chi\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big)\big)\big)\wedge C_{k+1}(\log\mid z_1\mid^2,\ldots,\log\mid z_{k+1}\mid^2)$$

$$-\Pi_k'^{l}(\overline{E})+\Pi_k'^{l}(\overline{\mathcal{O}}(-X)\otimes\overline{E})+\Pi_k'^{f}(u_t^*\overline{E})$$

$$=\widetilde{\mathbf{H}}_{k-1}\circ d\overline{E}+\mathbf{H}_k^{(1)}(\overline{E})-\Pi_k'^{l}(\overline{E})+\Pi_k'^{l}(\overline{\mathcal{O}}(-X)\otimes\overline{E})+\Pi_k'^{f}(u_t^*\overline{E})$$

$$-\frac{(-1)^{k+1}}{2(k+1)!(2\pi i)^{k+1}}\int_{(\mathbb{P}^1)^{k+1}}\mathrm{ch}_g^0\big(\mathrm{tr}_1\circ\lambda\big(\chi\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big)\big)\big)\wedge C_{k+1}(\log\mid z_1\mid^2,\ldots,\log\mid z_{k+1}\mid^2).$$

Similar to the tricks that we used frequently before, we set

$$\mathbf{H}_k^{(2')}(\overline{E})=\frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k}\int_{(\mathbb{P}^1)^k}C_{k+1}\Bigg(\frac{1}{(2\pi i)^{r_l}}\int_{W_{\mu_n}\times(\mathbb{P}^1)^k/S\times(\mathbb{P}^1)^k}T_g(W/X)$$

$$\bullet\mathrm{Td}_g(\overline{Tg})\mathrm{ch}_g^0\big(\mathrm{tr}_k\circ\lambda(\overline{E})\big),\log\mid z_1\mid^2,\ldots,\log\mid z_k\mid^2\Bigg)$$

$$+\frac{(-1)^{k+1}}{(k+1)!(2\pi i)^k}\int_{(\mathbb{P}^1)^k}C_{k+1}\Bigg(\frac{1}{(2\pi i)^{r_f}}\int_{X_{\mu_n}\times(\mathbb{P}^1)^k/S\times(\mathbb{P}^1)^k}\mathrm{Td}_g(\overline{\mathcal{N}})$$

$$\bullet\mathrm{Td}_g^{-1}(\overline{N}_{W/X})\mathrm{ch}_g^0\big(\mathrm{tr}_k\circ\lambda(u_t^*\overline{E})\big),\log\mid z_1\mid^2,\ldots,\log\mid z_k\mid^2\Bigg).$$

then our lemma follows from the Bimut–Ma immersion formula and the fact that there exists a homotopy between $\mathbf{H}_k^{(2')}(\overline{E})$ and $\mathbf{H}_k^{(2)}(\overline{E})$. So, we are done. □

Remark 4.14. Similar to Lemma 4.13, we consider other three divisors $W\xleftarrow[u_0]{}P\xrightarrow{p}S$, $W\xleftarrow[u_0]{}\widetilde{X}\xrightarrow{h_1}S$ and $W\xleftarrow[u_0]{}P\cap\widetilde{X}\xrightarrow{h_2}S$ and corresponding Koszul resolutions

$$0\to\overline{\mathcal{O}}(-P)\to\overline{\mathcal{O}}_W\to u_{0*}\overline{\mathcal{O}}_P\to 0,$$

$$0\to\overline{\mathcal{O}}(-\widetilde{X})\to\overline{\mathcal{O}}_W\to u_{0*}\overline{\mathcal{O}}_{\widetilde{X}}\to 0,$$

and

$$0\to\overline{\mathcal{O}}(-\widetilde{X})\otimes\overline{\mathcal{O}}(-P)\to\overline{\mathcal{O}}(-\widetilde{X})\oplus\overline{\mathcal{O}}(-P)\to\overline{\mathcal{O}}_W\to u_{0*}\overline{\mathcal{O}}_{\widetilde{X}\cap P}\to 0.$$

Then, for any element $x\in\widehat{K}_m(W,\mu_n)_{\mathbb{Q}}$, we have

$$p_*\circ u_0^*(x)-p_*(R_g(N_{W/P})\cdot u_0^*x)=l_*(x)-l_*\big(\overline{\mathcal{O}}(-P)\otimes x\big),$$

$$h_{1*}\circ u_0^*(x)-h_{1*}(R_g(N_{W/\widetilde{X}})\cdot u_0^*x)=l_*(x)-l_*\Big(\overline{\mathcal{O}}(-\widetilde{X})\otimes x\Big),$$

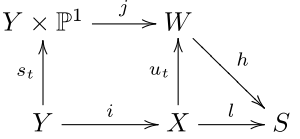
and

$$h_{2*}\circ u_0^*(x)-h_{2*}(R_g(N_{W/P\cap\widetilde{X}})\cdot u_0^*x)=l_*(x)-l_*(\overline{\mathcal{O}}(-P)\otimes x)-l_*(\overline{\mathcal{O}}(-\widetilde{X})\otimes x)$$
$$+l_*(\overline{\mathcal{O}}(-P)\otimes\overline{\mathcal{O}}(-\widetilde{X})\otimes x)$$

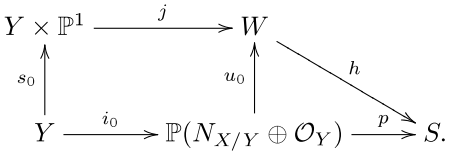
which hold in $\widehat{K}_m(S,\mu_n)_{\mathbb{Q}}$.

Now, we are ready to give the proof of Theorem 4.8.

Proof. Let x be an element in $\widehat{K}_m(Y,\mu_n)_{\mathbb{Q}}$, we consider the following two diagrams



with $\mathbb{Z}$-point $t\neq 0$ and



By Corollary 4.10 and the fact that s_t is a section of the obvious projection Pr from $Y\times\mathbb{P}^1$ to Y, we have $i_*(x)=u_t^*\circ j_*\circ Pr^*(x)$ and hence $l_*\circ i_*(x)=l_*\circ u_t^*\circ j_*\circ Pr^*(x)$. According to Lemma 4.13,

$$l_*\circ u_t^*(j_*Pr^*x)=h_*(j_*Pr^*x)-h_*(\overline{O}(-X)\otimes j_*Pr^*x)+l_*(R_g(N_{W/X})\cdot i_*x).$$

Similarly, we have

$$l_*\circ u_0^*(j_*Pr^*x)=h_*(j_*Pr^*x)-h_*(\overline{O}(-P)\otimes j_*Pr^*x)+p_*(R_g(N_{W/P})\cdot i_*x).$$

Note that the image $j(Y\times\mathbb{P}^1)$ does not meet $\widetilde{X}$, the localization sequence of the higher equivariant arithmetic K-groups implies that $u_0^*(j_*Pr^*x)$ vanishes in $\widehat{K}_m(\widetilde{X},\mu_n)_{\mathbb{Q}}$ and in $\widehat{K}_m(P\cap\widetilde{X},\mu_n)_{\mathbb{Q}}$, so that

$$h_*(\overline{O}(-X)\otimes j_*Pr^*x)=h_*(\overline{O}(-P)\otimes j_*Pr^*x).$$

This can be seen from the several identities mentioned in Remark 4.14. On the other hand,

$$\begin{aligned}R_g(N_{W/X})\cdot i_*x&=R_g(N_{W/X})\mathrm{ch}_g(i_*x)=R_g(N_{W/X})i_*\big(\mathrm{Td}_g^{-1}(N_{X/Y})\mathrm{ch}_g(x)\big)\\&=i_*\big(i^*R_g(N_{W/X})\mathrm{Td}_g^{-1}(N_{X/Y})\mathrm{ch}_g(x)\big)=0.\end{aligned}$$

The same reasoning gives that $R_g(N_{W/P})\cdot i_*x=0$ also. So, $l_*\circ i_*(x)$ is actually equal to $p_*\circ i_{0*}(x)$. Therefore, the statement in Theorem 4.8 follows from Corollary 4.7. □

4.3. *Proof of the statement*

In this subsection, we give a complete proof of Theorem 4.2. Denote by i the closed immersion $X_{\mu_n}\to X$, then the arithmetic concentration theorem (cf. [**34**, Theorem 5.2]) tells us that

$$i_*:\widehat{K}_m(X_{\mu_n},\mu_n)_\rho\cong\widehat{K}_m(X,\mu_n)_\rho$$

with inverse map $\otimes\lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}})\circ\tau$.

Then let x be any element in $\widehat{K}_m(X,\mu_n)$, we apply Theorem 4.8 to the morphisms i, f and $f_{\mu_n}=f\circ i$ and we compute

$$\begin{aligned}f_*(x)=&f_*\big(i_*\circ\otimes\lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}})\circ\tau(x)\big)\\=&f_*\circ i_*\big(\otimes\lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}})\circ\tau(x)\big)\\=&f_{\mu_n*}\big(\otimes\lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}})\circ\tau(x)\big)-f_{\mu_n*}\big(\otimes R_g(N_{X/X_{\mu_n}})\circ\otimes\lambda_{-1}^{-1}(\overline{N}_{X/X_{\mu_n}})\circ\tau(x)\big)\\=&f_{\mu_n*}(\Lambda_R\circ\tau(x))\end{aligned}$$

which holds in $\widehat{K}_m(Y,\mu_n)_\rho\otimes\mathbb{Q}$. This completes the proof of Theorem 4.2.

Appendix. *Remarks on the equivariant analytic torsion forms and the immersion formula*

by Xiaonan Ma

A.1. *Anomaly formula for the equivariant analytic torsion forms*

Let W, V be two μ_n-projective complex manifolds, and let $f : W \to V$ be an equivariant, holomorphic submersion with fiber X. Fix a μ_n-invariant Kähler metric on W and choose corresponding Kähler form ω as a Kähler fibration structure on f. We fix a primitive nth root of unity g as a generator of $\mu_n(\mathbb{C})$. In the following, ch_g and Td_g should stand for the usual Chern–Weil forms with the factor $2\pi i$ in their definitions. Note that they are denoted by ch'_g and Td'_g in the text.

Let (E, h^E) be a μ_n-equivariant hermitian vector bundle on W such that E is f-acyclic. Let $T_g(f, \omega, h^E) \in \bigoplus_{p\geqslant 0} A^{p,p}(V_{\mu_n})$ be the equivariant analytic torsion form [27, (2.27)] which satisfies the differential equation

$$\frac{\bar\partial\partial}{2\pi i} T_g(f, \omega, h^E) = \mathrm{ch}_g(f_*E, f_*h^E) - \int_{W_{\mu_n}/V_{\mu_n}} \mathrm{Td}_g(Tf, h^{Tf})\mathrm{ch}_g(E, h^E),$$

where h^{Tf} is the hermitian metric induced by ω on the holomorphic tangent bundle Tf. We shall write $T_g(\omega, h^E)$ for $T_g(f, \omega, h^E)$, if there is no ambiguity about the underlying map. The following result is [27, Theorem 2.13] which extends [7, Theorem 2.5; 11, Theorem 1.23; 12, Theorem 3.10].

THEOREM A.1 (Anomaly formula). *Let ω' be the form associated to another Kähler fibration structure on $f : W \to V$. Let h'^{Tf} be the metric on Tf induced by ω'. Then the following identity holds in $\bigoplus_{p\geqslant 0} A^{p,p}(V_{\mu_n})/(\mathrm{Im}\,\partial + \mathrm{Im}\,\bar\partial)$:*

$$\begin{aligned} T_g(\omega, h^E) - T_g(\omega', h^E) = &- \widetilde{\mathrm{ch}}_g(f_*E, h^{f_*E}, h'^{f_*E}) \\ &+ \int_{W_{\mu_n}/V_{\mu_n}} \widetilde{\mathrm{Td}}_g(Tf, h^{Tf}, h'^{Tf})\mathrm{ch}_g(E, h^E), \end{aligned}$$

where $(f_*E, h^{f_*E}, h'^{f_*E})$ and (Tf, h^{Tf}, h'^{Tf}) stand for the exact sequences of hermitian vector bundles

$$0 \longrightarrow (f_*E, h^{f_*E}) \xrightarrow{\mathrm{Id}} (f_*E, h'^{f_*E}) \longrightarrow 0 \longrightarrow 0$$

and

$$0 \longrightarrow (Tf, h^{Tf}) \xrightarrow{\mathrm{Id}} (Tf, h'^{Tf}) \longrightarrow 0 \longrightarrow 0\,.$$

We shall see that there is a natural way to write down explicitly some differential forms $\Delta^0(f, \overline{E}, \omega, \omega')$, $\Delta_0(f, \overline{E}, \omega, \omega')$ such that they are functorial in certain sense and they measure the difference of the anomaly formula.

$$\begin{aligned} \Delta &= \partial\Delta^0(f, \overline{E}, \omega, \omega') + \bar\partial\Delta_0(f, \overline{E}, \omega, \omega') \\ &= T_g(\omega, h^E) - T_g(\omega', h^E) + \widetilde{\mathrm{ch}}_g(f_*E, h^{f_*E}, h'^{f_*E}) - \int_{W_{\mu_n}/V_{\mu_n}} \widetilde{\mathrm{Td}}_g(Tf, h^{Tf}, h'^{Tf})\mathrm{ch}_g(E, h^E). \end{aligned}$$

To do so, we need to fix the construction of $\widetilde{\mathrm{ch}}_g(f_*E, h^{f_*E}, h'^{f_*E})$, $\widetilde{\mathrm{Td}}_g(Tf, h^{Tf}, h'^{Tf})$ at the differential form level, that is., without modulo $\mathrm{Im}\,\partial + \mathrm{Im}\,\bar\partial$. Let us fix the definition of $\widetilde{\mathrm{ch}}_g(f_*E, h^{f_*E}, h'^{f_*E})$ as the left side of [7, (2.42)], and the definition $\widetilde{\mathrm{Td}}_g(Tf, h^{Tf}, h'^{Tf})$ as the integral for 0 to 1 for the parameter c of the differential form as the part $\frac{\partial}{\partial b}\cdots$ via the last

term of [12, (3.67)], note that we can also fix the path of the metric as the segment direct connecting two metrics. Thus, we can write them under the notation in [27, (2.34), (2.56)] (cf. also the convention before Theorem 2.5 of this paper),

$$\widetilde{\mathrm{ch}}_g(f_*E, h^{f_*E}, h'^{f_*E}) = \int_0^1 \Phi \mathrm{Tr}_s[gQ_c^{H(X,E|_X)} \exp(-(\nabla_c^{H(X,E|_X)})^2)]dc,$$

$$\widetilde{\mathrm{Td}}_g(Tf, h^{Tf}, h'^{Tf}) = \int_0^1 \frac{\partial}{\partial b}\left[\mathrm{Td}\left(\frac{-R_c^{TX^g}}{2i\pi} - b(h^{TX^g})^{-1}\frac{\partial h^{TX^g}}{\partial c}\right)\right. \tag{A.1}$$

$$\left.\times \prod_{j=1}^{q} \frac{\mathrm{Td}}{e}\left(\frac{-R_c^{N_{X^g/X}^{\theta_j}}}{2i\pi} - b(h^{N_{X^g/X}^{\theta_j}})^{-1}\frac{\partial h^{N_{X^g/X}^{\theta_j}}}{\partial c} + i\theta_j\right)\right]_{b=0} dc.$$

Let V_1 be an equivariant closed submanifold of V, and let $W_1 = f^{-1}(V_1) \subset W$ be the closed submanifold of W with restricted Kähler metric. Then $f_1 : W_1 \to V_1$ is also an equivariant holomorphic submersion with compact fiber. Denote by j (respectively, i) the natural embedding $W_1 \to W$ (respectively, $V_1 \to V$) and by ω_1, ω_1' the induced Kähler forms $j^*\omega, j^*\omega'$. Let $\overline{E}$ be an f-acyclic hermitian bundle on W.

THEOREM A.2. *There is a natural way to write down explicitly differential forms* $\Delta^0(f, \overline{E}, \omega, \omega')$, $\Delta_0(f, \overline{E}, \omega, \omega')$ *such that* $\Delta = \partial\Delta^0(f, \overline{E}, \omega, \omega') + \bar{\partial}\Delta_0(f, \overline{E}, \omega, \omega')$ *and they are functorial in the following sense.*

$$i_{\mu_n}^* \Delta^0(f, \overline{E}, \omega, \omega') = \Delta^0(f_1, j^*\overline{E}, \omega_1, \omega_1') \tag{A.2}$$

and

$$i_{\mu_n}^* \Delta_0(f, \overline{E}, \omega, \omega') = \Delta_0(f_1, j^*\overline{E}, \omega_1, \omega_1'). \tag{A.3}$$

Proof. By the equivariant extension of [12, Definition 3.14, Theorems 3.16 and 3.17] (cf. [27, (2.34)]), there exist differential forms θ^1, θ^2 and θ^3 such that

$$\Delta + d\varrho = \bar{\partial}\theta^1 + \partial\theta^2 + \bar{\partial}\partial\theta^3 \tag{A.4}$$

and $d\varrho$ is from the last term of [12, (3.38)], in particular, ϱ is a local term from the small time heat kernel asymptotics of Bismut superconnection, θ^k $(k = 1, 2, 3)$ have universal expression in terms of g, ω, ω' and h^E via the Bismut superconnection. Thus, from [12, Definition 3.14, Theorem 3.16]] and [27, (2.34)], we know that if $i : V_1 \to V$ is a complex submanifold of V, when we consider the corresponding objects for the submersion f_1, each above term is the restriction of the corresponding term for the global submersion f. Thus, let Δ_1, θ_1^k be the corresponding terms associated to the fibration $f_1 : W_1 \to V_1$, then we have $\Delta_1 = i_{\mu_n}^*\Delta$ and $\theta_1^k = i_{\mu_n}^*\theta^k$ $(k = 1, 2, 3)$, $\varrho_1 = i_{\mu_n}^*\varrho$. So, write $\Delta^0(f, \overline{E}, \omega, \omega') = \theta^2 - \varrho$ and $\Delta_0(f, \overline{E}, \omega, \omega') = \theta^1 + \partial\theta^3 - \varrho$, we are done. □

A.2. *Functoriality of the equivariant analytic torsion forms*

Let W, V and S be three μ_n-equivariant projective complex manifolds with $S = S_{\mu_n}$. Suppose that $f : W \to V$ and $h : V \to S$ are two holomorphic submersions with compact fibers X, Y. Then $h \circ f$ is also a holomorphic submersion with compact fiber Z.

Let ω^W and ω^V be two μ_n-invariant Kähler forms on W and on V. As usual, ω^W and ω^V decide Kähler fibration structures on the morphisms f, h and g and they induce μ_n-invariant hermitian metrics associated with the Kähler forms ω^X, ω^Y and ω^Z on relative tangent

bundles Tf, Th and $T(h\circ f)$. Consider the following short exact sequence of hermitian vector bundles

$$\overline{T}(f,h,h\circ f):\quad 0\to\overline{Tf}\to\overline{T(h\circ f)}\to f^*\overline{Th}\to 0.$$

Denote by $\mathrm{Td}_g(\overline{T}(f,h,h\circ f))$ the equivariant secondary Todd form, it satisfies the differential equation

$$\frac{\bar\partial\partial}{2\pi i}\mathrm{Td}_g(\overline{T}(f,h,h\circ f))=\mathrm{Td}_g(\overline{T(h\circ f)})-f^*_{\mu_n}\mathrm{Td}_g(\overline{Th})\mathrm{Td}_g(\overline{Tf}).$$

Now, let $\overline{E}$ be a hermitian vector bundle on W, we shall assume that E is f-acyclic and $h\circ f$-acyclic. Then the Leray spectral sequence $E_2^{i,j}=R^ih_*(R^jf_*E)$ degenerates at E_2, so that $f_*E=R^0f_*(E)$ is h-acyclic and $(h\circ f)_*E\cong h_*f_*E$. Clearly, $(h\circ f)_*E$ and h_*f_*E carry in general different L^2 metrics (Note that for $\sigma\in((h\circ f)_*E)_b$, $b\in S$,

$$\begin{aligned}\|\sigma\|^2_{(h\circ f)_*E}&=(2\pi)^{-\dim Z}\int_{Z_b}|\sigma|^2\frac{(\omega^Z)^{\dim Z}}{(\dim Z)!},\\ \|\sigma\|^2_{h_*f_*E}&=(2\pi)^{-\dim Z}\int_{Y_b}\left(\int_X|\sigma|^2\frac{(\omega^X)^{\dim X}}{(\dim X)!}\right)\frac{(\omega^Y)^{\dim Y}}{(\dim Y)!},\end{aligned}\tag{A.5}$$

thus they are different in general). Consider the following short exact sequence of hermitian vector bundles

$$\overline{E}(f,h,h\circ f):\quad 0\to h_*f_*\overline{E}\to(h\circ f)_*\overline{E}\to 0\to 0.$$

The equivariant secondary Bott–Chern form $\widetilde{\mathrm{ch}}_g(\overline{E}(f,h,h\circ f))$ satisfies the differential equation

$$\frac{\bar\partial\partial}{2\pi i}\widetilde{\mathrm{ch}}_g(\overline{E}(f,h,h\circ f))=\mathrm{ch}_g((h\circ f)_*\overline{E})-\mathrm{ch}_g(h_*f_*\overline{E}).$$

THEOREM A.3. *Let notations and assumptions be as above. Then the following identity holds in* $\bigoplus_{p\geqslant 0}A^{p,p}(S)/(\mathrm{Im}\,\partial+\mathrm{Im}\,\bar\partial)$*:*

$$\begin{aligned}&T_g(h\circ f,\omega^W,h^E)-T_g(h,\omega^V,h^{f_*E})-\int_{V_{\mu_n}/S}\mathrm{Td}_g(\overline{Th})T_g(f,\omega^W,h^E)\\&\quad=\widetilde{\mathrm{ch}}_g(\overline{E}(f,h,h\circ f))-\int_{W_{\mu_n}/S}\widetilde{\mathrm{Td}}_g(\overline{T}(f,h,h\circ f))\mathrm{ch}_g(\overline{E}).\end{aligned}\tag{A.6}$$

Proof (a sketch). This is a natural extension of [**26**, Theorem 3.5] to the equivariant case, or the family extension of [**27**, Theorem 3.1] which is an equivariant extension of [**4**, Theorem 3.1]. To prove this extension, one may follow the same approach as in [**26**, Sections 4–9]. In fact, as a purely functional analysis argument, the [**26**, Theorems 4.5–4.7] can be extended formally to the equivariant case by introducing in the right place the operator g. The reason one can do this formal extension has been given in [**27**, Section 5]. For the equivariant extensions of [**26**, Theorems 4.8–4.11], one can show that their proofs are local on $f^{-1}(V_{\mu_n})$ and certain rescaling on Clifford variables which does not effect the action of g can be made (cf. [**26**, Section 7 b]). Replacing the equivariant local index technique in [**27**, Sections 7–9] by its equivariant relative local index, one gets the desired identity.

To help the readers, we will use directly the notation in [**26**, Section 4]. By the anomaly formula Theorem 2.11, we only need to establish Theorem A.3 for a special coupe of Kähler forms, thus we will assume that $\omega^W=\widetilde{\omega}^W+f^*\omega^V$ with $\widetilde{\omega}^W$ a Kähler form on W.

Let Δ be the rectangular domain in $\mathbb{R}^2$ with coordinates (u,T), defined by the four vertices $(1,\varepsilon),(T_0,\varepsilon),(T_0,A),(1,A)$, following [26, (4.7)], set

$$\theta_1^0=(2\pi i)^{-1/2}\int_\Delta \frac{2}{u}\frac{\partial}{\partial b}\big\{\varphi\mathrm{Tr}_s\big[g\big[B'_{3,u^2,T},N_{3,u^2,T}\big]\exp(-B^2_{3,u^2,T}-bM_{3,u^2,T})\big]\big\}_{b=0}dudT,$$

$$\theta_2^0=(2\pi i)^{-1/2}\int_\Delta \frac{2}{u}\frac{\partial}{\partial b}\big\{\varphi\mathrm{Tr}_s\big[g\big[B''_{3,u^2,T},N_{3,u^2,T}\big]\exp(-B^2_{3,u^2,T}-bM_{3,u^2,T})\big]\big\}_{b=0}dudT, \quad \text{(A.7)}$$

$$\theta_3^0=(2\pi i)^{-1}\int_\Delta \frac{2}{u}\frac{\partial}{\partial b}\big\{\varphi\mathrm{Tr}_s\big[gN_{3,u^2,T}\exp(-B^2_{3,u^2,T}-bM_{3,u^2,T})\big]\big\}_{b=0}dudT.$$

The only difference comparing with [26, (4.7)] is that in (A.7), we add the operator g as the first term in $\mathrm{Tr}_s[\cdots]$ in [26, (4.7)], that is, replace $\mathrm{Tr}_s[\cdots]$ by $\mathrm{Tr}_s[g\cdots]$. Note that $B_{3,u^2,T}$ is the Bismut superconnection assocaited with the submersion $h\circ f$ and the form $\omega_T^W=\frac{1}{T^2}\widetilde{\omega}^W+f^*\omega^V$, and $B'_{3,u^2,T},B''_{3,u^2,T}$ are holomorphic and anti-holomorphic part of $B_{3,u^2,T}$. Moreover, $N_{3,u^2,T}$ is a generalized number operator associated with ω_T^W.

The boundary of Δ composes as four oriented segments $\Gamma_1,\ldots,\Gamma_4$. Let I_k^0 be the integral of the one form on $\mathbb{R}^2$ with values in $\Lambda^\bullet(T^*_{\mathbb{R}}S)$ defined by replacing $\mathrm{Tr}_s[\cdots]$ by $\mathrm{Tr}_s[g\cdots]$ in [26, Definition 4.2], then we have the g-analogue of [26, (4.8)]:

$$\sum_{k=1}^4 I_k^0=\overline{\partial}\theta_1^0-\partial\theta_2^0-\overline{\partial}\partial\theta_3^0. \quad \text{(A.8)}$$

We study the terms I_k^0 and θ_j^0 in succession as $A\to+\infty$, $T_0\to+\infty$, $\varepsilon\to 0$: roughly, we get

- the term $-T_g(h,\omega^V,h^{f_*E})$ from I_1^0;
- a differential form version of $-\widetilde{\mathrm{ch}}_g(\overline{E}(f,h,h\circ f))$ (via [9, (1.58)] or [26, (4.17)] by replacing $\mathrm{Tr}_s[\cdots]$ by $\mathrm{Tr}_s[g\cdots]$) from I_2^0;
- $T_g(h\circ f,\omega^W,h^E)$ from I_3^0;
- $-\int_{V_{\mu_n}/S}\mathrm{Td}_g(\overline{Th})T_g(f,\omega^W,h^E)+\int_{W_{\mu_n}/S}\widetilde{\mathrm{Td}}_g(\overline{T}(f,h,h\circ f))\mathrm{ch}_g(\overline{E})$ (here we should use the differential form version of $\widetilde{\mathrm{Td}}_g(\overline{T}(f,h,h\circ f))$ from the term $\int_1^\infty\cdots$ in [4, (4.72)] by replacing Td therein by Td_g) from I_4^0.

Let $\theta_j^3(j=1,2,3)$ be the differential forms on S obtained from θ_j^0 by the above procedure, then the difference of two sides in (A.6) (by using the differential form versions of $\widetilde{\mathrm{ch}}_g(\overline{E}(f,h,h\circ f))$ and $\widetilde{\mathrm{Td}}_g(\overline{T}(f,h,h\circ f))$ as above) is

$$\Delta+d^S\Theta=\bar\partial\theta_1^3-\partial\theta_2^3-\bar\partial\partial\theta_3^3 \quad \text{(A.9)}$$

and θ_k^3 $(k=1,2,3)$ have universal expressions via the Bismut superconnection $B_{3,u^2,T}$, Θ is a combination of local terms from the small time heat kernel asymptotics of the Bismut superconnection for the fibration h and $h\circ f$ (cf. [26, (4.27) and (4.29); 27, (2.24) and (2.27)]). □

Let θ_k^3 $(k=1,2,3)$ be the form in (A.9) associated with the couple $\omega^M=\widetilde{\omega}^W+f^*\omega^V,\omega^V$. Set

$$\Delta^0(f,h,\omega^W,\omega^V,\overline{E})=-\theta_2^3-\Theta \quad \text{and } \Delta_0(f,h,\omega^W,\omega^V,\overline{E})=\theta_1^3-\partial\theta_3^3-\Theta. \quad \text{(A.10)}$$

Then when we fix the differential form versions of $\widetilde{\mathrm{ch}}_g(\overline{E}(f,h,h\circ f))$ and $\widetilde{\mathrm{Td}}_g(\overline{T}(f,h,h\circ f))$ as above, (A.10) measure the difference of the formula (A.6) at the differential form level from (A.9):

$$\begin{aligned}\Delta &= \partial\Delta^0(f,h,\omega^W,\omega^V,\overline{E}) + \bar\partial\Delta_0(f,h,\omega^W,\omega^V,\overline{E})\\ &= T_g(h\circ f,\omega^W,h^E) - T_g(h,\omega^V,h^{f_*E}) - \int_{V_{\mu_n}/S}\mathrm{Td}_g(\overline{Th})T_g(f,\omega^W,h^E)\\ &\quad -\widetilde{\mathrm{ch}}_g(\overline{E}(f,h,h\circ f)) + \int_{W_{\mu_n}/S}\widetilde{\mathrm{Td}}_g(\overline{T}(f,h,h\circ f))\mathrm{ch}_g(\overline{E}).\end{aligned} \tag{A.11}$$

Let S_1 be a closed submanifold of S, and let $V_1 = h^{-1}(S_1)\subset V$ (respectively, $W_1 = (h\circ f)^{-1}(S_1)\subset W$) be the closed submanifold of V (respectively, W) with restricted Kähler metric. Then $f_1 : W_1\to V_1$, $h_1 : V_1\to S_1$ and $h_1\circ f_1 : W_1\to S_1$ also form a triple of equivariant holomorphic submersions with compact fibers. Denote by j (respectively, i) the natural embedding $W_1\to W$ (respectively, $V_1\to V$) and by $\omega^{W_1},\omega^{V_1}$ the induced Kähler forms $j^*\omega^W, i^*\omega^V$. Denote by l the embedding $S_1\to S$. Let $\overline{E}$ be an f-acyclic and $h\circ f$-acyclic hermitian bundle on W.

THEOREM A.4. *The forms $\Delta^0(f,h,\omega^W,\omega^V,\overline{E})$ and $\Delta_0(f,h,\omega^W,\omega^V,\overline{E})$ are functorial in the following sense that*

$$l^*\Delta^0(f,h,\omega^W,\omega^V,\overline{E}) = \Delta^0(f_1,h_1,\omega^{W_1},\omega^{V_1},j^*\overline{E})$$

and

$$l^*\Delta_0(f,h,\omega^W,\omega^V,\overline{E}) = \Delta_0(f_1,h_1,\omega^{W_1},\omega^{V_1},j^*\overline{E}).$$

Proof. Note that the square of the Bismut superconnection is a second-order fiberwise elliptic operator with differential form coefficients [**5**, Theorem 3.6] (cf. also [**3**, Theorem 10.17]), in particular, its heat kernel along the fibers is well defined, and in (A.7), the terms $[B'_{3,u^2,T},N_{3,u^2,T}]$, $[B''_{3,u^2,T},N_{3,u^2,T}]$ are first-order differential operators along the fiber, the terms $N_{3,u^2,T}$, $M_{3,u^2,T}$ are tensors, thus we see clearly that when we consider the corresponding objects for the submersion $h_1\circ f_1$, each above term is the restriction of the corresponding term for the global submersion $h\circ f$.

We obtain that if $l : S_1\hookrightarrow S$ is a complex submanifold of S, and $\theta^0_{k,1}$, Θ_1 are the corresponding terms associated to the relevant fibrations, then we have

$$\Theta_1 = l^*\Theta,\quad \theta^0_{k,1} = l^*\theta^0_k \qquad (k=1,2,3). \tag{A.12}$$

Now, we make the procedure as $A\to+\infty$, $T_0\to+\infty$, $\varepsilon\to 0$, to get $\theta^3_{k,1}$, then from (A.12), we get $\theta^3_{k,1} = l^*\theta^3_k$ $(k=1,2,3)$. Combining it with (A.10), we get Theorem A.4. □

Now, for a general ω^W, as we can use the anomaly formula for the trip $(h\circ f,\omega^W,\omega^W + f^*\omega^V)$, in particular, its differential form version as in Section A.1, we can still define $\Delta^0(f,h,\omega^W,\omega^V,\overline{E})$ and $\Delta_0(f,h,\omega^W,\omega^V,\overline{E})$ such that Theorem A.4 and (A.11) still hold, again we need to fix a differential form version of $\widetilde{\mathrm{Td}}_g(\overline{T}(f,h,h\circ f))$.

A.3. *Immersion formula*

Let V,W be two μ_n-equivariant projective complex manifolds and let $i : W\hookrightarrow V$ be an equivariant closed immersion. Let S be a compact complex manifold with trivial μ_n-action, and let $f : W\to S$, $l : V\to S$ be two equivariant holomorphic submersions with fibers Y,X such that $f = l\circ i$. Assume that $\overline{\eta}$ is an equivariant hermitian bundle on W and $(\overline{\xi}.,v)$ is a

complex of equivariant hermitian bundles on V which provides a resolution of $i_*\overline{\eta}$ such that the metrics on $\xi.$ satisfy the Bismut's assumption (A). Let ω^W, ω^V be two Kähler fibrations on f and on l, respectively. We shall assume that ω^W is the pullback of ω^V, so that the Kähler metric on W is induced by the Kähler metric on V. Consider the following exact sequence

$$\overline{\mathcal{N}}: \quad 0 \to \overline{Tf} \to \overline{Tl}\,|_W \to \overline{N}_{X/Y} \to 0,$$

where $N_{X/Y}$ is endowed with the quotient metric. Then the equivariant secondary Todd form of $\overline{\mathcal{N}}$ satisfies the identity

$$\frac{\bar\partial\partial}{2\pi i}\widetilde{\mathrm{Td}}_g(\overline{\mathcal{N}}) = \mathrm{Td}_g(Tl\,|_W, h^{Tl}) - \mathrm{Td}_g(Tf, h^{Tf})\mathrm{Td}_g(\overline{N}_{X/Y}).$$

We suppose that in the resolution $\xi.$, ξ_j are all $l-$acyclic and moreover η is $f-$acyclic.

Let $T_g(\omega^V, h^\xi)$ be the equivariant analytic torsion forms associated with the family of relative Dolbeault double complexes $(\Omega(X, \xi|_X), \overline{\partial}^X + v)$. Let $h^{H(X,\xi|_X)}$ be the corresponding L_2 metric on the hypercohomology $H(X, \xi|_X)$ of $\xi|_X$.

Note that under our assumption, $H(X, \xi|_X) \simeq f_*\eta$. And we have the following exact sequence of hermitian vector bundles on S

$$\overline{\Xi}: \quad 0 \to l_*(\overline{\xi}_m) \to l_*(\overline{\xi}_{m-1}) \to \ldots \to l_*(\overline{\xi}_0) \to \overline{H(X, \xi|_X)} \to 0.$$

We can split $\overline{\Xi}.$ into a family of short exact sequence of hermitian bundles from $j = 1$ to m

$$\chi_j: \quad 0 \longrightarrow \mathrm{Ker}\, d_j \longrightarrow \overline{\Xi}_j \xrightarrow{d_j} \mathrm{Ker}\, d_{j-1} \longrightarrow 0$$

such that the kernel of every map d_{j-1} for $j = 2, \ldots, m$ carries the metric induced by $\overline{\Xi}_j$ and $\mathrm{Ker}\, d_0 = \overline{\Xi}_0 = \overline{H(X, \xi|_X)}, \mathrm{Ker}\, d_m = \overline{\Xi}_{m+1} = l_*(\overline{\xi}_m)$. We set $\widetilde{\mathrm{ch}}_g(\overline{\Xi}.) = \sum_{j=0}^{m+1}(-1)^j \widetilde{\mathrm{ch}}_g(\chi_j)$. Then it satisfies the differential equation

$$\frac{\bar\partial\partial}{2\pi i}\widetilde{\mathrm{ch}}_g(\overline{\Xi}.) = \mathrm{ch}_g(\overline{H(X, \xi|_X)}) - \sum_{j=0}^{m}(-1)^j \mathrm{ch}_g(l_*(\overline{\xi}_j)).$$

The following result is the combination of [14, Theorems 0.1 and 0.2] which is an equivariant extension of [8, Theorems 0.1 and 0.2], and a families extension of [7, Theorem 0.1; 13, Theorem 0.1].

Let R_g be the equivariant R-genus of Bismut [6].

THEOREM A.5 (Immersion formula). *The following identity holds in* $\bigoplus_{p\geqslant 0} A^{p,p}(S)/(\mathrm{Im}\,\partial + \mathrm{Im}\,\bar\partial)$.

$$T_g(\omega^V, h^\xi) - T_g(\omega^W, h^\eta) + \widetilde{\mathrm{ch}}_g(f_*\eta, h^{H(X,\xi|_X)}, h^{f_*\eta}) = \int_{V_{\mu_n}/S} \mathrm{Td}_g(\overline{Tl}) T_g(\overline{\xi}.)$$

$$- \int_{W_{\mu_n}/S} \widetilde{\mathrm{Td}}_g(\overline{\mathcal{N}})\mathrm{Td}_g^{-1}(\overline{N}_{X/Y})\mathrm{ch}_g(\overline{\eta}) + \int_{W_{\mu_n}/S} \mathrm{Td}_g(\overline{Tf}) R_g(\overline{N}_{X/Y})\mathrm{ch}_g(\overline{\eta}), \tag{A.13}$$

$$T_g(\omega^V, h^\xi) - \sum_{i=0}^{m}(-1)^i T_g(\omega^V, h^{\xi_i}) - \widetilde{\mathrm{ch}}_g(\overline{\Xi}.) = 0. \tag{A.14}$$

Again to understand (A.13) at the differential form level, that is, without modulo $\mathrm{Im}\,\partial + \mathrm{Im}\,\bar\partial$, then we need to fix first $\widetilde{\mathrm{ch}}_g(f_*\eta, h^{H(X,\xi|_X)}, h^{f_*\eta})$ and $\widetilde{\mathrm{Td}}_g(\overline{\mathcal{N}})$ as differential

forms, and $T_g(\overline{\xi}.)$ as a current. The natural and nice way is that we use [7, (7.33)] to replace $-\widetilde{\mathrm{Td}}_g(\overline{\mathcal{N}})\mathrm{Td}_g^{-1}(\overline{N}_{X/Y}) + \mathrm{Td}_g(\overline{Tf})R_g(\overline{N}_{X/Y})$ by the differential form $\mathbf{B}_g(\overline{\mathcal{N}})$ in [7, (7.24)]. Then we use the current $T_g(\overline{\xi}.)$ defined in [7, (6.30)] and $\widetilde{\mathrm{ch}}_g(f_*\eta, h^{H(X,\xi|_X)}, h^{f_*\eta})$ as the integral $\int_1^{+\infty}$ in [14, (3.24)].

Let $\Delta^0(f,l,i_*\overline{\eta},\overline{\xi}.)$ and $\Delta_0(f,l,i_*\overline{\eta},\overline{\xi}.)$ be the differential forms such that

$$\Delta := \partial\Delta^0(f,l,i_*\overline{\eta},\overline{\xi}.) + \bar{\partial}\Delta_0(f,l,i_*\overline{\eta},\overline{\xi}.)$$

measures the difference

$$T_g(\omega^V, h^\xi) - T_g(\omega^W, h^\eta) + \widetilde{\mathrm{ch}}_g(f_*\eta, h^{H(X,\xi|_X)}, h^{f_*\eta})$$
$$- \int_{V_{\mu_n}/S} \mathrm{Td}_g(\overline{Tl})T_g(\overline{\xi}.) - \int_{W_{\mu_n}/S} \mathbf{B}_g(\overline{\mathcal{N}})\mathrm{ch}_g(\overline{\eta}).$$

We claim that $\Delta^0(f,l,i_*\overline{\eta},\overline{\xi}.)$ and $\Delta_0(f,l,i_*\overline{\eta},\overline{\xi}.)$ can be written down explicitly and they admit certain functoriality.

Let S_1 be a closed submanifold of S, and let $W_1 = f^{-1}(S_1) \subset W$ (respectively, $V_1 = l^{-1}(S_1) \subset V$) be the closed submanifold of W (respectively, V) with restricted Kähler metric. Then $i_1 : W_1 \to V_1$, $l_1 : V_1 \to S_1$ and $f_1 : W_1 \to S_1$ also form a triple of equivariant morphisms such that $f_1 = l_1 \circ i_1$. Denote by j the embedding $S_1 \to S$.

THEOREM A.6. *There is a natural way to write down explicitly differential forms* $\Delta^0(f,l,i_*\overline{\eta},\overline{\xi}.)$ *and* $\Delta_0(f,l,i_*\overline{\eta},\overline{\xi}.)$ *such that* $\Delta := \partial\Delta^0(f,l,i_*\overline{\eta},\overline{\xi}.) + \bar{\partial}\Delta_0(f,l,i_*\overline{\eta},\overline{\xi}.)$ *and they are functorial in the following sense.*

$$j^*\Delta^0(f,l,i_*\overline{\eta},\overline{\xi}.) = \Delta^0(f_1,l_1,i_{1*}\overline{\eta}\,|_{W_1},\overline{\xi}.\,|_{V_1})$$

and

$$j^*\Delta_0(f,l,i_*\overline{\eta},\overline{\xi}.) = \Delta_0(f_1,l_1,i_{1*}\overline{\eta}\,|_{W_1},\overline{\xi}.\,|_{V_1}).$$

Proof. By the equivariant extension of [8, (6.109), (6.110), (6.158) and (6.170)] in [14, Definition 3.4], there exist universal smooth forms γ^3, δ^3 on S such that

$$\Delta + d^S\beta = \bar{\partial}\gamma^3 + \partial\delta^3.$$

Again β is a combination of local terms from the small time heat kernel asymptotics of the Bismut superconnection for the fibrations h and $h \circ f$ (cf. [8, Theorem 6.4, (6.36), (6.55); 27, (2.24), (2.27)]). More precisely, before we make the procedure as $A \to +\infty$, $T_0 \to +\infty$, $\varepsilon \to 0$, the forms γ, δ defined in [14, (3.13)] are double integrals of certain supertrace of the heat kernel of the square of the Bismut superconnection as in (A.7). Note that the square of the Bismut superconnection is a second-order fiberwise elliptic operator with differential form coefficients and when we consider the corresponding objects for the submersion l_1, each above term is the restriction of the corresponding term for the global submersion l, thus if $\Delta_1, \gamma_1^3, \delta_1^3, \beta_1$ are corresponding terms associated to the relevant fibrations i_1, l_1 and f_1, we have

$$\Delta_1 = j^*\Delta, \gamma_1^3 = j^*\gamma^3, \delta_1^3 = j^*\delta^3, \beta_1 = j^*\beta,$$

So, write $\Delta^0(f,l,i_*\overline{\eta},\overline{\xi}.) = \gamma^3 - \beta$ and $\Delta_0(f,l,i_*\overline{\eta},\overline{\xi}.) = \delta^3 - \beta$, we are done. □

We can do the same analysis for (A.14).

Note that we can relax our condition on $f : V \to S$ as follows: S is a (possible noncompact) complex manifold and $f : V \to S$ is a Kähler fibration in the sense of Bismut–Gillet–Soulé [10, Definition 1.4].

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