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Superconnection and family Bergman kernels

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Abstract
We establish an asymptotic version of Bismut's local family index theorem for the Bergman kernel. The key idea is to use the superconnection as in the local family index theorem.

0 Introduction

The recent study of the Bergman kernel in complex geometry mainly started with the paper of Tian [44], which was in turn inspired by a question of Yau. Since [44], the Bergman kernel has been studied extensively in [20, 27, 40, 46], establishing the diagonal asymptotic expansion for high powers of an ample line bundle. Moreover, the coefficients in the asymptotic expansion encode geometric information of the underlying complex projective manifolds. This asymptotic expansion plays a crucial role in the work of Donaldson [24], where the existence of Kähler metrics with constant scalar curvature is shown to be closely related to the Chow–Mumford stability.

In [22, 31, 32], Dai, Liu, Ma and Marinescu studied the asymptotic expansion of the (generalized) Bergman kernel of the spin^c Dirac operator and the renormalized Bochner–Laplacian associated to a positive line bundle on a compact symplectic manifold. As a by product, they gave a new proof of the results mentioned in the previous paragraph. They found also various applications therein, especially, as established in [32], the *full off-diagonal* asymptotic expansion implies Toeplitz operator type properties. Also Ma and Zhang [33, 35] generalized some of the above results to the context of geometric quantization.

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We refer the readers to the book [30] for a comprehensive study of the Bergman kernel, the Berezin–Toeplitz quantization and their applications. The point of view of the approach is from the local index theory, especially from the analytic localization techniques developed by Bismut–Lebeau [12, §11]. A simple principle of this approach is that the existence of the spectral gap of the operators implies the existence of the asymptotic expansion of the corresponding Bergman kernel if the manifold X is compact or not, or singular, or with boundary. Moreover, a general and algorithmic way to compute the coefficients in the expansion is presented.

The purpose of this paper is to establish an asymptotic version of Bismut's local family index theorem for the Bergman kernel. In the introduction, we only formulate the results in the fiberwise positive holomorphic line bundle case, while the main results hold also in the fiberwise symplectic case.

Let W, S be smooth compact complex manifolds with S being connected. Let $\pi : W \to S$ be a holomorphic submersion with compact fiber X and $\dim_{\mathbb{C}} X = n$.

Let $J^{T_{\mathbb{R}}X}$ be the complex structure on $T_{\mathbb{R}}X$, the relative real tangent bundle of π.

Let L, E be holomorphic vector bundles on W and the rank $\mathrm{rk}(L)$ of L is 1. Let h^L, h^E be Hermitian metrics on L, E. Let ∇^L, ∇^E be the Chern (i.e., holomorphic Hermitian) connections on (L, h^L), (E, h^E) with curvatures R^L, R^E. Set

$$\omega := \frac{\sqrt{-1}}{2\pi} R^L. \tag{0.1}$$

Then ω is a smooth real 2-form of complex type $(1, 1)$ on W.

We suppose that ω defines a fiberwise Kähler form along the fiber X, i.e.,

$$g^{T_{\mathbb{R}}X}(u, v) = \omega(u, J^{T_{\mathbb{R}}X} v) \tag{0.2}$$

defines a Riemannian metric on $T_{\mathbb{R}}X$. This simply means that (L, h^L) is a fiberwise positive line bundle on W. We denote by $h^{T^{(1,0)}X}$ the corresponding Hermitian metric on $T^{(1,0)}X$, the holomorphic relative tangent bundle of π.

For a differential form ϑ on S, we will denote by $\vartheta^{(i)}$ its component in $\Lambda^i(T^*_{\mathbb{R}}S)$.

By the Kodaira vanishing theorem and (0.2), there exists $p_0 \in \mathbb{N}$ such that for any $p > p_0$, $s \in S$, for the Dolbeault cohomology groups of $L^p \otimes E$ along the fiber X, we have

$$H^q(X_s, L^p \otimes E) = 0 \quad \text{for any } q > 0. \tag{0.3}$$

Then $H^0(X, L^p \otimes E)$ forms a holomorphic vector bundle on S for $p > p_0$. From now on, we always assume $p > p_0$.

By the Riemann–Roch–Grothendieck theorem, for $p > p_0$, we have (cf. (1.26))

$$\mathrm{ch}(H^0(X, L^p \otimes E)) = \int_X \mathrm{Td}(T^{(1,0)}X)\, \mathrm{ch}(E)\, \mathrm{ch}(L^p) \quad \text{in } H^\bullet(S, \mathbb{R}). \tag{0.4}$$

The component in $H^0(S,\mathbb{R})$ of (0.4) is the Riemann-Roch-Hirzebruch theorem,

$$\begin{aligned}\dim H^0(X, L^p\otimes E) &= \Big[\int_X \mathrm{Td}(T^{(1,0)}X)\,\mathrm{ch}(E)\,\mathrm{ch}(L^p)\Big]^{(0)}\\ &= \mathrm{rk}(E)\int_X \frac{c_1(L)^n}{n!}p^n + \int_X \Big(c_1(E)+\frac{\mathrm{rk}(E)}{2}c_1(T^{(1,0)}X)\Big)\frac{c_1(L)^{n-1}}{(n-1)!}p^{n-1}\\ &\quad +\mathscr{O}(p^{n-2}). \end{aligned}\tag{0.5}$$

For $s\in S$, let $P_{p,s}$ be the orthogonal projection from $\mathscr{C}^\infty(X_s, L^p\otimes E)$ onto $H^0(X_s, L^p\otimes E)$. Let $P_{p,s}(x,x')$ $(x,x'\in X_s, s\in S)$ be the smooth kernel of $P_{p,s}$ with respect to the Riemannian volume form $dv_X(x')$ (Note that $dv_X=(\omega^n)^{(0)}/n!$). Then $P_{p,s}(x,x')$ is smooth on $s\in S$, and we denote it simply by $P_p(x,x')$, especially, $P_p(x,x)\in \mathrm{End}(E_x)$.

The results of [20, 22, 27, 30, 31, 40, 44–46] tell us that there exist $b_r\in \mathscr{C}^\infty(X_s,\mathrm{End}(E))$, $(r\in\mathbb{N})$ such that for any $k,l\in\mathbb{N}$, there exists $C>0$ such that for any $p\in\mathbb{N}^*$, we have

$$\Big|\frac{1}{p^n}P_{p,s}(x,x)-\sum_{r=0}^{k}b_r(x)p^{-r}\Big|_{\mathscr{C}^l(X_s)}\le C\,p^{-k-1},\tag{0.6}$$

and the first two coefficients b_0,b_1 coincide with the local Riemann–Roch–Hirzebruch theorem, i.e., the leading term of the Chern–Weil representative of $\mathrm{Td}(T^{(1,0)}X)\,\mathrm{ch}(E)\,\mathrm{ch}(L^p)$ with respect to the metrics $h^{T^{(1,0)}X}, h^L, h^E$.

By (0.4), in $H^2(S,\mathbb{R})$, we have

$$\begin{aligned}c_1(H^0(X, L^p\otimes E)) &= \Big[\int_X \mathrm{Td}(T^{(1,0)}X)\,\mathrm{ch}(E)\,\mathrm{ch}(L^p)\Big]^{(2)}\\ &= \mathrm{rk}(E)\int_X \frac{c_1(L)^{n+1}}{(n+1)!}p^{n+1} + \int_X \Big(c_1(E)+\frac{\mathrm{rk}(E)}{2}c_1(T^{(1,0)}X)\Big)\frac{c_1(L)^{n}}{n!}p^{n}\\ &\quad +\mathscr{O}(p^{n-1}). \end{aligned}\tag{0.7}$$

Now, from the local index theory point of view [6], it is nature to ask whether the analogue of (0.6) still holds on the higher degree, so that one can refine (0.7) to an equality of differential forms via Chern–Weil representatives. We will prove the existence of the expansion of the curvature operator of the vector bundles $H^0(X, L^p\otimes E)$, and compute the first two coefficients in the expansion in this paper.

To define a canonical connection on $H^0(X, L^p\otimes E)$ via the connections ∇^L, ∇^E, we need to introduce a horizontal sub-bundle $T^H_{\mathbb{R}}W$ of $T_{\mathbb{R}}W$.

Let $T^H_{\mathbb{R}}W$ be a sub-bundle of $T_{\mathbb{R}}W$ such that $T^H_{\mathbb{R}}W$ is invariant by the complex structure on $T_{\mathbb{R}}W$ and

$$T_{\mathbb{R}}W = T^H_{\mathbb{R}}W\oplus T_{\mathbb{R}}X.\tag{0.8}$$

For $U \in T_{\mathbb{R}}S$, let $U^H \in T^H_{\mathbb{R}}W$ be the lift of U. Let $\nabla^{L^p\otimes E}$ be the connection on $L^p \otimes E$ induced by ∇^L, ∇^E. For $U \in T_{\mathbb{R}}S, \sigma \in \mathscr{C}^\infty(S, H^0(X, L^p \otimes E))$, we define

$$\nabla^{H^0(X,L^p\otimes E)}_U \sigma = P_p \nabla^{L^p\otimes E}_{U^H} P_p \sigma. \tag{0.9}$$

Then $\nabla^{H^0(X,L^p\otimes E)}$ is a holomorphic connection on $H^0(X, L^p \otimes E)$ with curvature $R^{H^0(X,L^p\otimes E)}$, but it need not to be a Hermitian connection with respect to the usual L^2 metric $h^{H^0(X,L^p\otimes E)}$ on $H^0(X, L^p \otimes E)$ (cf. (1.10)).

Let D_p be the Dirac operator associated with $L^p \otimes E$ (see (2.6) for details). Then by the Hodge theory and (0.3), $H^0(X, L^p \otimes E) = \operatorname{Ker}(D_p)$ for $p > p_0$. We now define another connection $\nabla^{\operatorname{Ker}(D_p)}$ which has a natural symplectic version. Let $\mathbf{k} \in T^*_{\mathbb{R}}W$ be such that for $U \in T_{\mathbb{R}}S$, $X \in T_{\mathbb{R}}X$,

$$\mathbf{k}(U^H) = \frac{1}{2}(\mathcal{L}_{U^H} dv_X)/dv_X, \quad \mathbf{k}(X) = 0, \tag{0.10}$$

where $\mathcal{L}_{U^H}$ is the Lie derivative of U^H. The canonical Hermitian connection $\nabla^{\operatorname{Ker}(D_p)}$ on $(H^0(X, L^p \otimes E), h^{H^0(X,L^p\otimes E)})$ is defined by

$$\nabla^{\operatorname{Ker}(D_p)}_U = P_p(\nabla^{L^p\otimes E}_{U^H} + \mathbf{k}(U^H))P_p \tag{0.11}$$

with curvature $R^{\operatorname{Ker}(D_p)}$, but $\nabla^{\operatorname{Ker}(D_p)}$ needs not to be holomorphic. Let

$$R^{H^0(X,L^p\otimes E)}(x, x'), R^{\operatorname{Ker}(D_p)}(x, x') \in \Lambda^2(T^*_{\mathbb{R}}S) \otimes (L^p \otimes E)_x \otimes (L^p \otimes E)^*_{x'} \tag{0.12}$$

$(x, x' \in X_s, s \in S)$ be the smooth kernels of the operators $R^{H^0(X,L^p\otimes E)}, R^{\operatorname{Ker}(D_p)}$ with respect to $dv_X(x')$. Then $R^{H^0(X,L^p\otimes E)}(x, x), R^{\operatorname{Ker}(D_p)}(x, x) \in \Lambda^2(T^*_{\mathbb{R}}S) \otimes \operatorname{End}(E_x)$.

Remark 0.1 If

$$T^H_{\mathbb{R}}W = \{u \in T_{\mathbb{R}}W : \omega(u, X) = 0 \text{ for any } X \in T_{\mathbb{R}}X\}, \tag{0.13}$$

then the triple $(\pi, g^{T_{\mathbb{R}}X}, T^H_{\mathbb{R}}W)$ defines a Kähler fibration in the sense of [10, Definition 1.4]. In this case, the connection $\nabla^{\operatorname{Ker}(D_p)}$ is the Chern connection on $(H^0(X, L^p \otimes E), h^{H^0(X,L^p\otimes E)})$, and

$$\mathbf{k} = 0, \quad \nabla^{\operatorname{Ker}(D_p)} = \nabla^{H^0(X,L^p\otimes E)}. \tag{0.14}$$

The following result is a special case of Theorem 1.8 where one finds also its symplectic version. Let $T \in \Lambda^2(T^*_{\mathbb{R}}W) \otimes T_{\mathbb{R}}X$ be the torsion tensor defined by (1.5).

Theorem 0.2 *There exist smooth sections* $b_{2,r}(x) \in \mathscr{C}^\infty(W, \pi^*(\Lambda^2(T^*_{\mathbb{R}}S)) \otimes \operatorname{End}(E_x))$, $(r \in \mathbb{N})$ *which are polynomials in* $R^{T_{\mathbb{R}}X}$, $R^{T^{(1,0)}X}$ (cf. Sect. 1.1), R^E

(resp. T, R^L), their derivatives of order $\leq 2r-2$ (resp. $2r-1$, $2r$) along the fiber X, with

$$b_{2,0} = -2\pi\sqrt{-1}\frac{(\omega^{n+1})^{(2)}}{(n+1)(\omega^n)^{(0)}}\operatorname{Id}_E, \tag{0.15}$$

such that for any $k, l \in \mathbb{N}$, there exists $C_{k,l} > 0$ such that for any $p \in \mathbb{N}$, $p > p_0$,

$$\Big|\frac{1}{p^{n+1}}R^{H^0(X,L^p\otimes E)}(x,x) - \sum_{r=0}^{k} b_{2,r}(x)p^{-r}\Big|_{\mathscr{C}^l(W)} \leq C_{k,l}\, p^{-k-1}. \tag{0.16}$$

For $R^{\mathrm{Ker}(D_p)}(x,x)$, we have the similar expansion as (0.16), with the same leading term $b_{2,0}$ in (0.15), and the corresponding $b_{2,r}(x)$ depends also on the derivatives of $d\mathbf{k}$ of order $\leq 2r-2$ along the fiber X.

Let $\{w_i\}$ be an orthonormal frame of $(T^{(1,0)}X, h^{T^{(1,0)}X})$. Let $\{g_\alpha\}$ be a frame of $T^{(1,0)}S$ with its dual frame $\{g^\alpha\}$. From (0.15), we get

$$b_{2,0} = 2\pi g^\alpha \wedge \overline{g}^\beta\Big[-\sqrt{-1}\omega(g_\alpha^H, \overline{g}_\beta^H) - \omega(g_\alpha^H, \overline{w}_j)\omega(\overline{g}_\beta^H, w_j)\Big]\operatorname{Id}_E. \tag{0.17}$$

Remark 0.3 From (0.16) and (0.17), the curvatures $R^{H^0(X,L^p\otimes E)}(x,x)$, $R^{\mathrm{Ker}(D_p)}(x,x)$ give us a natural approximation of the curvature on the space of Kähler metrics. Thus it should be naturally related to the existence problem of geodesics on the space of Kähler metrics (cf. [23, 36–38, 41]). Let (X,ω_0) be a compact Kähler manifold of dimension n, we suppose that there exists a holomorphic Hermitian line bundle (L, h^L) such that its first Chern form $c_1(L,h^L)$ is ω_0. Then the space of Kähler metrics in the cohomology class $[\omega_0]$ is

$$\mathcal{M} = \{\varphi : X \to \mathbb{R};\, c_1(L, e^{-2\pi\varphi}h^L) = \omega_0 + \sqrt{-1}\,\partial\overline{\partial}\varphi \text{ defines a Kähler form}\}/\sim, \tag{0.18}$$

where $\varphi_1 \sim \varphi_2$ if and only if $\varphi_1 = \varphi_2 + c$ for some constant c. For any complex manifold S of dimension 1 with maps $\phi : S \to \mathcal{M}$. Let p_1, p_2 be the natural projections from $W = X \times S$ onto X, S. We have the holomorphic Hermitian line bundle $(p_1^*L, e^{-2\pi\varphi_s}h^L)$ on W. In this case, if we take $T_{\mathbb{R}}^H W = p_2^* T_{\mathbb{R}} S$, $\varphi(x,s) = \phi_s(x)$, then (0.17) reads as

$$\begin{aligned} b_{2,0} &= 2\pi g^1 \wedge \overline{g}^1\Big[-\sqrt{-1}\omega(g_1,\overline{g}_1) - |\omega(g_1,\cdot)|^2_{h_s^{T^{(1,0)}X}}\Big] \\ &= 2\pi g^1 \wedge \overline{g}^1\Big[(\partial^S\overline{\partial}^S\varphi)(g_1,\overline{g}_1) - |(\overline{\partial}^X\partial^S\varphi)(g_1,\cdot)|^2_{h_s^{T^{(1,0)}X}}\Big]. \end{aligned} \tag{0.19}$$

Thus $b_{2,0} = 0$ is the geodesic equation in [37, (1.2)].

The second main result of this paper is as follows.

Theorem 0.4 *The curvature operators*

$$\frac{1}{p}R^{H^0(X,L^p\otimes E)},\ \frac{1}{p}R^{\mathrm{Ker}(D_p)}\in\Omega^2(S,\mathrm{End}(H^0(X,L^p\otimes E)))$$

are Toeplitz operators in the sense of Definition 1.16 for any $s\in S$, and their leading symbols coinside and equal to $b_{2,0}$, i.e., there exists $R_r\in\mathscr{C}^\infty(W,\pi^*(\Lambda^2(T^*_{\mathbb{R}}S))\otimes$ $\mathrm{End}(E))$, $(r\in\mathbb{N})$ such that for any $k\in\mathbb{N}$, when $p\to+\infty$, under the operator norm of the morphisms of vector bundles: $H^0(X,L^p\otimes E)\to\Lambda^2(T^*_{\mathbb{R}}S)\otimes H^0(X,L^p\otimes E)$ over S, we have

$$\frac{1}{p}R^{H^0(X,L^p\otimes E)}=\sum_{r=0}^{k}T_{R_r,p}\,p^{-r}+\mathcal{O}(p^{-k-1}),\ \textit{with}\ T_{R_r,p}=P_pR_rP_p,\ \ R_0=b_{2,0}. \tag{0.20}$$

Equation (0.20) for $k=0$ implies that there exists $C>0$ such that for any $s\in S$, $\sigma_1,\sigma_2\in H^0(X_s,L^p\otimes E)$, we have

$$\left|\left\langle\frac{\sqrt{-1}}{2\pi p}R_s^{H^0(X,L^p\otimes E)}\sigma_1,\sigma_2\right\rangle-\int_{X_s}\langle\sigma_1,\sigma_2\rangle_{L^p\otimes E}\frac{\omega^{n+1}}{(n+1)!}\right|\le\frac{C}{p}\|\sigma_1\|_{L^2}\|\sigma_2\|_{L^2}. \tag{0.21}$$

From (0.14), Eq. (0.21) gives an asymptotic exact local formula of the curvature estimate given in [1, §6]. Cf. also [2, 3] for further related works.

A simple corollary of Theorem 0.4 is as follows:

Corollary 0.5 *If (L,h^L) is positive on W, then for p large enough, $(H^0(X,L^p\otimes E),h^{H^0(X,L^p\otimes E)})$ is Nakano positive on S.*

In particular, if (F,h^F) is a Griffiths positive vector bundle on S (cf. [30, Def. 1.1.6]), then the projectivization $\mathbb{P}(F)$ of F with the hyperplane line bundle $\mathcal{O}(1)$ over $\mathbb{P}(F)$ is a positive line bundle on $\mathbb{P}(F)$ and for any $s\in S$,

$$\left(H^0(\mathbb{P}(F_s),\mathcal{O}(p)),h^{H^0(\mathbb{P}(F_s),\mathcal{O}(p))}\right)=(S^pF,h^{S^pF}), \tag{0.22}$$

the p-th symmetric product of (F,h^F). Thus from Corollary 0.5, for any holomorphic Hermitian vector bundle $(F',h^{F'})$ on S, $(S^pF\otimes F',h^{S^pF}\otimes h^{F'})$ is Nakano positive for p large enough.

Assume now (0.13) holds. For a differential form ϑ on W, we write ϑ^H, ϑ^X its components in $\pi^*(\Lambda(T^*_{\mathbb{R}}S))\otimes\mathbb{C}$, $\mathbb{C}\otimes\Lambda(T^*_{\mathbb{R}}X)$ under the decomposition $\Lambda(T^*_{\mathbb{R}}W)=\pi^*(\Lambda(T^*_{\mathbb{R}}S))\widehat{\otimes}\Lambda(T^*_{\mathbb{R}}X)$ via (0.8). Then by (0.13), we have

$$\omega=\omega^X+\omega^H\quad\text{with}\ \ \omega^H=g^\alpha\wedge\overline{g}^\beta\omega(g^H_\alpha,\overline{g}^H_\beta). \tag{0.23}$$

Moreover $R^{T_{\mathbb{R}}X}$ coinsides with $R^{T^{(1,0)}X}$ and is the curvature of the Chern connection $\nabla^{T^{(1,0)}X}$ on $(T^{(1,0)}X, h^{T^{(1,0)}X})$. Let Δ_X be the (positive) Laplacian along the fiber $(X, g^{T_{\mathbb{R}}X})$.

Theorem 0.6 *If* (0.13) *holds, then for* $b_{2,0}$, $b_{2,1}$ *in* (0.16), R_1 *in* (0.20), *we have*

$$
\begin{aligned}
b_{2,0} &= -2\pi\sqrt{-1}\omega^H,\\
b_{2,1} &= \left(\left(\frac{1}{2}\operatorname{Tr}[R^{T^{(1,0)}X}] + R^E + \frac{\sqrt{-1}}{4}\Delta_X\omega^H\right)\omega^n\right)^{(2)} \Big/ (\omega^n)^{(0)},\\
R_1 &= \left(R^E + \frac{1}{2}\operatorname{Tr}[R^{T^{(1,0)}X}]\right)^H - \frac{\sqrt{-1}}{4}\Delta_X\omega^H.
\end{aligned}
\tag{0.24}
$$

Remark 0.7 i) If we take the trace on E and the integral along X for (0.16), from (0.15) and (0.24), we refine (0.7) on the level of differential form in the spirit of local index theory. Note that for $p > p_0$, on the determinant line bundle $\lambda_p = \det H^0(X, L^p \otimes E)$ over S, the Quillen metric $\|\ \ \|_Q$ [11, Definition 1.5] is the product of the L^2-metric $\|\ \ \|_{L^2}$ and the associated analytic torsion τ_p. The curvature formula of Bismut–Gillet–Soulé [11, Theorem 1.27] expresses its first Chern form $c_1(\lambda_p, \|\ \ \|_Q)$ as the Chern–Weil representatives of the right hand side of the first line of (0.7). Comparing with these two results, we know that

$$
\overline{\partial}\partial \log \tau_p = \mathscr{O}(p^{n-1}) \quad \text{on } S. \tag{0.25}
$$

Recently, by extending Bismut–Vasserot's result [15], Finski [25] obtained a full asymptotics of $\log \tau_p$ as $p \to +\infty$ which refines (0.25).

ii) As explained in Remark 1.9, from Theorems 1.7 and 1.8, we get immediately the existence of the same type asymptotic expansion as (0.16) for $\frac{1}{p^{n+k}}(R^{\operatorname{Ker}(D_p)})^k(x,x)$ for $k > 1$ with leading term $b_{2,0}^k$. Also by [30, Theorem 7.4.1] and Theorem 0.4, we know that $\frac{1}{p^k}(R^{\operatorname{Ker}(D_p)})^k$ is a Toeplitz operator with leading symbol $b_{2,0}^k$.

The last result of our paper is

Theorem 0.8 *For any* $f \in \mathscr{C}^\infty(W, \operatorname{End}(E))$, $U \in \mathscr{C}^\infty(S, T_{\mathbb{R}}S)$, $\nabla_U^{\operatorname{End}(D_p)} T_{f,p}$, $\nabla_U^{H^0(X,L^p\otimes E)} T_{f,p}$ *are Toeplitz operators with leading symbol* $\nabla_{U^H}^{\operatorname{End}(E)} f$.

We will combine the superconnection framework [6] with the local index technique developed for Bergman kernels [22, 30] to prove our results. One of the important features of the superconnection formalism is that the superconnection itself has derivatives along the horizontal direction, but its curvature is a second order elliptic differential operator along the fiber X. This allows us to work directly on each fiber without taking derivatives along the horizontal direction. This is also one of the key points in the local family index theory [6]. By combining with the formal power series trick in [31], we get in fact a general and algorithmic way to compute the coefficients in the expansion.

This paper is organized as follows. In Sect. 1, we establish a general asymptotic expansion for the curvature of the kernel bundle of a family of spin^c Dirac operators, Theorem 1.8. Then as a consequence, we show that the curvature operator is a Toeplitz operator, thus establishing Theorems 0.2 and 0.4. We establish also Theorem 1.19, as a symplectic version of Theorem 0.8. In Sect. 2, in the holomorphic situation, we explain our result in detail, and establish Corollary 0.5 and Theorem 0.6.

Some results of this paper have been announced in [34]. We will not try to update the complete references. We simply point out our results have been used in the study of the asymptotics of the analytic torsion in the recent works [14, 39].

Notation: When we work in the holomorphic situation, we will add a subscript $\mathbb{R}$ for the corresponding real objects. Thus TX is the holomorphic relative tangent bundle of π, and $T_{\mathbb{R}}X$ is the corresponding real bundle.

For an operator A, we denote $\mathrm{Spec}(A)$ its spectrum, $\mathrm{Ker}(A)$ its kernel and $\mathrm{Coker}(A)$ its cokernel. As in [5, §1.3], for two operators A, B with $\mathbb{Z}_2$-grading, $[A, B]$ means their supercommutator $[A, B] = AB - (-1)^{\deg A\cdot\deg B} BA$. For $\mathbb{Z}_2$-graded algebras $\mathcal{A}, \mathcal{B}$ with identity, we denote by $\mathcal{A}\widehat{\otimes}\mathcal{B}$ the $\mathbb{Z}_2$-graded tensor product of $\mathcal{A}$ and $\mathcal{B}$ with product

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) := (-1)^{\deg a_2\cdot\deg b_1} a_1a_2 \otimes b_1b_2. \tag{0.26}$$

When an index variable appears twice in a single term, it means that we are summing over all its possible values.

1 Asymptotic expansion of family Bergman kernels

In this Section, we establish a general off-diagonal asymptotic expansion for the curvature of the kernel bundle of a family of spin^c Dirac operators in Theorem 1.8. We work in the fiberwise symplectic case in Sects. 1.1–1.7.

This Section is organized as follows. In Sect. 1.1, as a motivation of our work, we explain Bismut's superconnection and his local family index theorem. This part gives us the inspiration how to get a family version of Bergman kernels, especially, how to use the superconnection. In Sect. 1.2, we review the results in [22] and explain how they depend on the parameters. In Sect. 1.3, we explain a general off-diagonal asymptotic expansion for the curvature of the kernel bundle of a family of spin^c Dirac operators in Theorem 1.8. In Sect. 1.4, we explain how to introduce the superconnection here to solve our problem. In Sect. 1.5, we explain the Taylor expansion of the rescaled curvature of the superconnection, and the spectrum of the limit operator. In Sect. 1.6, we give a way to compute the coefficients in the expansion by combining with the formal power series trick in [31, §1.5] (cf. [30, §4.1.7]). Especially, we compute the leading coefficient. In Sect. 1.7, we explain the curvature operator as a Toeplitz operator. In Sect. 1.8, we establish Theorems 0.2, 0.4 and 0.8.

1.1 Local family index theorem

Let W, S be two smooth manifolds. Let $\pi : W \to S$ be a smooth submersion with compact fiber X and $\dim_{\mathbb{R}} X = 2n$. Let TX be the relative tangent bundle of the fibration π. Let g^{TX} be a metric on TX.

Let E be a complex vector bundle on W with a Hermitian metric h^E. Let ∇^E be a Hermitian connection on (E, h^E).

Let $T^H W$ be a sub-bundle of TW such that

$$TW = T^H W \oplus TX. \tag{1.1}$$

Let P^{TX} be the projection from TW onto TX. For $U \in TS$, let $U^H \in T^H W$ be the lift of U, i.e., $d\pi(U^H) = U$. We denote by $\mathcal{L}_{U^H}$ the Lie derivative of U^H.

Definition 1.1 [6, Definition 1.6] The canonical metric connection ∇^{TX} on $(TX \to W, g^{TX})$ is defined by the following properties.

a) On each fiber X, ∇^{TX} restricts to the Levi–Civita connection of (TX, g^{TX}).
b) If $U \in TS$, then

$$\nabla^{TX}_{U^H} = \mathcal{L}_{U^H} + \frac{1}{2}(g^{TX})^{-1}(\mathcal{L}_{U^H} g^{TX}). \tag{1.2}$$

Let R^{TX} be the curvature of ∇^{TX}.

Let g^{TS} be a Riemannian metric on TS. Let $g^{TW} = \pi^* g^{TS} \oplus g^{TX}$ be the induced metric on TW via (1.1). Let ∇^{TW}, ∇^{TS} be the Levi-Civita connections on (TW, g^{TW}), (TS, g^{TS}). Then by [10, Theorem 1.2] (cf. [8, Theorems 1.1 and 1.2]), we get

$$\nabla^{TX} = P^{TX} \nabla^{TW}. \tag{1.3}$$

Set

$${}^0\nabla^{TW} = \pi^* \nabla^{TS} \oplus \nabla^{TX}, \quad S = \nabla^{TW} - {}^0\nabla^{TW}. \tag{1.4}$$

Then ${}^0\nabla^{TW}$ is a Euclidean connection on TW and $S \in T^*W \otimes \mathrm{End}(TW)$. Let T be the torsion of the connection ${}^0\nabla^{TW}$. Then by [8, Theorem 1.1], for $U, V \in TS$, $X, Y \in TX$, we have

$$\begin{aligned} T(U^H, V^H) &= -P^{TX}[U^H, V^H], \quad T(X, Y) = 0, \\ T(U^H, X) &= \frac{1}{2}(g^{TX})^{-1}(\mathcal{L}_{U^H} g^{TX}) X. \end{aligned} \tag{1.5}$$

Moreover, from [6, (1.28)], for $U, V \in TS$, $X, Y \in TX$, we have

$$\langle T(U^H, X), Y\rangle = \langle T(U^H, Y), X\rangle = \langle S(X)U^H, Y\rangle,$$
$$\langle S(X)U^H, V^H\rangle = \frac{1}{2}\langle T(U^H, V^H), X\rangle. \tag{1.6}$$

From now on, we suppose that there exists an almost complex structure J^{TX} on TX and

$$g^{TX}(J^{TX}u, J^{TX}v) = g^{TX}(u, v). \tag{1.7}$$

The almost complex structure J^{TX} induces a splitting

$$TX \otimes_{\mathbb{R}} \mathbb{C} = T^{(1,0)}X \oplus T^{(0,1)}X,$$

where $T^{(1,0)}X$ and $T^{(0,1)}X$ are the eigenbundles of J^{TX} corresponding to the eigenvalues $\sqrt{-1}$ and $-\sqrt{-1}$ respectively. We denote by $P^{T^{(1,0)}X}$ the projection from $TX \otimes_{\mathbb{R}} \mathbb{C}$ to $T^{(1,0)}X$. Let $T^{*(1,0)}X$ and $T^{*(0,1)}X$ be the corresponding dual bundles.

For any $v \in TX \otimes_{\mathbb{R}} \mathbb{C}$ with decomposition $v = v_{1,0} + v_{0,1} \in T^{(1,0)}X \oplus T^{(0,1)}X$, let $v^*_{1,0} \in T^{*(0,1)}X$ be the metric dual of $v_{1,0}$. Then

$$c(v) := \sqrt{2}(v^*_{1,0} \wedge -i_{v_{0,1}}) \tag{1.8}$$

defines the Clifford action of v on $\Lambda(T^{*(0,1)}X)$, where $\wedge$ and i denote the exterior and interior multiplications respectively.

Let $\nabla^{T^{(1,0)}X} = P^{T^{(1,0)}X}\nabla^{TX}P^{T^{(1,0)}X}$ be the Hermitian connection on $T^{(1,0)}X$ induced by ∇^{TX} with curvature $R^{T^{(1,0)}X}$. Let $\nabla^{\det}$ be the connection on the determinant line $\det(T^{(1,0)}X) := \Lambda^n(T^{(1,0)}X)$ induced by $\nabla^{T^{(1,0)}X}$.

By [26, pp.397–398], ∇^{TX} and $\nabla^{\det}$ induce canonically a Clifford connection ∇^{Cliff} on $\Lambda(T^{*(0,1)}X)$ with curvature R^{Cliff} (cf. also [28, §2], [30, §1.3]).

Let $\{e_j\}$ be an orthonormal basis of TX. Then

$$R^{\mathrm{Cliff}} = \frac{1}{4}\sum_{i,j}\langle R^{TX}e_i, e_j\rangle c(e_i)c(e_j) + \frac{1}{2}\operatorname{Tr}\left[R^{T^{(1,0)}X}\right]. \tag{1.9}$$

Let $\nabla^{\Lambda(T^{*(0,1)}X)\otimes E}$ be the connection on $\Lambda(T^{*(0,1)}X) \otimes E$ induced by ∇^{Cliff} and ∇^E.

Let $\langle \cdot, \cdot\rangle_{\Lambda(T^{*(0,1)}X)\otimes E}$ be the metric on $\Lambda(T^{*(0,1)}X) \otimes E$ induced by g^{TX} and h^E. Let dv_X be the Riemannian volume form of (TX, g^{TX}). The L^2-scalar product on $\Omega^{0,\bullet}(X, E) = \oplus_{q=0}^n \Omega^{0,q}(X, E)$, the space of smooth sections of $\Lambda(T^{*(0,1)}X)\otimes E = \oplus_{q=0}^n \Lambda^q(T^{*(0,1)}X) \otimes E$ on X, is given by

$$\langle s_1, s_2\rangle = \int_X \langle s_1(x), s_2(x)\rangle_{\Lambda(T^{*(0,1)}X)\otimes E}\, dv_X(x). \tag{1.10}$$

We denote the corresponding norm by $\|\cdot\|_{L^2}$.

Definition 1.2 The spin^c Dirac operator D is defined by

$$D := \sum_{j=1}^{2n} c(e_j)\nabla_{e_j}^{\Lambda(T^{*(0,1)}X)\otimes E} : \Omega^{0,\bullet}(X, E) \longrightarrow \Omega^{0,\bullet}(X, E)\,. \tag{1.11}$$

Clearly, D is a formally self–adjoint, first order elliptic differential operator on $\Omega^{0,\bullet}(X, E)$, which interchanges $\Omega^{0,\mathrm{even}}(X, E)$ and $\Omega^{0,\mathrm{odd}}(X, E)$. Let D_+ be the restriction of D on $\Omega^{0,\mathrm{even}}(X, E)$.

Assumption The rank of $\mathrm{Ker}(D_s)$ is locally constant on $s \in S$.

Then $\mathrm{Ker}(D)$ forms a smooth vector bundle on S. Let $h^{\mathrm{Ker}(D)}$ be the metric on $\mathrm{Ker}(D)$ induced by the scalar product $\langle\ \ \rangle$ in (1.10) on $\Omega^{0,\bullet}(X, E)$.

For $s \in S$, let P_s be the orthogonal projection from $(\Omega^{0,\bullet}(X_s, E), \langle\ \ \rangle)$ onto $\mathrm{Ker}(D_s)$, then P_s is smooth on $s \in S$. Set

$$P^{\perp} = 1 - P. \tag{1.12}$$

Let $\mathbf{k} \in T^*W$ be defined by for $U \in TS$, $X \in TX$,

$$\mathbf{k}(U^H) = \frac{1}{2}(\mathcal{L}_{U^H} dv_X)/dv_X, \quad \mathbf{k}(X) = 0. \tag{1.13}$$

For $U \in TS$, if s is a smooth section of $\Omega^{0,\bullet}(X, E)$ over S, set

$$\nabla_U^{\Omega} s = \nabla_{U^H}^{\Lambda(T^{*(0,1)}X)\otimes E} s + \mathbf{k}(U^H)s. \tag{1.14}$$

Then ∇^{Ω} is a Hermitian connection on the infinite dimensional vector bundle $\Omega^{0,\bullet}(X, E)$ over S. Let R^{Ω} be the curvature of the connection ∇^{Ω}, then by (1.5) and (1.14), for $U, V \in TS$,

$$R^{\Omega}(U, V) = (R^{\mathrm{Cliff}} + R^E)(U^H, V^H) + d\mathbf{k}(U^H, V^H) - \nabla_{T(U^H,V^H)}^{\Lambda(T^{*(0,1)}X)\otimes E}. \tag{1.15}$$

Then ∇^{Ω} induces a Hermitian connection $\nabla^{\mathrm{Ker}(D)}$ on $(\mathrm{Ker}\, D, h^{\mathrm{Ker}(D)})$ by

$$\nabla^{\mathrm{Ker}(D)} = P\nabla^{\Omega}P. \tag{1.16}$$

The curvature $R^{\mathrm{Ker}(D)}$ of $\nabla^{\mathrm{Ker}(D)}$ is

$$R^{\mathrm{Ker}(D)} := (\nabla^{\mathrm{Ker}(D)})^2 \in \Lambda^2(T^*S) \otimes \mathrm{End}(\mathrm{Ker}(D)). \tag{1.17}$$

Let $R^{\mathrm{Ker}(D)}(x, x')$, $\exp(-R^{\mathrm{Ker}(D)})(x, x')$ $(x, x' \in X_s, s \in S)$ be the smooth kernel of $R^{\mathrm{Ker}(D)}$, $\exp(-R^{\mathrm{Ker}(D)})$ with respect to $dv_X(x')$.

Let $\{f_\alpha\}$ be a basis of TS, and $\{f^\alpha\}$ its dual basis. For $u>0$, let $\psi_u : \Lambda(T^*S) \to \Lambda(T^*S)$ defined by

$$\psi_u \vartheta = u^{-\deg \vartheta/2}\vartheta. \tag{1.18}$$

For Q an operator along the fiber with values in $\Lambda(T^*S)$, we will denote by

$$Q = \sum_{i=0}^{\dim_{\mathbb{R}} S} Q^{(i)}, \quad \text{with } Q^{(i)} \in \Lambda^i(T^*S)\widehat{\otimes}\,\mathrm{End}(\Omega^{0,\bullet}(X,E)). \tag{1.19}$$

We express now the curvature operator $R^{\mathrm{Ker}(D)}$ by using superconnections. Let

$$B^{(2)} \in \mathscr{C}^\infty(W, \pi^*(\Lambda^2(T^*S))\widehat{\otimes}\,\mathrm{End}(\Lambda(T^{*(0,1)}X)\otimes E))$$

such that it changes the parity of $\Lambda(T^{*(0,1)}X)$. For $u>0$, set B, B_u the superconnections on $\mathscr{C}^\infty(S, \Lambda(T^*S)\widehat{\otimes}\Omega^{0,\bullet}(X,E))$ defined by

$$B = D + \nabla^\Omega + B^{(2)}, \quad B_u = \psi_u \sqrt{u}\, B \psi_u^{-1} = \sqrt{u} D + \nabla^\Omega + \frac{1}{\sqrt{u}} B^{(2)}. \tag{1.20}$$

Then B_u^2 is a second order elliptic operator along the fiber X, and from (1.20),

$$(B^2)^{(0)} = D^2, \quad B^2 = D^2 + (B^2)^{(>0)}, \quad (B^2)^{(2)} = R^\Omega + [D, B^{(2)}], \tag{1.21}$$

and by [6, Theorem 2.5] and (1.5), we get

$$\begin{aligned}(B^2)^{(1)} &= [D, \nabla^\Omega] \\ &= f^\alpha \wedge c(e_i)\Big[(R^{\mathrm{Cliff}} + R^E)(f_\alpha^H, e_i) - e_i \mathbf{k}(f_\alpha^H) - \nabla^{\Lambda(T^{*(0,1)}X)\otimes E}_{T(f_\alpha^H, e_i)}\Big].\end{aligned} \tag{1.22}$$

By (1.21) and (1.22), for $\lambda \notin \mathrm{Spec}(D_s^2)$, we have

$$\begin{aligned}&(\lambda - B^2)^{-1} = (\lambda - D^2)^{-1} + (\lambda - D^2)^{-1} \sum_{j=1}^{\dim_{\mathbb{R}} S} \Big((B^2)^{(>0)}(\lambda - D^2)^{-1}\Big)^j, \\ &P(B^2)^{(1)}P = 0.\end{aligned} \tag{1.23}$$

From (1.16), (1.21), (1.22), (1.23) and the residue formula, if $\mu < \inf_{s\in S}\{\lambda > 0, \lambda \in \mathrm{Spec}(D_s^2)\}$, we get the following important formula for the curvature operator via the resolvent of the superconnection B,

$$\begin{aligned}R^{\mathrm{Ker}(D)} &= PR^\Omega P - P\nabla^\Omega P^\perp \nabla^\Omega P \\ &= PR^\Omega P - P(B^2)^{(1)}((B^2)^{(0)})^{-1}P^\perp (B^2)^{(1)}P \\ &= \frac{1}{2\pi\sqrt{-1}}\Big[\int_{|\lambda|=\mu} (\lambda - B^2)^{-1}\lambda d\lambda\Big]^{(2)}.\end{aligned} \tag{1.24}$$

In the rest of this paper, all estimates and convergences are uniformly with respect to any compact subset of S. For simplicity, we will assume S is compact from now on.

We explain now that the connection $\nabla^{\mathrm{Ker}(D)}$ is natural in the family index theory. Let $\exp(-B_u^2)(x,x')$ $(x,x' \in X_s, s\in S)$ be the smooth kernel of $\exp(-B_u^2)$ with respect to $dv_X(x')$. By [5, Theorem 9.19], for any $l\in\mathbb{N}$ there exists $C_l>0$ such that for any $u>1$, we have

$$\left| e^{-B_u^2}(x,x') - \exp(-R^{\mathrm{Ker}(D)})(x,x') \right|_{\mathscr{C}^l(W\times_S W)} \leq C_l u^{-1/2}, \tag{1.25}$$

where $W\times_S W$ is the fiberwise product of W over S. We recall finally Bismut's local family index theorem. For any Hermitian (complex) vector bundle (F,h^F) with Hermitian connection ∇^F and curvature R^F on W, set

$$\begin{aligned} \mathrm{ch}(F,\nabla^F) &:= \mathrm{Tr}\left[\exp\left(\frac{-R^F}{2\pi\sqrt{-1}}\right)\right], \quad c_1(F,\nabla^F) := \mathrm{Tr}\left[\frac{-R^F}{2\pi\sqrt{-1}}\right],\\ \mathrm{Td}(F,\nabla^F) &:= \det\left(\frac{R^F/(2\pi\sqrt{-1})}{\exp(R^F/(2\pi\sqrt{-1}))-1}\right), \\ \hat{A}(TX,\nabla^{TX}) &:= \left(\det\left(\frac{R^{TX}/(2\pi\sqrt{-1})}{\sinh(R^{TX}/(2\pi\sqrt{-1}))}\right)\right)^{1/2}. \end{aligned} \tag{1.26}$$

They are closed differential forms on W and their cohomology classes do not depend on the choice of the metric h^F and the connections ∇^F, ∇^{TX}. The corresponding cohomology classes are called the Chern character of F, the first Chern class of F, the Todd class of F, the Hirzebruch $\hat{A}$-class of TX and we denote them by $\mathrm{ch}(F)$, $c_1(F)$, $\mathrm{Td}(F)$, $\hat{A}(TX)$.

Let N_X be the number operator on $\Lambda(T^{*(0,1)}X)$, i.e., N_X acts on $\Lambda^k(T^{*(0,1)}X)$ by multiplication by k. For $\vartheta\in\Lambda(T^*S)$, $Q\in\mathrm{End}(\Omega^{0,\bullet}(X_s,E))$, we define the supertrace Tr_s by

$$\mathrm{Tr}_s[\vartheta\wedge Q] = \vartheta\,\mathrm{Tr}[(-1)^{N_X}Q]. \tag{1.27}$$

To get Bismut's local family index theorem, we need to introduce the Bismut superconnection $\mathcal{B}_u$ as following

$$\mathcal{B}_u = \sqrt{u}D + \nabla^{\Omega} + \frac{1}{\sqrt{u}}\mathcal{B}^{(2)}, \text{ with } \mathcal{B}^{(2)} = -\frac{1}{8}\langle T(f_\alpha^H, f_\beta^H), e_i\rangle f^\alpha\wedge f^\beta\wedge c(e_i). \tag{1.28}$$

Theorem 1.3 (Bismut [6]). *For $u>0$, the differential form $\mathrm{Tr}_s[\exp(-\mathcal{B}_u^2)]$ is closed on S and its cohomology class does not depend on $u>0$ and is equal to the Chern character of the index bundle $\mathrm{Ind}(D_+) = \mathrm{Ker}(D_+) - \mathrm{Coker}(D_+)$. Moreover, uniformly on W,*

$$\lim_{u\to 0} \mathrm{Tr}_s[\exp(-\mathcal{B}_u^2)(x,x)]dv_X(x)$$
$$= \left\{ \hat{A}(TX,\nabla^{TX}) e^{c_1(\det(T^{(1,0)}X),\nabla^{\det})}\, \mathrm{ch}(E,\nabla^E) \right\}_x^{Max}, \tag{1.29}$$

here $\{\}^{Max}$ *means the maximal degree part of the fiber* X.

After integrating (1.29) along the fiber X, we get the Atiyah-Singer family index theorem

$$\mathrm{ch}(\mathrm{Ind}(D_+)) = \int_X \hat{A}(TX) e^{c_1(\det(T^{(1,0)}X))}\, \mathrm{ch}(E) \quad \text{in } H^\bullet(S,\mathbb{R}). \tag{1.30}$$

1.2 Asymptotic expansion of Bergman kernels

As explained in Sect. 1.1, we will suppose that W, S are compact.

Let L be a complex line bundle on W with Hermitian metric h^L. Let ∇^L be a Hermitian connection on (L,h^L) with curvature R^L. We suppose that

$$\omega := \frac{\sqrt{-1}}{2\pi} R^L, \tag{1.31}$$

defines a fiberwise symplectic form along the fiber X, and $\omega(\cdot, J^{TX}\cdot)$ defines a J^{TX}-invariant metric on TX.

Set

$$\mu_0 = \inf_{u\in T_x^{(1,0)}X,\, x\in W} R_x^L(u,\overline{u})/|u|^2_{g^{TX}} > 0. \tag{1.32}$$

Let $\{w_i\}$ be an orthonormal frame of $(T^{(1,0)}X, g^{TX})$. Set

$$\omega_d = -\sum_{l,m} R^L(w_l,\overline{w}_m)\,\overline{w}^m \wedge i_{\overline{w}_l}\,, \qquad \tau(x) = \sum_j R^L(w_j,\overline{w}_j)\,. \tag{1.33}$$

Let $\mathbf{J} : TX \to TX$ be the skew–adjoint linear map which satisfies the relation

$$\omega(u,v) = g^{TX}(\mathbf{J}u, v) \tag{1.34}$$

for $u,v \in TX$. Then J^{TX} commutes with $\mathbf{J}$ and $J^{TX} = \mathbf{J}(-\mathbf{J}^2)^{-1/2}$.

We will add a subscript p to denote the corresponding objects in Sect. 1.1 associated with $L^p \otimes E$. Especially D_p is the fiberwise Dirac operator in (1.11) associated with $L^p \otimes E$, and ∇^{E_p} be the connection on

$$E_p := \Lambda(T^{*(0,1)}X) \otimes L^p \otimes E \tag{1.35}$$

induced by ∇^{Cliff}, ∇^L and ∇^E. Let R^{E_p} be the curvature of ∇^{E_p}, then

$$R^{E_p} = R^{\text{Cliff}} + p\,R^L + R^E. \tag{1.36}$$

The following result was obtained in [28, Theorems 1.1 and 2.5] by applying the Lichnerowicz formula (cf. also [15, Theorem 1] in the holomorphic case).

Theorem 1.4 *There exists $C_L > 0$ such that for any $p \in \mathbb{N}$ and any $s \in \Omega^{0,>0}(X, L^p \otimes E) = \bigoplus_{q\geq 1} \Omega^{0,q}(X, L^p \otimes E)$,*

$$\|D_p s\|^2_{L^2} \geq (2p\mu_0 - C_L)\|s\|^2_{L^2}\,. \tag{1.37}$$

Moreover $\operatorname{Spec}(D_p^2) \subset \{0\} \cup [2p\mu_0 - C_L, +\infty[$.

For (1.37), for p large enough, $D_p^2|_{\Omega^{0,\text{odd}}(X,L^p\otimes E)}$ is invertible (cf. also [17, 19]). Thus there exists $p_0 > 0$ such that for $p > p_0$, $\operatorname{Ker}(D_p)$ is a vector bundle on S. Especially, the assumption in Sect. 1.1 is verified for $p > p_0$.

For $s \in S$, let $P_{p,s}$ be the orthogonal projection from $\Omega^{0,\bullet}(X_s, L^p \otimes E)$ onto $\operatorname{Ker}(D_{p,s})$, and $P_{p,s}(x, x')$ $(x, x' \in X_s)$ be the smooth kernel of $P_{p,s}$ with respect to the Riemannian volume form $dv_X(x')$.

Let a^X be the injectivity radius of (X, g^{TX}), and $\varepsilon \in]0, a^X/4[$. We denote by $B^X(x, \epsilon)$ and $B^{T_xX}(0, \epsilon)$ the open balls in X and T_xX with center x and radius ϵ, respectively. Then the fiberwise exponential map $T_xX \ni Z \to \exp^X_x(Z) \in X$ is a diffeomorphism from $B^{T_xX}(0, \epsilon)$ on $B^X(x, \epsilon)$ for $\epsilon \leq a^X$. From now on, we identify $B^{T_xX}(0, \epsilon)$ with $B^X(x, \epsilon)$ for $\epsilon \leq a^X$.

Let $f : \mathbb{R} \to [0, 1]$ be a smooth even function such that

$$f(v) = \begin{cases} 1 & \text{for} \quad |v| \leq \varepsilon/2, \\ 0 & \text{for} \quad |v| \geq \varepsilon. \end{cases} \tag{1.38}$$

Set

$$F(a) = \Big(\int_{-\infty}^{+\infty} f(v)dv\Big)^{-1} \int_{-\infty}^{+\infty} e^{iva} f(v)dv. \tag{1.39}$$

Then the even function $F(a)$ lies in Schwartz space $\mathcal{S}(\mathbb{R})$ and $F(0) = 1$.

Let $F(D_p)(x, x')$, $(x, x' \in X)$ be the smooth kernels of $F(D_p)$ with respect to $dv_X(x')$.

Let $d^X(x, x')$ $(x, x' \in X_s, s \in S)$ be the Riemannian distance on (X_s, g^{TX}).

The following result is an easy extension of [22, Prop. 4.1].

Proposition 1.5 *For any $l, m \in \mathbb{N}$, $\varepsilon > 0$, there exists $C_{l,m,\varepsilon} > 0$ such that for $p \geq 1$, $x, x' \in X$,*

$$\begin{aligned} |F(D_p)(x, x') - P_p(x, x')|_{\mathscr{C}^m(W\times_S W)} &\leq C_{l,m,\varepsilon}\, p^{-l}, \\ |P_p(x, x')|_{\mathscr{C}^m(W\times_S W)} &\leq C_{l,m,\varepsilon}\, p^{-l} \quad \text{if } d(x, x') \geq \varepsilon. \end{aligned} \tag{1.40}$$

Here the $\mathscr{C}^m$ norm is induced by ∇^L, ∇^E and ∇^{Cliff}.

Proof For $a \in \mathbb{R}$, set

$$\phi_p(a) = 1_{[\sqrt{p\mu_0}, +\infty[}(|a|)F(a). \tag{1.41}$$

Then by Theorem 1.4, for $p > C_L/\mu_0$,

$$F(D_p) - P_p = \phi_p(D_p). \tag{1.42}$$

To prove (1.40), we only need to prove the analogue of [22, (4.16)]: for $U_1, \cdots, U_k$ vector fields on S, $l, m, m' \in \mathbb{N}$, there exists $C > 0$ such that

$$\| D_p^m (\nabla^{E_p}_{U_1^H} \cdots \nabla^{E_p}_{U_k^H} \phi_p(D_p)) D_p^{m'} s\|_{L^2} \leq C p^{-l} \|s\|_{L^2}. \tag{1.43}$$

Now

$$\begin{aligned} D_p^m (\nabla^{E_p}_{U_1^H} \phi_p(D_p)) D_p^{m'} = \nabla^{E_p}_{U_1^H} (D_p^m \phi_p(D_p) D_p^{m'}) - [\nabla^{E_p}_{U_1^H}, D_p^m] \phi_p(D_p) D_p^{m'} \\ - D_p^m \phi_p(D_p) [\nabla^{E_p}_{U_1^H}, D_p^{m'}]. \end{aligned} \tag{1.44}$$

Let Γ_p be the union of the contour (which are parallel to the axis) from $+\infty + \sqrt{-1}$ to $\sqrt{p\mu_0} + \sqrt{-1}$, then to $\sqrt{p\mu_0} - \sqrt{-1}$ then to $+\infty - \sqrt{-1}$, and the contour from $-\infty - \sqrt{-1}$ to $-\sqrt{p\mu_0} - \sqrt{-1}$, then to $-\sqrt{p\mu_0} + \sqrt{-1}$ then to $-\infty + \sqrt{-1}$. Then

$$\begin{aligned} \nabla^{E_p}_{U_1^H} (D_p^m \phi_p(D_p) D_p^{m'}) &= \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_p} \lambda^{m+m'} F(\lambda) \nabla^{E_p}_{U_1^H} (\lambda - D_p)^{-1} d\lambda \\ &= \frac{1}{2\pi\sqrt{-1}} \int_{\Gamma_p} \lambda^{m+m'} F(\lambda) (\lambda - D_p)^{-1} [\nabla^{E_p}_{U_1^H}, D_p] (\lambda - D_p)^{-1} d\lambda. \end{aligned} \tag{1.45}$$

Observe that $[\nabla^{E_p}_{U_1^H}, D_p]$ is a first order differential operator along the fiber X (cf. (1.5), (1.22)), thus from [22, (4.7), (4.14)] and (1.45), we get

$$\|\nabla^{E_p}_{U_1^H} (D_p^m \phi_p(D_p) D_p^{m'}) s\|_{L^2} \leq C p^{-l} \|s\|_{L^2}. \tag{1.46}$$

Observe that $[\nabla^{E_p}_{U_1^H}, D_p^m]$, $[\nabla^{E_p}_{U_1^H}, D_p^{m'}]$ are differential operators along the fiber X (cf. (1.22)), thus from [22, (4.15), (4.16)] and (1.46), we get (1.43) for $k = 1$. For $k > 1$, by the same argument, we get (1.43).

By the finite propagation speed of solutions of hyperbolic equations [21, §7.8], [43, §4.4], (cf. also [30, Appendix D.2]), $F(D_p)(x, x')$ only depends on the restriction of D_p to $B^X(x, \varepsilon)$, and

$$F(D_p)(x, x') = 0 \quad \text{if } d^X(x, x') \geq \varepsilon. \tag{1.47}$$

The proof of Proposition 1.5 is completed. □

We denote by $I_{\mathbb{C}\otimes E}$ the orthogonal projection from $\mathbf{E} := \Lambda(T^{*(0,1)}X) \otimes E$ onto $\mathbb{C} \otimes E$. Let $\nabla^{\mathrm{End}(\mathbf{E})}$ be the connection on $\mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)$ induced by ∇^{Cliff} and ∇^E.

We will use the normal coordinates along the fiber X now. For $x_0 \in X_s$, $s \in S$, we identify L_Z, E_Z and $(E_p)_Z$ for $Z \in B^{T_{x_0}X}(0, \varepsilon)$ to L_{x_0}, E_{x_0} and $(E_p)_{x_0}$ by parallel transport with respect to the connections ∇^L, ∇^E and ∇^{E_p} along the curve $\gamma_Z : [0, 1] \ni u \to \exp^X_{x_0}(uZ)$. Under this identification and (1.40), we will view $P_p(x, x')$ as a smooth section $P_{p,x_0}(Z, Z')$, $(Z, Z' \in B^{T_{x_0}X}(0, \varepsilon))$, of $\pi_1^*(\mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E))$ on $TX \times_W TX$ with the projection $\pi_1 : TX \times_W TX \to W$ from the fiberwise product of TX on W. And $\nabla^{\mathrm{End}(\mathbf{E})}$ induces naturally a $\mathscr{C}^m$-norm for the parameter $x_0 \in W$.

Let dv_{TX} be the Riemannian volume form on $(T_{x_0}X, g^{T_{x_0}X})$. Let $\kappa(Z)$ be the smooth positive function defined by the equation

$$dv_X(Z) = \kappa(Z) dv_{TX}(Z), \tag{1.48}$$

with $\kappa(0) = 1$.

We denote by $\det_{\mathbb{C}}$ for the determinant function on the complex bundle $T^{(1,0)}X$, and $|\mathbf{J}_{x_0}| = (-\mathbf{J}^2_{x_0})^{1/2}$. For $U \in T_{x_0}X$, denote by ∇_U the ordinary differentiation operator on $T_{x_0}X$ in the direction U. Let $\{e_i\}$ be an orthonormal basis of $(T_{x_0}X, g^{T_{x_0}X})$.

On $T_{x_0}X \simeq \mathbb{R}^{2n}$, where the identification is given by

$$(Z_1, \cdots, Z_{2n}) \in \mathbb{R}^{2n} \longrightarrow \sum_i Z_i e_i \in T_{x_0}X, \tag{1.49}$$

set (with τ in (1.33))

$$\mathscr{L} = -\sum_j \Big(\nabla_{e_j} + \frac{1}{2} R^L_{x_0}(Z, e_j)\Big)^2 - \tau_{x_0}. \tag{1.50}$$

Let $\mathscr{P}(Z, Z')$ be the Bergman kernel of $\mathscr{L}$, i.e., the smooth kernel of the orthogonal projection from $L^2(\mathbb{R}^{2n}, \mathbb{C})$ onto $\mathrm{Ker}(\mathscr{L})$. Then for $Z, Z' \in T_{x_0}X$, (cf. [31, (1.81)])

$$\begin{aligned}\mathscr{P}(Z, Z') = &\det{}_{\mathbb{C}}(|\mathbf{J}_{x_0}|)\\ &\times \exp\Big(-\frac{\pi}{2}\big\langle |\mathbf{J}_{x_0}|(Z - Z'), (Z - Z')\big\rangle - \pi\sqrt{-1}\big\langle \mathbf{J}_{x_0} Z, Z'\big\rangle\Big).\end{aligned} \tag{1.51}$$

If $\alpha = (\alpha_1, \cdots, \alpha_{2n})$ is a multi-index, set

$$|\alpha| = \sum_{j=1}^{2n} \alpha_j, \quad \partial^\alpha := \frac{\partial^{|\alpha|}}{\partial Z^\alpha} = \frac{\partial^{\alpha_1}}{\partial Z_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_{2n}}}{\partial Z_{2n}^{\alpha_{2n}}}, \quad Z^\alpha = Z_1^{\alpha_1} \cdots Z_{2n}^{\alpha_{2n}}. \tag{1.52}$$

Theorem 1.6 *There exist* $J_r(Z, Z') \in \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}$ $(x_0 \in X_s, s \in S, r \in \mathbb{N})$, *polynomials in* Z, Z' *with the same parity as* r *and with* $\deg J_r \leq 3r$, *whose* *coefficients are polynomials in* R^{TX}, $R^{T^{(1,0)}X}$, R^E *(and* R^L*) and their derivatives of order* $\leq r-2$ (and $\leq r$) *along the fiber X, and reciprocals of linear combinations of eigenvalues of* $\mathbf{J}$ *at* x_0*, such that by setting*

$$\mathscr{F}_{r,x_0}(Z,Z') = J_r(Z,Z')\mathscr{P}(Z,Z'), \quad J_0(Z,Z') = I_{\mathbb{C}\otimes E}\ , \tag{1.53}$$

the following statement holds: there exists $C''>0$*, such that for any* $k,m,m'\in\mathbb{N}$*, there exist* $N\in\mathbb{N}$, $C>0$ *such that for* $\alpha,\alpha'\in\mathbb{N}^{2n}$, $|\alpha|+|\alpha'|\leq m$*,* $Z,Z'\in T_{x_0}X$*,* $|Z|,|Z'|\leq\varepsilon$*,* $x_0\in X$*,* $p\geq p_0$*,*

$$\begin{aligned}&\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha\partial Z'^{\alpha'}}\left(\frac{1}{p^n}P_p(Z,Z')-\sum_{r=0}^{k}\mathscr{F}_r(\sqrt{p}Z,\sqrt{p}Z')\kappa^{-\frac{1}{2}}(Z)\kappa^{-\frac{1}{2}}(Z')p^{-\frac{r}{2}}\right)\right|_{\mathscr{C}^{m'}(W)}\\&\quad\leq Cp^{-(k+1-m)/2}(1+|\sqrt{p}Z|+|\sqrt{p}Z'|)^N\exp(-\sqrt{C''\mu_0}\sqrt{p}|Z-Z'|)\\&\qquad+\mathscr{O}(p^{-\infty}).\end{aligned} \tag{1.54}$$

The term $\mathscr{O}(p^{-\infty})$ *means that for any* $l,l_1\in\mathbb{N}$*, there exists* $C_{l,l_1}>0$ *such that its* $\mathscr{C}^{l_1}$*-norm is dominated by* $C_{l,l_1}p^{-l}$*.*

Proof Actually, in [22, Theorem 4.18′], they only explain for the family of data (g^{TX}, h^L, ∇^L, h^E, ∇^E) run over a set which are bounded in $\mathscr{C}^s$ and with g^{TX} bounded below. Here the complex structure J^{TX} can also be changed, still as explained after [22, (4.122)], the constants in [22, Theorems 4.11 and 4.15] will be uniformly bounded, especially, in [22, Theorem 4.11], we need to replace $\mathscr{C}^{m'}(X)$ therein by $\mathscr{C}^{m'}(W)$ as in (1.54). Finally, if we go through the argument in [22, Theorem 4.11], we can precise N in (1.54) by $2(n+k+m'+1)+m$ (cf. also [29], [30, (4.2.2)]). □

1.3 Family Bergman kernels

Recall that $R^{\mathrm{Ker}(D_p)}$ is the curvature operator in (1.17) associated with D_p acting on $\Omega^{0,\bullet}(X,L^p\otimes E)$.

Theorem 1.7 *For any* $l,m\in\mathbb{N}$ *and* $\varepsilon>0$*, there exists* $C_{l,m,\varepsilon}>0$ *such that for* $p>p_0$*,* $x,x'\in X$*,* $d^X(x,x')>\varepsilon$*,*

$$|R^{\mathrm{Ker}(D_p)}(x,x')|_{\mathscr{C}^m(W\times_S W)}\leq C_{l,m,\varepsilon}\,p^{-l}. \tag{1.55}$$

Proof Let $R_p^\Omega\in\Lambda^2(T^*S)\otimes\mathrm{End}(\Omega^{0,\bullet}(X,L^p\otimes E))$ be the curvature of the connection ∇_p^Ω. Then for $U,V\in TS$, by (1.15) and (1.36), we have

$$R_p^\Omega(U,V)=R^{E_p}(U^H,V^H)+d\mathbf{k}(U^H,V^H)-\nabla^{E_p}_{T(U^H,V^H)}. \tag{1.56}$$

By (1.24), we have

$$R^{\mathrm{Ker}(D_p)} = P_p R_p^{\Omega} P_p - P_p \nabla_p^{\Omega} P_p^{\perp} \nabla_p^{\Omega} P_p. \tag{1.57}$$

As $P_p^2 = P_p$, we get

$$P_p[\nabla_{U^H}^{E_p}, P_p] + [\nabla_{U^H}^{E_p}, P_p]P_p = [\nabla_{U^H}^{E_p}, P_p]. \tag{1.58}$$

Thus $[\nabla_{U^H}^{E_p}, P_p]$ exchanges $\mathrm{Ker}(D_p)$ and $(\mathrm{Ker}(D_p))^{\perp}$, the orthogonal complement of $\mathrm{Ker}(D_p)$, i.e.,

$$P_p[\nabla_{U^H}^{E_p}, P_p]P_p = 0. \tag{1.59}$$

Then from (1.57) and (1.59), we get

$$R^{\mathrm{Ker}(D_p)} = P_p R_p^{\Omega} P_p - P_p[\nabla_p^{\Omega}, P_p]P_p^{\perp}[\nabla_p^{\Omega}, P_p]P_p. \tag{1.60}$$

Now, from Proposition 1.5, Theorem 1.6, (1.56), we get Theorem 1.7. □

From (1.55), to understand the asymptotic expansion of $R^{\mathrm{Ker}(D_p)}(x, x')$ when $p \to +\infty$, we only need to restrict ourselves to $d^X(x, x') < \varepsilon$ for any $\varepsilon > 0$.

We will use the normal coordinates along the fiber X as above. Under this identification and (1.55), we will view $R^{\mathrm{Ker}(D_p)}(x, x')$ as a smooth section $R_{x_0}^{\mathrm{Ker}(D_p)}(Z, Z')$, $(Z, Z' \in B^{T_{x_0}X}(0, \varepsilon))$, of $\Lambda^2(T^*S) \otimes \pi_1^*(\mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E))$ on $TX \times_W TX$.

The following result is the first main result of this paper.

Theorem 1.8 *There exist $\mathscr{J}_r(Z, Z') \in \Lambda^2(T^*S) \otimes \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}$ $(x_0 \in X_s, s \in S, r \in \mathbb{N})$, polynomials in Z, Z' with the same parity as r and with $\deg \mathscr{J}_r \leq 3r$, whose coefficients are polynomials in R^{TX}, $R^{T^{(1,0)}X}$, R^E (and T, R^L) and their derivatives of order $\leq r-2$ (and $\leq r-1$, $\leq r$) and reciprocals of linear combinations of eigenvalues of $\mathbf{J}$ at x_0, such that by setting*

$$\mathscr{Q}_{r,x_0}(Z, Z') = \mathscr{J}_r(Z, Z')\mathscr{P}(Z, Z'),$$
$$\mathscr{J}_0(Z, Z') = -2\pi\sqrt{-1}\frac{(\omega^{n+1})^{(2)}}{(n+1)(\omega^n)^{(0)}} I_{\mathbb{C}\otimes E}\ , \tag{1.61}$$

the following statement holds: There exists $C'' > 0$ such that for any $k, m, m' \in \mathbb{N}$, there exist $N \in \mathbb{N}$ and $C > 0$ with

$$\left|\frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^{\alpha}\partial Z'^{\alpha'}}\left(\frac{1}{p^{n+1}}R^{\mathrm{Ker}(D_p)}(Z, Z') - \sum_{r=0}^{k}\mathscr{Q}_r(\sqrt{p}Z, \sqrt{p}Z')\kappa^{-\frac{1}{2}}(Z)\kappa^{-\frac{1}{2}}(Z')p^{-\frac{r}{2}}\right)\right|_{\mathscr{C}^{m'}(W)}$$
$$\leq Cp^{-(k+1-m)/2}(1 + |\sqrt{p}Z| + |\sqrt{p}Z'|)^N \exp(-\sqrt{C''\mu_0}\sqrt{p}|Z - Z'|) + \mathscr{O}(p^{-\infty}), \tag{1.62}$$

for any $\alpha, \alpha' \in \mathbb{N}^{2n}$*, with* $|\alpha| + |\alpha'| \le m$*, any* $Z, Z' \in T_{x_0}X$ *with* $|Z|, |Z'| \le \varepsilon$ *and any* $x_0 \in W$*,* $p \ge 1$*.*

In particular, set $b_{2,r}(x_0) = \mathcal{Q}_{2r,x_0}(0,0)$*, then* $b_{2,r} \in \mathscr{C}^\infty(W, \pi^*(\Lambda^2(T^*S)) \otimes \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E))$ *and for any* $k, l \in \mathbb{N}$*, there exists* $C_{k,l} > 0$ *such that for any* $p \in \mathbb{N}$*,* $p > p_0$*, we have*

$$\left| \frac{1}{p^{n+1}} R^{\mathrm{Ker}(D_p)}(x,x) - \sum_{r=0}^{k} b_{2,r}(x) p^{-r} \right|_{\mathscr{C}^l(W)} \le C_{k,l}\, p^{-k-1}. \tag{1.63}$$

Remark 1.9 From Theorems 1.7 and 1.8, we get immediately (cf. the argument of the proof of [32, Lemma 4.6]) the same type asymptotic expansion as in (1.55), (1.62) for $\frac{1}{p^{n+q}}(R^{\mathrm{Ker}(D_p)})^q(Z,Z')$ for $q > 1$.

Proof Let $\Gamma^{\mathrm{Cliff}}, \Gamma^L, \Gamma^E$ be the respective connection forms of $\nabla^{\mathrm{Cliff}}, \nabla^L, \nabla^E$ computed with respect to some frame of $\Lambda(T^{*(0,1)}X)$, L, E. Observe that

$$\nabla^{E_p} = d + \Gamma^{\mathrm{Cliff}} + p\,\Gamma^L + \Gamma^E. \tag{1.64}$$

By Proposition 1.5, Theorem 1.6, (1.36), (1.51), (1.60), and a rough computation, we know that there exist polynomials $\mathscr{J}_r(Z,Z') \in \Lambda^2(T^*S) \otimes \mathrm{End}(\Lambda(T^{*(0,1)}X) \otimes E)_{x_0}$ ($x_0 \in X_s, s \in S, r \ge -2$), such that under the notation in (1.62), we have

$$\begin{aligned} &\left| \frac{\partial^{|\alpha|+|\alpha'|}}{\partial Z^\alpha \partial Z'^{\alpha'}} \left(\frac{1}{p^{n+1}} R^{\mathrm{Ker}(D_p)}(Z,Z') - \sum_{r=-2}^{k} \mathcal{Q}_r(\sqrt{p}Z, \sqrt{p}Z')\kappa^{-\frac{1}{2}}(Z)\kappa^{-\frac{1}{2}}(Z')p^{-\frac{r}{2}} \right) \right|_{\mathscr{C}^{m'}(W)} \\ &\quad \le Cp^{-(k+1-m)/2}(1 + |\sqrt{p}Z| + |\sqrt{p}Z'|)^N \exp(-\sqrt{C''\mu_0}\sqrt{p}|Z - Z'|) \\ &\qquad + \mathscr{O}(p^{-\infty}). \end{aligned} \tag{1.65}$$

But we could not get the precise information on $\mathscr{J}_r$ as stated in Theorem.

It should also be possible to prove

$$\mathscr{J}_{-2} = \mathscr{J}_{-1} = 0, \quad \mathscr{J}_0 = -2\pi\sqrt{-1}\frac{(\omega^{n+1})^{(2)}}{(n+1)(\omega^n)^{(0)}} I_{\mathbb{C}\otimes E} \tag{1.66}$$

directly from the expansion (1.54), but it seems that it is quite complicate, and this does not give us a clear way to compute the coefficients $\mathscr{J}_r$.

In subsections 1.4 and 1.5, we will give a proof of Theorem 1.8 by introducing the superconnection in local family index theory, and in this way, we get also a general way to compute the coefficients $\mathscr{J}_r$. □

1.4 Superconnection and family Bergman kernels

For $x_0 \in X_{s_0}, s_0 \in S$, let $\mathcal{U}$ be an open neighborhood of $s_0 \in S$ such that $\pi^{-1}(\mathcal{U}) \simeq \mathcal{U} \times X_{s_0}$. Let $\mathcal{U}_1 \subset \mathcal{U}$ be an open neighborhood of $s_0 \in S$ such that $\overline{\mathcal{U}_1} \subset \mathcal{U}$.

We identify L_Z, E_Z and $(E_p)_Z$ for $Z \in B^{T_{(s,x_0)}X}(0,\varepsilon)$ to $L_{(s,x_0)}, E_{(s,x_0)}$ and $(E_p)_{(s,x_0)}$ by parallel transport with respect to the connections ∇^L, ∇^E and ∇^{E_p} along the curve $\gamma_Z : [0,1] \ni u \to uZ$. Let $\{e_i\}$ be an oriented orthonormal basis of $T_{(s,x_0)}X$. We also denote by $\{e^i\}$ the dual basis of $\{e_i\}$. Let $\widetilde{e}_i(Z)$ be the parallel transport of e_i with respect to ∇^{TX} along the above curve.

Now, for $\varepsilon > 0$ small enough, we will extend the geometric objects on $B^{T_{(s,x_0)}X}(0,\varepsilon)|_{\mathcal{U}}$ to $\mathcal{U} \times \mathbb{R}^{2n} \simeq T_{(s,x_0)}X|_{\mathcal{U}}$ (here we identify $(Z_1, \cdots, Z_{2n}) \in \mathbb{R}^{2n}$ to $\sum_i Z_i e_i \in T_{(s,x_0)}X =: X_0$) such that D_p is the restriction of a spin^c Dirac operator on $\mathbb{R}^{2n}$ associated with a Hermitian line bundle with positive curvature. In this way, we can replace $\pi^{-1}(\mathcal{U})$ by $\mathcal{U} \times \mathbb{R}^{2n}$.

First of all, we denote L_0, E_0 the bundles $L|_{\mathcal{U}\times\{x_0\}}$, $E|_{\mathcal{U}\times\{x_0\}}$ lifted on $W_0 = \mathcal{U} \times \mathbb{R}^{2n}$. And we still denote by ∇^L, ∇^E, h^L etc. the connections and metrics on L_0, E_0 on $B^{T_{(s,x_0)}X}(0,4\varepsilon)|_{\mathcal{U}}$ induced by the above identification. Then h^L, h^E are identified with the metrics $h^{L_0} = h^{L_{(s,x_0)}}$, $h^{E_0} = h^{E_{(s,x_0)}}$. Let $\mathcal{R} = \sum_i Z_i e_i = Z$ be the radial vector field on $\mathcal{U} \times \mathbb{R}^{2n}$.

Let $\rho : \mathbb{R} \to [0,1]$ be a smooth even function such that

$$\rho(v) = 1 \ \text{ if } \ |v| < 2; \quad \rho(v) = 0 \ \text{ if } |v| > 4. \tag{1.67}$$

Let $\varphi_\varepsilon : \mathcal{U} \times \mathbb{R}^{2n} \to \mathcal{U} \times \mathbb{R}^{2n}$ be the map defined by $\varphi_\varepsilon(s,Z) = (s, \rho(|Z|/\varepsilon)Z)$. Let $g_s^{TX_0}(Z) = g^{TX}(\varphi_\varepsilon(s,Z))$ be the metric on TX_0. Set $T^H_{(s,Z)}W_0 = T^H_{\varphi_\varepsilon(s,Z)}W$.

Let $\nabla^{E_0} = \varphi_\varepsilon^*\nabla^E$, then ∇^{E_0} is the extension of ∇^E on $B^{T_{(s,x_0)}X}(0,\varepsilon)|_{\mathcal{U}}$. Let ∇^{L_0} be the Hermitian connection on (L_0, h^{L_0}) defined by

$$\nabla^{L_0}|_{(s,Z)} = \varphi_\varepsilon^*\nabla^L + \frac{1}{2}(1-\rho^2(|Z|/\varepsilon))R^L_{(s,x_0)}(\mathcal{R}, \cdot). \tag{1.68}$$

Then for ε small enough, by [22, (4.24)], the curvature R^{L_0} of ∇^{L_0} is non-degenerate along $\mathbb{R}^{2n}$ and $R^{L_0}_{(s,Z)} = R^L_{(s,x_0)}$ and $T^H_{(s,Z)}W_0 = T^H_{(s,x_0)}W$ for $|Z| > 4\varepsilon$.

Let J_0 be the almost complex structure on TX_0 compatible with $\frac{\sqrt{-1}}{2\pi}R^{L_0}$ and such that g^{TX_0} is J_0-invariant (If we define $A \in \mathrm{End}(TX_0)$ by $g^{TX_0}(AX,Y) = \frac{\sqrt{-1}}{2\pi}R^{L_0}(X,Y)$, then $J_0 = A(-A^2)^{-1/2}$). Thus we have $J = J_0$ for $|Z| < 2\varepsilon$ and $J_0(Z) = J_{x_0}$ for $|Z| > 4\varepsilon$.

Then R^{L_0} is positive in the sense of (1.32) for ε small enough, and the corresponding constant μ_0 for R^{L_0} is bigger than $\frac{4}{5}\mu_0$. From now on, we fix ε as above.

Let $T^{*(0,1)}X_0$ be the anti-holomorphic cotangent bundle of (X_0, J_0). Let $\nabla^{\mathrm{Cliff}_0}$ be the Clifford connection on $\Lambda(T^{*(0,1)}X_0)$ induced by the connection ∇^{TX_0} on (TX_0, g^{TX_0}) as in Section 1.1 for the fibration $\mathcal{U} \times X_0 \to \mathcal{U}$. Let R^{E_0}, R^{TX_0}, R^{Cliff_0} be the corresponding curvatures on E_0, TX_0 and $\Lambda(T^{*(0,1)}X_0)$.

We identify $\Lambda(T^{*(0,1)}X_0)_Z$ with $\Lambda(T^{*(0,1)}_{x_0}X)$ by using the parallel transport with respect to the connection $\nabla^{\mathrm{Cliff}_0}$ along the curve γ_Z. Let S_L be a unit section of $L|_{\mathcal{U}\times\{x_0\}}$ over $\mathcal{U} \times \{x_0\}$. Using S_L and the above discussion, we get an isometry $E_{0,p} := \Lambda(T^{*(0,1)}X_0) \otimes E_0 \otimes L_0^p \simeq (\Lambda(T^{*(0,1)}X) \otimes E)_{x_0} =: \mathbf{E}_{x_0}$.

Let $D_p^{X_0}$ (resp. $\nabla^{E_{0,p}}$) be the Dirac operator on X_0 (resp. the connection on $E_{0,p}$) associated with the above data by the construction in Section 1.2. By the argument in [28, p. 656-657], we know that Theorem 1.5 still holds for $D_p^{X_0}$. In particular, there exists $C > 0$ such that

$$\operatorname{Spec}(D_p^{X_0})^2 \subset \{0\} \cup \Big[\frac{8}{5} p\mu_0 - C, +\infty\Big[. \tag{1.69}$$

Let $P_{0,p}$ be the orthogonal projection from $\Omega_0^{0,\bullet}(X_0, L_0^p \otimes E_0) \simeq \mathscr{C}_0^\infty(X_0, \mathbf{E}_{x_0})$ on $\operatorname{Ker}(D_p^{X_0})$, and let $P_{0,p}(x,x')$ be the smooth kernel of $P_{0,p}$ with respect to the Riemannian volume form $dv_{X_0}(x')$.

Proposition 1.10 *For any $l, m \in \mathbb{N}$, there exists $C_{l,m} > 0$ such that for $x, x' \in B^{T_{(s,x_0)}X}(0,\varepsilon)$,*

$$\Big|(P_{0,p} - P_p)(x,x')\Big|_{\mathscr{C}^m(\mathcal{U}_1\times\mathbb{R}^{2n}\times\mathbb{R}^{2n})} \leq C_{l,m} p^{-l}. \tag{1.70}$$

Proof Using (1.39) and (1.69), we know that $P_{0,p} - F(D_p)$ verifies (1.40) for $x, x' \in B^{T_{(s,x_0)}X}(0,\varepsilon)|_{\mathcal{U}_1}$, thus we get (1.70). □

Set

$$R^{\operatorname{Ker}(D_p^{X_0})} = P_{0,p} R_{0,p}^{\Omega} P_{0,p} - P_{0,p} \nabla_{0,p}^{\Omega} P_{0,p}^{\perp} \nabla_{0,p}^{\Omega} P_{0,p}, \text{ with } P_{0,p}^{\perp} = 1 - P_{0,p}. \tag{1.71}$$

Let $R^{\operatorname{Ker}(D_p^{X_0})}(x,x') \in \Lambda^2(T^*S) \otimes \operatorname{End}(\Lambda(T^{*(0,1)}X) \otimes E))_{(s,x_0)}$ be the smooth kernel of the operator $R^{\operatorname{Ker}(D_p^{X_0})}$ with respect to $dv_{X_0}(x')$. As all geometric data on $\mathcal{U} \times B^{T_{(s,x_0)}X}(0,2\varepsilon)$ inherit from the corresponding geometric data on W, thus $\nabla_{0,p}^{\Omega}$, $D_{0,p}$ are the same as ∇_p^{Ω}, D_p on $\mathcal{U} \times B^{T_{(s,x_0)}X}(0,2\varepsilon)$. By replacing $P_{0,p}$, P_p by $F(D_p)$ as in (1.40) and (1.70), from (1.24) and (1.71), we get that for any $l, m \in \mathbb{N}$, there exists $C > 0$ such that for $x, x' \in B^{T_{(s,x_0)}X}(0,\varepsilon)$,

$$\Big|(R^{\operatorname{Ker}(D_p^{X_0})} - R^{\operatorname{Ker}(D_p)})(x,x')\Big|_{\mathscr{C}^m(\mathcal{U}_1\times\mathbb{R}^{2n}\times\mathbb{R}^{2n})} \leq C_{l,m} p^{-l}. \tag{1.72}$$

As we know from (1.24) that the term $B^{(2)}$ does not play any role in the construction of $R^{\operatorname{Ker}(D)}$, thus we will choose the superconnection with $B^{(2)} = 0$, more precisely, set

$$B_p = D_p + \nabla_p^{\Omega}, \quad B_{0,p} = D_{0,p} + \nabla_{0,p}^{\Omega}. \tag{1.73}$$

Then

$$B_p = B_{0,p} \quad \text{on } B^{T_{(s,x_0)}X}(0,2\varepsilon)|_{\mathcal{U}}, \quad P_{0,p}(B_{0,p}^2)^{(1)} P_{0,p} = 0. \tag{1.74}$$

From (1.69) and (1.74), as in (1.24), for $p > 5C/4\mu_0$, we have

$$R^{\mathrm{Ker}(D_p^{X_0})} = P_{0,p}R^{\Omega}_{0,p}P_{0,p} - P(B^2_{0,p})^{(1)}((B^2_{0,p})^{(0)})^{-1}P^{\perp}_{0,p}(B^2_{0,p})^{(1)}P_{0,p}$$
$$= \frac{1}{2\pi\sqrt{-1}}\left[\int_{|\lambda|=p\mu_0}(\lambda - B^2_{0,p})^{-1}\lambda d\lambda\right]^{(2)} \tag{1.75}$$
$$= \frac{p}{2\pi\sqrt{-1}}\left[\int_{|\lambda|=\mu_0}\left(\lambda - \frac{1}{p}B^2_{0,p}\right)^{-1}\lambda d\lambda\right]^{(2)}.$$

From (1.75) and as explained in (1.21) and (1.22), $B^2_{0,p}$ is a second order elliptic operator along $\mathbb{R}^{2n}$, we know that to study the asymptotics of $R^{\mathrm{Ker}(D_p^{X_0})}$, we only need to work fiberwisely. Now, we will only work on the fiber X_{s_0} with center x_0.

To define an L^2-norm, we fix a metric g^{TS} on TS, and let $h^{\Lambda\otimes\mathbf{E}}$ be the metric on $\Lambda(T^*S)\widehat{\otimes}\mathbf{E}$ induced by g^{TS}, g^{TX} and h^E. Let $\langle\, ,\, \rangle_0$ be the scalar product on $\mathscr{C}^\infty_0(\mathbb{R}^{2n}, \Lambda(T^*S)\,\widehat{\otimes}\mathbf{E}_{x_0})$ induced by $h^{\Lambda\otimes\mathbf{E}}_{x_0}$ and dv_{TX} as in (1.10).

We denote by $\mathcal{R} = \sum_i Z_i e_i = Z$ the radial vector field on $\mathbb{R}^{2n}$. For $\sigma \in \mathscr{C}^\infty(\mathbb{R}^{2n}, \Lambda(T^*S)\widehat{\otimes}\mathbf{E}_{x_0})$, $Z \in \mathbb{R}^{2n}$, and for $t = \frac{1}{\sqrt{p}}$, set

$$\begin{aligned}&(S_t\sigma)(Z) = \sigma(Z/t), \quad \nabla_t = S_t^{-1}t\kappa^{1/2}\nabla^{E_{0,p}}\kappa^{-1/2}S_t,\\ &\nabla_{0,\cdot} = \nabla_{\cdot} + \frac{1}{2}R^L_{x_0}(\mathcal{R},\cdot)\,, \quad \mathscr{L}_t = S_t^{-1}\kappa^{1/2}t^2B^2_{0,p}\kappa^{-1/2}S_t.\end{aligned} \tag{1.76}$$

By (1.19), (1.69), (1.73) and (1.76), we have

$$\begin{aligned}&\mathscr{L}_t = \mathscr{L}_t^{(0)} + \mathscr{L}_t^{(1)} + \mathscr{L}_t^{(2)},\\ &\mathrm{Spec}(\mathscr{L}_t) = \mathrm{Spec}(\mathscr{L}_t^{(0)}) \subset \{0\} \cup \left[\frac{8}{5}\mu_0 - Ct^2, +\infty\right[.\end{aligned} \tag{1.77}$$

From (1.77), set

$$\mathscr{P}_t = \frac{1}{2\pi\sqrt{-1}}\left[\int_{|\lambda|=\mu_0}(\lambda - \mathscr{L}_t)^{-1}\lambda\, d\lambda\right]^{(2)}. \tag{1.78}$$

Let $\mathscr{P}_t(Z, Z')(Z, Z' \in \mathbb{R}^{2n})$ be the smooth kernel of the operator $\mathscr{P}_t$ with respect to $dv_{T_{(s,x_0)}X}(Z')$. Then by (1.75), (1.76) and (1.78) as in [30, (1.6.66)], we get with $t = \frac{1}{\sqrt{p}}$,

$$R^{\mathrm{Ker}(D_p^{X_0})}(Z, Z') = t^{-2n-2}\kappa^{-1/2}(Z)\mathscr{P}_t(Z/t, Z'/t)\kappa^{-1/2}(Z'). \tag{1.79}$$

From (1.72) and (1.79), to study the asymptotic expansion of $R^{\mathrm{Ker}(D_p)}(x, x')$, we need only to study the asymptotic expansion of $\mathscr{P}_t(Z, Z')$ which involves superconnections.

1.5 Taylor expansion of $\mathscr{L}_t$ and the spectrum of $\mathscr{L}_0$

Set (with ω_d, $\mathscr{L}$ in (1.33), (1.50))

$$\begin{aligned}&\mathscr{L}_0^{(0)} := \mathscr{L} - 2\omega_{d,x_0}, \quad \mathscr{L}_0^{(1)} := \mathcal{O}_0^{(1)} := f^\alpha \wedge c(e_i)\, R^L_{x_0}(f^H_\alpha, e_i),\\ &\mathscr{L}_0^{(2)} := \mathcal{O}_0^{(2)} := \frac{1}{2} f^\alpha \wedge f^\beta R^L_{x_0}(f^H_\alpha, f^H_\beta), \quad \mathscr{L}_0 = \mathscr{L}_0^{(0)} + \mathscr{L}_0^{(1)} + \mathscr{L}_0^{(2)}.\end{aligned} \tag{1.80}$$

Let $\mathcal{O}_1^{(0)} := \mathscr{Q}_1, \mathcal{O}_2^{(0)} := \mathscr{Q}_2$ be given in [29, Theorem 2.2]. Let $(\partial^\alpha R^L)_{x_0}$ be the tensor $(\partial^\alpha R^L)_{x_0}(e_i, e_j) = \partial^\alpha (R^L(e_i, e_j))_{x_0}$. Set also

$$\begin{aligned}\mathcal{O}_1^{(1)} =& f^\alpha \wedge c(e_l)\Big[-\nabla_{0, T(f^H_\alpha, e_l)_{x_0}} + \nabla_Z (R^L(f^H_\alpha, \cdot))_{x_0}(e_l)\Big],\\ \mathcal{O}_2^{(1)} =& f^\alpha \wedge c(e_l)\Big\{\Big[\frac{1}{4}\langle R^{TX} e_i, e_j\rangle c(e_i)c(e_j) + \frac{1}{2}\operatorname{Tr}\Big[R^{T^{(1,0)}X}\Big] + R^E\Big]_{x_0}(f^H_\alpha, e_l)\\ &+ \frac{1}{2}(\nabla\nabla(R^L(f^H_\alpha, \widetilde{e}_l)))_{x_0,(Z,Z)} - e_l \mathbf{k}(f^H_\alpha)_{x_0}\\ &- \langle(\nabla^{TX}_Z T(f^H_\alpha, \cdot))(e_l), e_i\rangle_{x_0} \nabla_{0,e_i} - \frac{1}{3}(\partial_k R^L)_{x_0} Z_k (\mathcal{R}, T(f^H_\alpha, e_l)_{x_0})\Big\},\end{aligned} \tag{1.81}$$

$$\begin{aligned}\mathcal{O}_1^{(2)} =& \frac{1}{2} f^\alpha \wedge f^\beta\Big[-\nabla_{0, T(f^H_\alpha, f^H_\beta)_{x_0}} + \nabla_Z (R^L(f^H_\alpha, f^H_\beta))_{x_0}\Big],\\ \mathcal{O}_2^{(2)} =& \frac{1}{2} f^\alpha \wedge f^\beta\Big\{\Big[\frac{1}{4}\langle R^{TX} e_i, e_j\rangle c(e_i)c(e_j) + \frac{1}{2}\operatorname{Tr}\Big[R^{T^{(1,0)}X}\Big] + R^E\Big]_{x_0}(f^H_\alpha, f^H_\beta)\\ &+ \frac{1}{2}(\nabla\nabla(R^L(f^H_\alpha, f^H_\beta)))_{x_0,(Z,Z)} + d\mathbf{k}(f^H_\alpha, f^H_\beta)_{x_0}\\ &- \langle \nabla^{TX}_Z (T(f^H_\alpha, f^H_\beta)), e_i\rangle_{x_0} \nabla_{0,e_i} - \frac{1}{3}(\partial_k R^L)_{x_0} Z_k(\mathcal{R}, T(f^H_\alpha, f^H_\beta)_{x_0})\Big\}.\end{aligned}$$

The operator $\mathscr{L}_t$ is the rescaled operator in (1.76), which we now develop in Taylor series.

Theorem 1.11 *There exist polynomials $\mathcal{A}_{i,j,r}$ (resp. $\mathcal{B}_{i,r}$, $\mathcal{C}_r$) ($r \in \mathbb{N}, i, j \in \{1, \cdots, 2n\}$) in Z with the following properties:*

- *their coefficients are polynomials in R^{TX} (resp. $d\mathbf{k}$, T, R^{TX}, $R^{T^{(1,0)}X}$, R^L, R^E) and their derivatives along the fiber X at x_0 up to order $r-2$ (resp. $r-2$, $r-1$, $r-2$, $r-2$, r, $r-2$),*
- *$\mathcal{A}_{i,j,r}$ is a monomial in Z of degree r, the degree in Z of $\mathcal{B}_{i,r}$ (resp. $\mathcal{C}_r$) has the same parity with $r-1$ (resp. r),*
- *if we denote by*

$$\mathcal{O}_r = \mathcal{A}_{i,j,r}\nabla_{e_i}\nabla_{e_j} + \mathcal{B}_{i,r}\nabla_{e_i} + \mathcal{C}_r, \tag{1.82}$$

then

$$\mathscr{L}_t = \mathscr{L}_0 + \sum_{r=1}^{m} t^r \mathcal{O}_r + \mathscr{O}(t^{m+1}), \tag{1.83}$$

and there exists $m' \in \mathbb{N}$ such that for any $k \in \mathbb{N}$, $|t| \leq 1$ the derivatives of order $\leq k$ of the coefficients of the operator $\mathscr{O}(t^{m+1})$ are dominated by $Ct^{m+1}(1+|Z|)^{m'}$. Moreover $\mathscr{L}_0$, $\mathcal{O}_1$, $\mathcal{O}_2$ are given by (1.80) and (1.81).

Proof Now, by using (1.21), (1.22) and (1.36), we have

$$\begin{aligned}
&(B_p^2)^{(0)} = D_p^2, \quad (B_p^2)^{(2)} = R_p^{\Omega},\\
&(B_p^2)^{(1)} = f^\alpha \wedge c(e_i)\Big[R^{E_p}(f_\alpha^H, e_i) - e_i \mathbf{k}(f_\alpha^H) - \nabla^{E_p}_{T(f_\alpha^H, e_i)}\Big].
\end{aligned} \tag{1.84}$$

By (1.74), (1.84), we have established (1.83) for $\mathscr{L}_t^{(0)}$ in [22, Theorem 4.6], (cf. also [29, Theorem 2.2]), moreover ' $\mathscr{L}_0^{(0)}$, $\mathcal{O}_1^{(0)}$, $\mathcal{O}_2^{(0)}$ were also computed in [29, Theorem 2.2].

By (1.9), (1.15), (1.36), (1.56) and (1.84),

$$\begin{aligned}
\mathscr{L}_t^{(1)} =& f^\alpha \wedge c(\widetilde{e}_i)\Big\{t^2\Big[\frac{1}{4}\langle R^{TX}\widetilde{e}_l, \widetilde{e}_m\rangle c(\widetilde{e}_l)c(\widetilde{e}_m) + \frac{1}{2}\operatorname{Tr}[R^{T^{(1,0)}X}]\Big](f_\alpha^H, \widetilde{e}_i)_{tZ}\\
&+ t^2 R^E(f_\alpha^H, \widetilde{e}_i)_{tZ} + R^L(f_\alpha^H, \widetilde{e}_i)_{tZ} - t^2\widetilde{e}_i\mathbf{k}(f_\alpha^H)_{tZ} - t\nabla_{t,T(f_\alpha^H,\widetilde{e}_i)_{tZ}}\Big\},\\
\mathscr{L}_t^{(2)} =& \frac{1}{2} f^\alpha \wedge f^\beta \Big\{t^2\Big[\frac{1}{4}\langle R^{TX}\widetilde{e}_l, \widetilde{e}_m\rangle c(\widetilde{e}_l)c(\widetilde{e}_m) + \frac{1}{2}\operatorname{Tr}[R^{T^{(1,0)}X}]\Big](f_\alpha^H, f_\beta^H)_{tZ}\\
&+t^2 R^E(f_\alpha^H, f_\beta^H)_{tZ} + R^L(f_\alpha^H, f_\beta^H)_{tZ} + t^2 d\mathbf{k}(f_\alpha^H, f_\beta^H)_{tZ} - t\nabla_{t,T(f_\alpha^H, f_\beta^H)_{tZ}}\Big\}.
\end{aligned} \tag{1.85}$$

On $B^{T_{x_0}X}(0, 2\varepsilon/t)$, by [22, (4.46), (4.48)] (cf. [30, (1.2.30), (4.1.34)]), we have

$$\nabla_{t,e_i}|_Z = \nabla_{e_i} + \Big(\frac{1}{2}R^L_{x_0} + \frac{t}{3}(\partial_k R^L)_{x_0} Z_k\Big)(\mathcal{R}, e_i) + \mathscr{O}(t^2). \tag{1.86}$$

Moreover, as we use the normal coordinates, we have (cf. [30, Lemma 1.2.3])

$$\widetilde{e}_i(Z) = e_i - \frac{1}{6}\sum_j \Big\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_j\Big\rangle e_j + \mathscr{O}(|Z|^3). \tag{1.87}$$

By the definition of ∇^{Cliff}, for $X, Y \in \mathscr{C}^\infty(X, TX)$, we have

$$[\nabla^{\text{Cliff}}_X, c(Y)] = c(\nabla^{TX}_X Y). \tag{1.88}$$

Note that on $B^{T_{x_0}X}(0, 2\varepsilon)$, we trivialize $\Lambda(T^{*(0,1)}X)$ by using ∇^{Cliff} along the curve $u \to uZ$ and $\nabla^{TX}_Z \widetilde{e}_j = 0$, we get as in [5, Lemma 4.13], $c(\widetilde{e}_j)$ is the constant endomorphism $c(e_j)$. From (1.85), we get the expansion (1.83) for $\mathscr{L}_t^{(1)}$, $\mathscr{L}_t^{(2)}$. Especially, their leading terms are $\mathscr{L}_0^{(1)}$, $\mathscr{L}_0^{(2)}$ in (1.80). From (1.85), (1.86) and (1.87), we get the coefficients of the expansions for $\mathscr{L}_t^{(1)}$, $\mathscr{L}_t^{(2)}$ in (1.81). □

Now we discuss the eigenvalues and eigenfunctions of $\mathscr{L}_0^{(0)}$ in a more precise way. We choose $\{w_i\}_{i=1}^n$, an orthonormal basis of $T_{x_0}^{(1,0)}X$, such that

$$-2\pi\sqrt{-1}\mathbf{J}_{x_0} = \mathrm{diag}(a_1, \cdots, a_n) \in \mathrm{End}(T_{x_0}^{(1,0)}X), \tag{1.89}$$

with $0 < a_1 \le a_2 \le \cdots \le a_n$, and let $\{w^j\}_{j=1}^n$ be its dual basis. Then $e_{2j-1} = \frac{1}{\sqrt{2}}(w_j + \overline{w}_j)$ and $e_{2j} = \frac{\sqrt{-1}}{\sqrt{2}}(w_j - \overline{w}_j)$, $j = 1, \ldots, n$ form an orthonormal basis of $T_{x_0}X$. We use the coordinates on $T_{x_0}X \simeq \mathbb{R}^{2n}$ induced by $\{e_i\}$ as in (1.49) and in what follows we also introduce the complex coordinates $z = (z_1, \cdots, z_n)$ on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Thus $Z = z + \overline{z}$, and $w_i = \sqrt{2}\frac{\partial}{\partial z_i}$, $\overline{w}_i = \sqrt{2}\frac{\partial}{\partial \overline{z}_i}$. We will also identify z to $\sum_i z_i \frac{\partial}{\partial z_i}$ and $\overline{z}$ to $\sum_i \overline{z}_i \frac{\partial}{\partial \overline{z}_i}$ when we consider z and $\overline{z}$ as vector fields. Remark that

$$\left|\frac{\partial}{\partial z_i}\right|^2 = \left|\frac{\partial}{\partial \overline{z}_i}\right|^2 = \frac{1}{2}, \quad \text{so that } |z|^2 = |\overline{z}|^2 = \frac{1}{2}|Z|^2 . \tag{1.90}$$

It is very useful to rewrite $\mathscr{L}_0^{(0)}$ by using the creation and annihilation operators. Set

$$b_i = -2\nabla_{0,\frac{\partial}{\partial z_i}}, \quad b_i^+ = 2\nabla_{0,\frac{\partial}{\partial \overline{z}_i}}, \quad b = (b_1, \cdots, b_n) . \tag{1.91}$$

Then by (1.76) and (1.89), we have

$$b_i = -2\frac{\partial}{\partial z_i} + \frac{1}{2}a_i\overline{z}_i , \quad b_i^+ = 2\frac{\partial}{\partial \overline{z}_i} + \frac{1}{2}a_i z_i , \tag{1.92}$$

and for any polynomial $g(z, \overline{z})$ on z and $\overline{z}$,

$$\begin{aligned}
&[b_i, b_j^+] = b_i b_j^+ - b_j^+ b_i = -2a_i\delta_{ij} ,\\
&[b_i, b_j] = [b_i^+, b_j^+] = 0 ,\\
&[g(z,\overline{z}), b_j] = 2\frac{\partial}{\partial z_j}g(z,\overline{z}), \quad [g(z,\overline{z}), b_j^+] = -2\frac{\partial}{\partial \overline{z}_j}g(z,\overline{z}) .
\end{aligned} \tag{1.93}$$

By (1.33) and (1.89), $\tau_{x_0} = \sum_i a_i$. Thus from (1.50), (1.80), (1.89) and (1.91)-(1.93),

$$\mathscr{L} = \sum_j b_j b_j^+, \quad \mathscr{L}_0^{(0)} = \sum_j b_j b_j^+ + 2\sum_j a_j \overline{w}^j \wedge i_{\overline{w}_j} . \tag{1.94}$$

The following result was established in [31, Theorem 1.15] (cf. [30, Theorem 4.1.20]).

Theorem 1.12 *The spectrum of the restriction of $\mathscr{L}$ on $L^2(\mathbb{R}^{2n})$ is given by*

$$\operatorname{Spec}(\mathscr{L}|_{L^2(\mathbb{R}^{2n})}) = \Big\{2\sum_{i=1}^{n}\alpha_i a_i \,:\, \alpha=(\alpha_1,\cdots,\alpha_n)\in\mathbb{N}^n\Big\} \tag{1.95}$$

and an orthogonal basis of the eigenspace of $2\sum_{i=1}^{n}\alpha_i a_i$ *is given by*

$$\begin{aligned} &b^{\alpha}\Big(z^{\beta}\exp\big(-\frac{1}{4}\sum_i a_i|z_i|^2\big)\Big), \quad \textit{with } b^{\alpha}=\prod_{j=1}^{n}b_j^{\alpha_j},\\ &z^{\beta}=\prod_{j=1}^{n}z_j^{\beta_j},\ \ \beta\in\mathbb{N}^n. \end{aligned} \tag{1.96}$$

From Theorem 1.12, we know $\mathscr{P}(Z,Z')$ in (1.51) is the smooth kernel of the orthogonal projection from $L^2(\mathbb{R}^{2n})$ onto $\operatorname{Ker}(\mathscr{L}|_{L^2(\mathbb{R}^{2n})})$. Moreover, from (1.94), we have

$$\begin{aligned} &\operatorname{Ker}(\mathscr{L}|_{L^2(\mathbb{R}^{2n})}) = \cap_j \operatorname{Ker}(b_j^+),\\ &\mathscr{P}(Z,Z') = \frac{1}{(2\pi)^n}\Big(\prod_{i=1}^{n}a_i\Big)\ \exp\Big(-\frac{1}{4}\sum_i a_i\big(|z_i|^2+|z_i'|^2-2z_i\overline{z}_i'\big)\Big). \end{aligned} \tag{1.97}$$

Let $P^N(Z,Z')$ be the smooth kernel of the orthogonal projection P^N from $L^2(\mathbb{R}^{2n},\Lambda(T^*S)\widehat{\otimes}\mathbf{E}_{x_0})$ onto $\operatorname{Ker}(\mathscr{L}_0^{(0)})$. Set $P^{N^{\perp}}=1-P^N$.

Recall that we denote by $I_{\mathbb{C}\otimes E}$ the orthogonal projection from $\mathbf{E}:=\Lambda(T^{*(0,1)}X)\otimes E$ onto $\mathbb{C}\otimes E$. Then by (1.94), we have

$$P^N(Z,Z') = \mathscr{P}(Z,Z')I_{\mathbb{C}\otimes E}. \tag{1.98}$$

From (1.80), we get

$$\begin{aligned} \mathscr{O}_0^{(1)} &= f^{\alpha}\wedge\Big(c(\overline{w}_j)R^L_{x_0}(f^H_{\alpha},w_j)+c(w_j)R^L_{x_0}(f^H_{\alpha},\overline{w}_j)\Big)\\ &= \sqrt{2}f^{\alpha}\wedge\Big(-i_{\overline{w}_j}R^L_{x_0}(f^H_{\alpha},w_j)+\overline{w}^jR^L_{x_0}(f^H_{\alpha},\overline{w}_j)\Big). \end{aligned} \tag{1.99}$$

From (1.98) and (1.99), we get

$$P^N\mathscr{O}_0^{(1)}P^N = 0. \tag{1.100}$$

1.6 Evaluation of $\mathcal{Q}_r$: a proof of Theorem 1.8

Let $\mathcal{P}_t$ be the orthogonal projection from $\mathscr{C}_0^\infty(X_0, \Lambda(T^*S)\widehat{\otimes}\mathbf{E}_{x_0})$ onto the kernel of $\mathscr{L}_t^{(0)}$ with respect to $\langle\ ,\ \rangle_0$. From (1.19), (1.23), (1.73) and (1.76), we have

$$\begin{aligned}\left[(\lambda-\mathscr{L}_t)^{-1}\right]^{(2)} =&(\lambda-\mathscr{L}_t^{(0)})^{-1}\mathscr{L}_t^{(1)}(\lambda-\mathscr{L}_t^{(0)})^{-1}\mathscr{L}_t^{(1)}(\lambda-\mathscr{L}_t^{(0)})^{-1}\\ &+(\lambda-\mathscr{L}_t^{(0)})^{-1}\mathscr{L}_t^{(2)}(\lambda-\mathscr{L}_t^{(0)})^{-1},\\ \mathcal{P}_t\mathscr{L}_t^{(1)}\mathcal{P}_t =&0.\end{aligned} \tag{1.101}$$

The following equation is an analogue of [31, (1.55)]: by (1.77), (1.78), (1.101) and the residue formula, we have for any $k\geq 1$,

$$\begin{aligned}\mathscr{P}_t =& \frac{1}{2\pi k\sqrt{-1}}\int_{|\lambda|=\mu_0}\lambda^k\sum_{i=1}^k(\lambda-\mathscr{L}_t^{(0)})^{-i}\\ &\qquad\left[\mathscr{L}_t^{(2)}+\mathscr{L}_t^{(1)}(\lambda-\mathscr{L}_t^{(0)})^{-1}\mathscr{L}_t^{(1)}\right](\lambda-\mathscr{L}_t^{(0)})^{-k+i-1}d\lambda\\ =&\frac{1}{2\pi k\sqrt{-1}}\left[\int_{|\lambda|=\mu_0}\lambda^k(\lambda-\mathscr{L}_t)^{-k}d\lambda\right]^{(2)}.\end{aligned} \tag{1.102}$$

We define first the Sobolev norm $\|\ \ \|_{t,m}$ for $m\in\mathbb{N}$ on $\mathscr{C}_0^\infty(\mathbb{R}^{2n}, \Lambda(T^*S)\widehat{\otimes}\mathbf{E}_{x_0})$ by using ∇_{t,e_j} and $\langle\ ,\ \rangle_0$ as in [30, (4.1.36)]. Note that $\mathscr{L}_t^{(0)}$ is L_2^t in [22, (4.37)], by (1.77) and (1.85), we know that the analogue of [22, Theorem 4.7] holds for $\mathscr{L}_t$: There exist $C_1, C_2, C_3>0$ such that for $t\in]0,1]$ and any $s, s'\in\mathscr{C}_0^\infty(\mathbb{R}^{2n}, \Lambda(T^*S)\widehat{\otimes}\mathbf{E}_{x_0})$, we have

$$\begin{aligned}&\mathrm{Re}\,\langle\mathscr{L}_t s, s\rangle_{t,0}\geq C_1\|s\|_{t,1}^2-C_2\|s\|_{t,0}^2,\\ &|\left\langle\mathscr{L}_t s, s'\right\rangle_{t,0}|\leq C_3\|s\|_{t,1}\|s'\|_{t,1}.\end{aligned} \tag{1.103}$$

Thus [22, Theorems 4.8-4.10] hold for $\mathscr{L}_t$. From (1.102), we can proceed as in the proof of [30, Theorems 4.1.13-4.1.18] and get that there exist functions $\mathcal{Q}_r$ on Z, Z' such that for $t\in]0,1]$, $q>0$, $Z, Z'\in T_{(s,x_0)}X$, $|Z|, |Z'|\leq q$, we have

$$\left|\mathscr{P}_t(Z,Z')-\sum_{r=0}^k\mathcal{Q}_r(Z,Z')t^r\right|_{\mathscr{C}^{m'}(W)}\leq Ct^{k+1}. \tag{1.104}$$

Comparing with (1.65), (1.72), (1.79) and (1.104), we get in (1.65),

$$\mathcal{Q}_{-2}(Z,Z')=\mathcal{Q}_{-1}(Z,Z')=0. \tag{1.105}$$

Remark 1.13 A direct alternate way to obtain (1.62) (i.e., (1.65) and (1.105)) is to follow the strategy of [22, §4] (cf. [30, §4.2]) by using (1.77). We explain more details

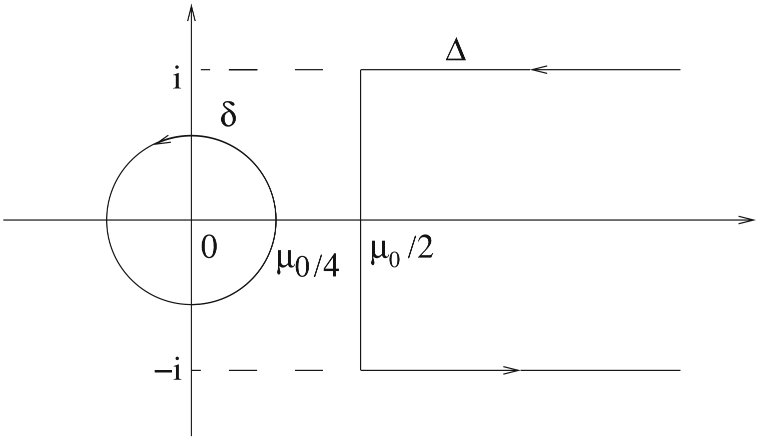

Fig. 1 Contour

now. Let δ be the counterclockwise oriented circle in $\mathbb{C}$ of center 0 and radius $\mu_0/4$, and let Δ be the oriented path in $\mathbb{C}$ defined by Fig. 1.

Let $e^{-u\mathscr{L}_t}$ be the heat operator associated with $\mathscr{L}_t$ for $u > 0$. By (1.77), (1.78), (1.101) and the residue formula, we have

$$\begin{aligned} \mathscr{P}_t &= \frac{1}{2\pi\sqrt{-1}}\Big[\int_{|\lambda|=\mu_0/4} e^{-u\lambda}\lambda(\lambda-\mathscr{L}_t)^{-1}\Big]^{(2)} d\lambda, \\ \Big[\mathscr{L}_t e^{-u\mathscr{L}_t}\Big]^{(2)} &= \frac{1}{2\pi\sqrt{-1}}\Big[\int_{\delta\cup\Delta} e^{-u\lambda}\lambda(\lambda-\mathscr{L}_t)^{-1}\Big]^{(2)} d\lambda, \\ \Big[\mathscr{L}_t^2 e^{-u\mathscr{L}_t}\Big]^{(2)} &= \frac{1}{2\pi\sqrt{-1}}\Big[\int_{\Delta} e^{-u\lambda}\lambda^2(\lambda-\mathscr{L}_t)^{-1}\Big]^{(2)} d\lambda. \end{aligned} \tag{1.106}$$

Set

$$F_u(\mathscr{L}_t) = \frac{1}{2\pi\sqrt{-1}}\Big[\int_{\Delta} e^{-u\lambda}\lambda(\lambda-\mathscr{L}_t)^{-1}\Big]^{(2)} d\lambda. \tag{1.107}$$

Then from (1.77), (1.106) and (1.107), we get

$$\begin{aligned} \mathscr{P}_t &= \lim_{u\to+\infty}\Big[\mathscr{L}_t e^{-u\mathscr{L}_t}\Big]^{(2)}, \\ F_u(\mathscr{L}_t) &= \Big[\mathscr{L}_t e^{-u\mathscr{L}_t}\Big]^{(2)} - \mathscr{P}_t = \int_u^{+\infty}\Big[\mathscr{L}_t^2 e^{-u_1\mathscr{L}_t}\Big]^{(2)} du_1. \end{aligned} \tag{1.108}$$

From (1.106), in particular, the integral of the third equation is taken only along Δ, we get the analogue of [30, Theorem 4.2.5] for $\Big[\mathscr{L}_t e^{-u\mathscr{L}_t}\Big]^{(2)}$ and $\Big[\mathscr{L}_t^2 e^{-u\mathscr{L}_t}\Big]^{(2)}$. Combining it with (1.108), we get the analogue of [30, Corollary 4.2.6]. Then the argument of [30, Theorems 4.2.7, 4.2.8] gives a direct proof of (1.62).

Now we concentrate to compute $\mathcal{Q}_r$. Let $f(\lambda, t)$ be a formal power series with values in $\mathrm{End}(L^2(\mathbb{R}^{2n}, \Lambda(T^*S)\widehat{\otimes}\mathbf{E}_{x_0}))$

$$f(\lambda, t) = \sum_{r=0}^{\infty} t^r f_r(\lambda), \quad f_r(\lambda) \in \mathrm{End}(L^2(\mathbb{R}^{2n}, \Lambda(T^*S)\widehat{\otimes}\mathbf{E}_{x_0})). \tag{1.109}$$

By (1.83), consider the equation of formal power series for $|\lambda| = \mu_0$,

$$\left(-\mathscr{L}_0^{(0)} + \lambda - \sum_{r=1}^{\infty} t^r \mathcal{O}_r^{(0)}\right) f(\lambda, t) = \mathrm{Id}_{L^2(\mathbb{R}^{2n}, \Lambda(T^*S)\widehat{\otimes}\mathbf{E}_{x_0})}. \tag{1.110}$$

Then for $r \in \mathbb{N}$, we have

$$f_r(\lambda) = (\lambda - \mathscr{L}_0^{(0)})^{-1} \sum_{j=1}^{r} \mathcal{O}_j^{(0)} f_{r-j}(\lambda). \tag{1.111}$$

Especially, we have

$$\begin{aligned} f_0(\lambda) =&(\lambda - \mathscr{L}_0^{(0)})^{-1} = \frac{1}{\lambda} P^N + (\lambda - \mathscr{L}_0^{(0)})^{-1} P^{N^\perp}, \\ f_1(\lambda) =&(\lambda - \mathscr{L}_0^{(0)})^{-1} \mathcal{O}_1^{(0)} (\lambda - \mathscr{L}_0^{(0)})^{-1}, \\ f_2(\lambda) =&(\lambda - \mathscr{L}_0^{(0)})^{-1} \left[\mathcal{O}_1^{(0)} (\lambda - \mathscr{L}_0^{(0)})^{-1} \mathcal{O}_1^{(0)} + \mathcal{O}_2^{(0)}\right] (\lambda - \mathscr{L}_0^{(0)})^{-1}. \end{aligned} \tag{1.112}$$

Then by (1.101), as in (1.110), we have the following equation as formal power series

$$\left[(\lambda - \mathscr{L}_t)^{-1}\right]^{(2)} = \sum_{r=0}^{\infty} \left(\sum_{\sum_i r_i = r} f_{r_1} \mathcal{O}_{r_2}^{(1)} f_{r_3} \mathcal{O}_{r_4}^{(1)} f_{r_5} + \sum_{\sum_i j_i = r} f_{j_1} \mathcal{O}_{j_2}^{(2)} f_{j_3} \right)(\lambda)\, t^r. \tag{1.113}$$

By the same argument as in [31, (1.110)] (cf. [30, (4.1.91)]), (1.102) and (1.113), we get

$$\begin{aligned} \mathcal{Q}_r =& \frac{1}{2\pi\sqrt{-1}} \sum_{\sum_i r_i = r} \int_{|\lambda|=\mu_0} f_{r_1}(\lambda) \mathcal{O}_{r_2}^{(1)} f_{r_3}(\lambda) \mathcal{O}_{r_4}^{(1)} f_{r_5}(\lambda) \lambda \, d\lambda \\ &+ \frac{1}{2\pi\sqrt{-1}} \sum_{\sum_i j_i = r} \int_{|\lambda|=\mu_0} f_{j_1}(\lambda) \mathcal{O}_{j_2}^{(2)} f_{j_3}(\lambda) \lambda \, d\lambda. \end{aligned} \tag{1.114}$$

From Theorems 1.11, 1.12, (1.94), (1.114) and the residue formula, we can get $\mathcal{Q}_r$ by using the operators $(\mathscr{L}_0^{(0)})^{-1}$, P^N, $P^{N^\perp}$, $\mathcal{O}_k$ $(k \le r)$. This gives us a direct method to compute $\mathcal{Q}_r$ in view of Theorem 1.12.

From Theorem 1.11 and (1.114), we get the properties of the coefficients $\mathscr{J}_r(Z, Z')$. To finish the proof of Theorem 1.8, we need to compute $\mathscr{J}_0(Z, Z')$.

From Theorem 1.12, (1.100), (1.112) and (1.114), we get

$$\begin{aligned}\mathscr{Q}_0 &= \frac{1}{2\pi\sqrt{-1}}\int_{|\lambda|=\mu_0}(\lambda-\mathscr{L}_0^{(0)})^{-1}\mathcal{O}_0^{(1)}(\lambda-\mathscr{L}_0^{(0)})^{-1}\mathcal{O}_0^{(1)}(\lambda-\mathscr{L}_0^{(0)})^{-1}\lambda\, d\lambda\\ &\quad+\frac{1}{2\pi\sqrt{-1}}\int_{|\lambda|=\mu_0}(\lambda-\mathscr{L}_0^{(0)})^{-1}\mathcal{O}_0^{(2)}(\lambda-\mathscr{L}_0^{(0)})^{-1}\lambda\, d\lambda\\ &= P^N\mathcal{O}_0^{(2)}P^N - P^N\mathcal{O}_0^{(1)}P^{N^\perp}(\mathscr{L}_0^{(0)})^{-1}\mathcal{O}_0^{(1)}P^N. \end{aligned} \tag{1.115}$$

Thus from Theorem 1.12, (1.80), (1.94), (1.98), (1.99) and (1.115), we get

$$\begin{aligned}\mathscr{Q}_0(Z,Z') &= \frac{1}{2}f^\alpha\wedge f^\beta R^L_{x_0}(f^H_\alpha, f^H_\beta)P^N(Z,Z')\\ &\quad+2\left(P^N f^\alpha i_{\overline{w}_j}R^L_{x_0}(f^H_\alpha, w_j)(\mathscr{L}_0^{(0)})^{-1}f^\beta\wedge\overline{w}^k R^L_{x_0}(f^H_\beta,\overline{w}_k)P^N\right)(Z,Z')\\ &= f^\alpha\wedge f^\beta\left[\frac{1}{2}R^L_{x_0}(f^H_\alpha, f^H_\beta)-\frac{1}{a_j}R^L_{x_0}(f^H_\alpha, w_j)R^L_{x_0}(f^H_\beta,\overline{w}_j)\right]P^N(Z,Z')\\ &= -2\pi\sqrt{-1}\frac{(\omega^{n+1})^{(2)}}{(n+1)(\omega^n)^{(0)}}P^N(Z,Z'). \end{aligned}\tag{1.116}$$

The proof of Theorem 1.8 is completed.

Remark 1.14 For $A\in\mathscr{C}^\infty(W,\Lambda^3(T^*X))$, we replace the operator D_p by the modified Dirac operator $D_p^{c,A}$ in [7], [30, §1.3.3], certainly, we still have the same Theorems 1.6, 1.8. Especially, if the fiber X is holomorphic and L, E are holomorphic along the fiber X, let $\overline{\partial}^{L^p\otimes E,*}$ be the adjoint of the fiberwise Dolbeault operator $\overline{\partial}^{L^p\otimes E}$ along the fiber X, then we can take

$$D_p=\sqrt{2}(\overline{\partial}^{L^p\otimes E}+\overline{\partial}^{L^p\otimes E,*}), \tag{1.117}$$

as D_p is a modified Dirac operator by [7] (cf. [30, Theorem 1.4.5]).

Remark 1.15 As R^L is non-degenerate along the fiber X, we have a natural choice of the horizontal bundle T^HW in (1.1). Namely, set

$$T^HW=\{u\in TW:\omega(u,X)=0 \text{ for any } X\in TX\}. \tag{1.118}$$

Then from Theorem 1.12, (1.80), (1.81) and (1.118), we get

$$\mathcal{O}_0^{(1)}=0,\quad P^N\mathcal{O}_1^{(1)}P^N=0. \tag{1.119}$$

In this case, we have a simpler formula for $\mathscr{Q}_2$ from (1.112), (1.114) and (1.119),

$$\begin{aligned}
\mathcal{Q}_2 = & \frac{1}{2\pi\sqrt{-1}}\int_{|\lambda|=\mu_0}(\lambda-\mathscr{L}_0^{(0)})^{-1}\mathcal{O}_1^{(1)}(\lambda-\mathscr{L}_0^{(0)})^{-1}\mathcal{O}_1^{(1)}(\lambda-\mathscr{L}_0^{(0)})^{-1}\lambda\, d\lambda \\
& +\frac{1}{2\pi\sqrt{-1}}\int_{|\lambda|=\mu_0}\Big\{(\lambda-\mathscr{L}_0^{(0)})^{-1}\mathcal{O}_2^{(2)}(\lambda-\mathscr{L}_0^{(0)})^{-1} \\
& +(\lambda-\mathscr{L}_0^{(0)})^{-1}\mathcal{O}_1^{(0)}(\lambda-\mathscr{L}_0^{(0)})^{-1}\mathcal{O}_1^{(2)}(\lambda-\mathscr{L}_0^{(0)})^{-1} \\
& +(\lambda-\mathscr{L}_0^{(0)})^{-1}\mathcal{O}_1^{(2)}(\lambda-\mathscr{L}_0^{(0)})^{-1}\mathcal{O}_1^{(0)}(\lambda-\mathscr{L}_0^{(0)})^{-1} \\
& +(\lambda-\mathscr{L}_0^{(0)})^{-1}\mathcal{O}_1^{(0)}(\lambda-\mathscr{L}_0^{(0)})^{-1}\mathcal{O}_0^{(2)}(\lambda-\mathscr{L}_0^{(0)})^{-1}\mathcal{O}_1^{(0)}(\lambda-\mathscr{L}_0^{(0)})^{-1} \\
& +(\lambda-\mathscr{L}_0^{(0)})^{-1}\Big[\mathcal{O}_1^{(0)}(\lambda-\mathscr{L}_0^{(0)})^{-1}\mathcal{O}_1^{(0)}+\mathcal{O}_2^{(0)}\Big](\lambda-\mathscr{L}_0^{(0)})^{-1}\mathcal{O}_0^{(2)}(\lambda-\mathscr{L}_0^{(0)})^{-1} \\
& +(\lambda-\mathscr{L}_0^{(0)})^{-1}\mathcal{O}_0^{(2)}(\lambda-\mathscr{L}_0^{(0)})^{-1} \\
& \Big[\mathcal{O}_1^{(0)}(\lambda-\mathscr{L}_0^{(0)})^{-1}\mathcal{O}_1^{(0)}+\mathcal{O}_2^{(0)}\Big](\lambda-\mathscr{L}_0^{(0)})^{-1}\Big\}\lambda\, d\lambda.
\end{aligned} \tag{1.120}$$

By [29, Theorem 2.3] (or [30, (4.1.94)]), we know that

$$P^N\mathcal{O}_1^{(0)}P^N=0. \tag{1.121}$$

Observe that from (1.80),

$$(\lambda-\mathscr{L}_0^{(0)})^{-1}\mathcal{O}_0^{(2)}(\lambda-\mathscr{L}_0^{(0)})^{-1}=(\lambda-\mathscr{L}_0^{(0)})^{-2}\mathcal{O}_0^{(2)}=\mathcal{O}_0^{(2)}(\lambda-\mathscr{L}_0^{(0)})^{-2}. \tag{1.122}$$

By Theorem 1.12, (1.119)-(1.122) and the residue formula, we get under the assumption (1.118)

$$\begin{aligned}
\mathcal{Q}_2 = & -P^N\mathcal{O}_1^{(1)}(\mathscr{L}_0^{(0)})^{-1}P^{N^\perp}\mathcal{O}_1^{(1)}P^N+P^N\mathcal{O}_2^{(2)}P^N \\
& -P^N\mathcal{O}_1^{(0)}(\mathscr{L}_0^{(0)})^{-1}P^{N^\perp}\mathcal{O}_1^{(2)}P^N-(\mathscr{L}_0^{(0)})^{-1}\mathcal{O}_1^{(0)}P^N\mathcal{O}_1^{(2)}P^N \\
& -P^N\mathcal{O}_1^{(2)}(\mathscr{L}_0^{(0)})^{-1}\mathcal{O}_1^{(0)}P^N-P^N\mathcal{O}_1^{(2)}P^N\mathcal{O}_1^{(0)}(\mathscr{L}_0^{(0)})^{-1}P^{N^\perp} \\
& +P^N\mathcal{O}_1^{(0)}\mathcal{O}_0^{(2)}(\mathscr{L}_0^{(0)})^{-2}P^{N^\perp}\mathcal{O}_1^{(0)}P^N+(\mathscr{L}_0^{(0)})^{-1}P^{N^\perp}\mathcal{O}_1^{(0)}\mathcal{O}_0^{(2)}P^N\mathcal{O}_1^{(0)}(\mathscr{L}_0^{(0)})^{-1}P^{N^\perp} \\
& +(\mathscr{L}_0^{(0)})^{-1}P^{N^\perp}\Big[\mathcal{O}_1^{(0)}(\mathscr{L}_0^{(0)})^{-1}\mathcal{O}_1^{(0)}-\mathcal{O}_2^{(0)}\Big]P^N\mathcal{O}_0^{(2)} \\
& +\mathcal{O}_0^{(2)}P^N\Big[\mathcal{O}_1^{(0)}(\mathscr{L}_0^{(0)})^{-1}\mathcal{O}_1^{(0)}-\mathcal{O}_2^{(0)}\Big](\mathscr{L}_0^{(0)})^{-1}P^{N^\perp}.
\end{aligned} \tag{1.123}$$

1.7 The curvature as a Toeplitz operator

First, we describe the formalism discovered by Berezin [4] and Boutet de Monvel–Guillemin [18] on the definition of Toeplitz operators, and further pursued by Bordemann–Meinrenken–Schlichenmaier [16, 42], and Ma–Marinescu [30, 32].

Let (X, J, ω) be a compact symplectic manifold of real dimension $2n$, with compatible almost complex structure J and g^{TX} a J-invariant metric. Let (L, h^L, ∇^L) be a prequantum line bundle over X as in (1.31). We consider a Hermitian vector bundle (E, h^E, ∇^E) on X with Hermitian connection ∇^E, and the space $\big(L^2(X, E_p), \langle\cdot,\cdot\rangle\big)$

introduced in (1.10). Let P_p be the orthogonal projection from $L^2(X, E_p)$ onto $\mathrm{Ker}(D_p)$ as in Section 1.2.

A section $g \in \mathscr{C}^\infty(X, \mathrm{End}(E))$ defines a vector bundle morphism $\mathrm{Id}_{\Lambda(T^{*(0,1)}X)\otimes L^p} \otimes g$ of $E_p := \Lambda(T^{*(0,1)}X) \otimes L^p \otimes E$, which we still denote by g.

In [32, Definition 4.1] (cf. [30, Definition 8.1.8]), Ma-Marinescu defined a vector space of Toeplitz operators. The following definition is a natural extension of [32, Definition 4.1] by twisting a finite dimensional algebra $\mathcal{A}$.

Definition 1.16 A Toeplitz operator with coefficients in a finite dimensional algebra $\mathcal{A}$ over $\mathbb{C}$ is a sequence $\{T_p\} = \{T_p\}_{p\in\mathbb{N}}$ of linear operators

$$T_p : \mathcal{A} \otimes L^2(X, E_p) \longrightarrow \mathcal{A} \otimes L^2(X, E_p)\,, \tag{1.124}$$

with the properties:

(i) For any $p \in \mathbb{N}$, we have

$$T_p = P_p\, T_p\, P_p\,. \tag{1.125}$$

(ii) There exist a sequence $g_l \in \mathcal{A} \otimes \mathscr{C}^\infty(X, \mathrm{End}(E))$ such that for all $k \geq 0$ there exists $C_k > 0$ with

$$\Big\| T_p - P_p \Big(\sum_{l=0}^{k} p^{-l} g_l \Big) P_p \Big\| \leq C_k\, p^{-k-1}, \tag{1.126}$$

where $\|\cdot\|$ denotes the operator norm on the space of bounded operators.

The full symbol of $\{T_p\}$ is the formal series $\sum_{l=0}^{\infty} \hbar^l g_l \in \mathcal{A} \otimes \mathscr{C}^\infty(X, \mathrm{End}(E))[[\hbar]]$ and the principal symbol of $\{T_p\}$ is g_0.

For any $f \in \mathcal{A} \otimes \mathscr{C}^\infty(X, \mathrm{End}(E))$,

$$T_{f,\,p} := P_p\, f\, P_p : \mathcal{A} \otimes L^2(X, E_p) \longrightarrow \mathcal{A} \otimes L^2(X, E_p) \tag{1.127}$$

is a Toeplitz operator and called as Berezin-Toeplitz quantization of f. Then we can express (1.126) symbolically by

$$T_p = \sum_{l=0}^{k} T_{g_l,p}\, p^{-l} + \mathcal{O}(p^{-k-1}). \tag{1.128}$$

Then we can reformulate [32, Theorem 1.1] (cf. [30, Theorem 8.1.10]) as

Theorem 1.17 *The space of Toeplitz operators with coefficients in a finite dimensional algebra $\mathcal{A}$ over $\mathbb{C}$ forms an algebra. Let $f, g \in \mathcal{A} \otimes \mathscr{C}^\infty(X, \mathrm{End}(E))$. Then the product* *of the Toeplitz operators $T_{f,\,p}$ and $T_{g,\,p}$ is a Toeplitz operator, more precisely, it admits the asymptotic expansion in the sense of (1.128) for any $k \in \mathbb{N}$:*

$$T_{f,\,p}\,T_{g,\,p} = \sum_{r=0}^{k} p^{-r}\,T_{C_r(f,g),\,p} + \mathcal{O}(p^{-k-1}), \tag{1.129}$$

where C_r are bidifferential operators and $C_r(f,g) \in \mathcal{A} \otimes \mathscr{C}^\infty(X, \mathrm{End}(E))$ and $C_0(f,g) = fg$.

By the characterization of Toeplitz operators via the expansion of their kernels [32, Theorem 4.9, (4.30)] (cf. [30, Theorem 8.1.9, (8.1.18)]), Theorems 1.7, 1.8 and (1.116) imply the following result:

Theorem 1.18 *The curvature operators $\frac{1}{p}R^{\mathrm{Ker}(D_p)} \in \Omega^2(S, \mathrm{End}(\mathrm{Ker}(D_p)))$ in Section 1.3 is a Toeplitz operator with coefficients in $\mathcal{A} = \Lambda^{2*}(T_s^*S)$ for any $s \in S$, with its leading symbol R_0 being $b_{2,0}$ in (0.15).*

We have also

Theorem 1.19 *For any $f \in \mathscr{C}^\infty(W, \mathrm{End}(E))$, $U \in \mathscr{C}^\infty(S, TS)$, $\nabla_U^{\mathrm{End}(D_p)} T_{f,p}$ is a Toeplitz operator with leading symbol $\nabla_{U^H}^{\mathrm{End}(E)} f$.*

Proof From (1.14), (1.16) and (1.127), we get

$$\begin{aligned}\nabla_U^{\mathrm{End}(D_p)} T_{f,p} &= P_p[\nabla_U^{\Omega}, P_p] f P_p + P_p[\nabla_U^{\Omega}, f] P_p + P_p f [\nabla_U^{\Omega}, P_p] P_p,\\ [\nabla_U^{\Omega}, f] &= \nabla_{U^H}^{\mathrm{End}(E)} f.\end{aligned} \tag{1.130}$$

We need to show that $P_p[\nabla_U^{\Omega}, P_p] f P_p$ and $P_p f [\nabla_U^{\Omega}, P_p] P_p$ are Toeplitz operators. We use $\nabla^{T^{(1,0)}X}$ to trivialise $T^{(1,0)}X|_{\mathcal{U}\times\{x_0\}}$ near (s_0, x_0) in Section 1.4, then the normal coordinate along X in Section 1.4 and (1.89)-(1.90) is identified as $\mathcal{U} \times \mathbb{R}^{2n}$ with canonical almost complex structure and metric on $\mathbb{R}^{2n}$. By Theorem 1.6, $[\nabla_U^{\Omega}, P_p]$ has the same type expansion as in (1.54) by replacing $\mathscr{F}_r$ by $\mathscr{F}_r'$ with

$$\mathscr{F}_r'(Z, Z') = J_r'(Z, Z')\mathscr{P}(Z, Z'), \tag{1.131}$$

and $J_r'(Z, Z')$ is a polynomial in Z, Z' with the same parity as r and

$$\begin{aligned}J_0'(Z, Z') &= \Big(\nabla_{U^H} \log \det\nolimits_{\mathbb{C}}(|\mathbf{J}|) - \frac{\pi}{2}\big\langle(\nabla_{U^H}|\mathbf{J}|)(Z - Z'), Z - Z'\big\rangle\\ &\qquad -\pi\sqrt{-1}\big\langle(\nabla_{U^H}\mathbf{J})Z, Z'\big\rangle\Big) I_{\mathbb{C}\otimes E}\\ &= \Big\{ \mathrm{Tr}\,|_{T^{(1,0)}X}[\mathbf{J}^{-1}\nabla_{U^H}^{T^{(1,0)}X}\mathbf{J}] + \pi\sqrt{-1}\Big\langle(\nabla_{U^H}^{T^{(1,0)}X}\mathbf{J})(z - z'), \overline{z} - \overline{z}'\Big\rangle\\ &\qquad -\pi\sqrt{-1}\Big(\Big\langle(\nabla_{U^H}^{T^{(1,0)}X}\mathbf{J})z, \overline{z}'\Big\rangle - \Big\langle(\nabla_{U^H}^{T^{(1,0)}X}\mathbf{J})z', \overline{z}\Big\rangle\Big)\Big\} I_{\mathbb{C}\otimes E}.\end{aligned} \tag{1.132}$$

From the argument in the proof of [30, Lemma 7.2.4] and Theorem 1.6 for $m' = 0$, we know that $P_p[\nabla^\Omega_U, P_p]f P_p$ has the same type expansion as in (1.54) and the leading term is given by

$$\mathscr{P}(J'_0\mathscr{P})f(x_0)\mathscr{P} = f(x_0)\mathscr{P}(J'_0\mathscr{P})\mathscr{P} = 0, \tag{1.133}$$

here we understand $J'_0\mathscr{P}$ as an operator on $\mathbb{C}^n$ with kernel $(J'_0\mathscr{P})(Z, Z')$ with respect to the volume form $dv_{TX}(Z')$. Note that we can get (1.133) by a direct computation from the kernel calculus in [30, §7.1]: put $b_{i\bar{j}} = \left\langle (\nabla^{T^{(1,0)}X}_{U^H}\mathbf{J})\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right\rangle$, then

$$J'_0\mathscr{P} = \pi\sqrt{-1}\left\{-4b_{j\bar{j}}a_j^{-1} + b_{i\bar{j}}\left[(z_i - z'_i)\frac{b_j}{a_j} - z_i\bar{z}'_j + z'_i(\frac{b_j}{a_j} + \bar{z}'_j)\right]\right\}\mathscr{P}(Z, Z')I_{\mathbb{C}\otimes E}.$$

Thus $\mathscr{P}(J'_0\mathscr{P}) = \pi\sqrt{-1}\Big[-2b_{j\bar{j}}a_j^{-1} - b_{i\bar{j}}(z_i - z'_i)\bar{z}'_j\Big]\mathscr{P}(Z, Z')I_{\mathbb{C}\otimes E}$, and we get (1.133). Here is an argument without computation: Observe that

$$\mathscr{F}'_0 = \nabla_{U^H}\mathscr{P}, \quad \mathscr{P}^2 = \mathscr{P}. \tag{1.134}$$

From the second equation of (1.134), we get $\mathscr{P}(\nabla_{U^H}\mathscr{P}) + (\nabla_{U^H}\mathscr{P})\mathscr{P} = \nabla_{U^H}\mathscr{P}$, thus

$$\mathscr{P}\mathscr{F}'_0\mathscr{P} = \mathscr{P}(\nabla_{U^H}\mathscr{P})\mathscr{P} = 0. \tag{1.135}$$

By the characterization of Toeplitz operators via the expansion of their kernels [32, Theorem 4.9, (4.30)] (cf. [30, Theorem 8.1.9, (8.1.18)]) as above, we know that $P_p[\nabla^\Omega_U, P_p]f P_p$ is a Toeplitz operator and its asymptotic expansion starts from p^{-1}. Same argument shows that $P_p f[\nabla^\Omega_U, P_p]P_p$ is a Toeplitz operator with principal symbol 0.

The proof of Theorem 1.19 is completed. □

1.8 A proof of Theorems 0.2, 0.4 and 0.8.

From (0.2), in (1.34), $\mathbf{J} = J^{T_\mathbb{R}X}$, thus $a_j = 2\pi$ in (1.89), and $\mathscr{P}(0, 0) = 1$ in (1.97).

By Theorems 1.8 and 1.18, we get Theorems 0.2 and 0.4 for $\frac{1}{p}R^{\mathrm{Ker}(D_p)}$. When we take $Z = Z' = 0$ in (1.62), we get (0.16) and

$$b_{2,r} = (\mathscr{J}_{2r}\mathscr{P})(0, 0). \tag{1.136}$$

Note that in the holomorphic Kähler situation (0.2), even we work on the full degree of $\Lambda(T^{*(0,1)}X)$, but our connection $\nabla^{\Lambda(T^{*(0,1)}X)}$ along the fiber X becomes the Chern connection which preserves the $\mathbb{Z}$-grading of $\Lambda(T^{*(0,1)}X)$ on $B^{T_{x_0}X}(0, 2\varepsilon)$. From (0.12) and our trivialization, we get $\mathscr{J}_r(Z, Z') \in \Lambda^2(T^*_\mathbb{R}S) \otimes \mathrm{End}(E)_{x_0}$ and $b_{2,r} \in \mathscr{C}^\infty(W, \pi^*(\Lambda^2(T^*_\mathbb{R}S)) \otimes \mathrm{End}(E))$. Thus we get also (0.15) from (1.61) (cf. (1.116)).

In the proof, we wrote for the Hermitian connection ∇_p^{Ω}, however, we only use it as a motivation from the local index theory, all arguments here go through for the curvature $R^{H^0(X,L^p\otimes E)}$ from (0.9). Thus we get also Theorems 0.2 and 0.4 for $\frac{1}{p}R^{H^0(X,L^p\otimes E)}$.

Finally, from Theorem 1.17, (0.9) and (0.11), Theorem 0.8 is a special case of Theorem 1.19.

2 An analogue of Bismut's local family index theorem for Bergman kernels

This Section is organized as follows. In Section 2.1, we recall some results on the Kähler fibration. In Section 2.2, we establish Corollary 0.5 and Theorem 0.6.

In this Section, we will use the notation in Introduction. We denote by $\langle\cdot,\cdot\rangle$ the $\mathbb{C}$-bilinear form on $T_{\mathbb{R}}X\otimes_{\mathbb{R}}\mathbb{C}$ induced by the metric $g^{T_{\mathbb{R}}X}$ in (0.2).

2.1 Kähler fibration

Let W, S be compact complex manifolds. Let $\pi : W\to S$ be a holomorphic submersion with compact fiber X and $\dim_{\mathbb{C}} X = n$. In this section, we denote by TW, TS, TX the corresponding holomorphic tangent bundles, and $T_{\mathbb{R}}W$, $T_{\mathbb{R}}S$, $T_{\mathbb{R}}X$ the associated real tangent bundles. Let $J^{T_{\mathbb{R}}X}$ be the almost complex structure on the relative real tangent bundle $T_{\mathbb{R}}X$.

Let $T^H_{\mathbb{R}}W$ be a sub-bundle of $T_{\mathbb{R}}W$ such that (0.8) holds.

Let $g^{T_{\mathbb{R}}X}$ be a $J^{T_{\mathbb{R}}X}$-invariant metric on $T_{\mathbb{R}}X$. Let r^X be the scalar curvature of $(X, g^{T_{\mathbb{R}}X})$.

Definition 2.1 [10, Def. 1.4] The triple $(\pi, g^{T_{\mathbb{R}}X}, T^H_{\mathbb{R}}W)$ is said to define a Kähler fibration if there exists a smooth closed real 2-form ω^W of complex type (1, 1) on W such that

- $T^H_{\mathbb{R}}W$ and $T_{\mathbb{R}}X$ are orthogonal with respect to ω^W.
- If $X, Y \in T_{\mathbb{R}}X$,

$$\omega^W(X,Y) = g^{T_{\mathbb{R}}X}(J^{T_{\mathbb{R}}X}X, Y). \tag{2.1}$$

We suppose now that the triple $(\pi, g^{T_{\mathbb{R}}X}, T^H_{\mathbb{R}}W)$ defines a Kähler fibration.

We will denote by ω^H, ω^X the restrictions of ω to $T^H_{\mathbb{R}}W$, $T_{\mathbb{R}}X$. We extend ω^H, ω^X to $T_{\mathbb{R}}W$ by taking the convention that if $X\in T_{\mathbb{R}}X$ and $U\in T_{\mathbb{R}}S$, then $i_X\omega^H = 0$ and $i_{U^H}\omega^X = 0$. Therefore

$$\omega = \omega^H + \omega^X. \tag{2.2}$$

The Riemannian volume form dv_X on $(X, g^{T_{\mathbb{R}}X})$ is given by

$$dv_X = (\omega^X)^n/n!. \tag{2.3}$$

Note that $T^{(1,0)}X$ in Section 1.1 is identified naturally as the holomorphic relative tangent bundle TX of the fibration π. Let $h^{T^{(1,0)}X}$ be the Hermitian metric on $T^{(1,0)}X$ induced by $g^{T_{\mathbb{R}}X}$. We still denote by ∇^{TX} the connection on $T_{\mathbb{R}}X$ with curvature R^{TX} defined in Definition 1.1 associated with $(\pi, g^{T_{\mathbb{R}}X}, T^H_{\mathbb{R}}W)$. By [10, Theorem 1.7], ∇^{TX} preserves $T^{(1,0)}X$ and $T^{(0,1)}X$, and it is the Chern connection $\nabla^{T^{(1,0)}X}$ on $(T^{(1,0)}X, h^{T^{(1,0)}X})$, and for $U, V \in T_{\mathbb{R}}S$, we have

$$\begin{aligned} &\mathbf{k}(U^H) = 0, \quad \mathcal{L}_{U^H}\omega^X = 0, \\ &\nabla^{TX}_{\cdot}\omega^X = 0, \quad d^X(\omega^H(U^H, V^H)) + i_{T(U^H,V^H)}\omega^X = 0, \end{aligned} \tag{2.4}$$

where we denote by d^X the exterior differential operator along the fiber X.

Let E be a holomorphic vector bundle on W. Let h^E be a Hermitian metric on E. Let ∇^E be the Chern connection on (E, h^E) with curvature R^E.

Let $\nabla^{\Lambda(T^{*(0,1)}X)}, \nabla^{\Lambda(T^{*(0,1)}X)\otimes E}$ be the connections on $\Lambda(T^{*(0,1)}X), \Lambda(T^{*(0,1)}X)\otimes E$ induced by ∇^{TX} and ∇^E with curvatures $R^{\Lambda(T^{*(0,1)}X)}$, $R^{\Lambda(T^{*(0,1)}X)\otimes E}$. Then $\nabla^{\Lambda(T^{*(0,1)}X)}$ is the Clifford connection ∇^{Cliff} on $\Lambda(T^{*(0,1)}X)$ in Section 1.

Let $\{w_i\}$ be an orthonormal basis of $T^{(1,0)}X$, by the above discussion and (1.9), we have

$$\begin{aligned} &R^{TX} = R^{T^{(1,0)}X}, \quad R^{\text{Cliff}} = R^{\Lambda(T^{*(0,1)}X)} = \langle R^{TX}w_i, \overline{w}_j\rangle \overline{w}^j \wedge i_{\overline{w}_i}, \\ &\operatorname{Tr}[R^{T^{(1,0)}X}] = \langle R^{TX}w_k, \overline{w}_k\rangle, \quad r^X = 2\langle R^{TX}(w_j, \overline{w}_j)w_k, \overline{w}_k\rangle. \end{aligned} \tag{2.5}$$

Let $\overline{\partial}^{E,*}$ be the formal adjoint of the Dolbeault operator $\overline{\partial}^E$ along the fibers X with respect to (1.10), then

$$D = \sqrt{2}(\overline{\partial}^E + \overline{\partial}^{E,*}) \tag{2.6}$$

is the Dirac operator along the fiber X (cf. [30, Theorem 1.4.5]). Moreover,

$$D^2 = 2\left(\overline{\partial}^E\overline{\partial}^{E,*} + \overline{\partial}^{E,*}\overline{\partial}^E\right) \tag{2.7}$$

preserves the $\mathbb{Z}$-grading of $\Omega^{0,\bullet}(X, E)$.

For $s \in X$, let $H^\bullet(X_s, E)$ be the Dolbeault cohomology of E along the fiber X_s.

By the Hodge theory, for any $q \in \mathbb{N}$, $s \in S$, we have

$$\operatorname{Ker}(D_s|_{\Omega^{0,q}}) = H^q(X_s, E). \tag{2.8}$$

Assumption The rank of $H^\bullet(X_s, E)$ is locally constant for $s \in S$.

By the **Assumption**, $H^\bullet(X_s, E)$ $(s \in S)$ form a smooth vector bundle $H^\bullet(X, E)$ on S, and it is the direct image of the sheaf of the holomorphic sections of E for the map π. Thus $H^\bullet(X, E)$ is canonically a holomorphic vector bundle on S.

Recall that P is the orthogonal projection from $\Omega^{0,\bullet}(X,E)$ onto $\mathrm{Ker}(D)$. The L^2-product on $\Omega^{0,\bullet}(X,E)$ induces naturally a metric $h^{H^\bullet(X,E)}$ on $H^\bullet(X,E)$ by (2.8). We denote also by $\nabla^{H^\bullet(X,E)}$ the connection on $H^\bullet(X,E)$ defined by (1.16) and (2.8). By (1.13), (2.3) and (2.4), we know that for $U\in T_{\mathbb{R}}S$,

$$\nabla_U^{H^\bullet(X,E)} = P\nabla_{U^H}^{\Lambda(T^{*(0,1)}X)\otimes E}P. \tag{2.9}$$

The following result was established by [13, Theorem 3.2] (cf. [11, Theorem 3.11]).

Theorem 2.2 *The connection $\nabla^{H^\bullet(X,E)}$ is the Chern connection on $(H^\bullet(X,E), h^{H^\bullet(X,E)})$.*

2.2 Family Bergman kernels: a proof of Corollary 0.5 and Theorem 0.6

Let L be a holomorphic line bundle on W. Let h^L be a Hermitian metric on L. Let ∇^L be the Chern connection on (L,h^L) with curvature R^L.

We suppose that $\omega := \frac{\sqrt{-1}}{2\pi}R^L$ defines a fiberwise Kähler form along the fiber X.

Let $h^{T^{(1,0)}X}$ be the associated Kähler metric on $T^{(1,0)}X$ as in (0.2). Let $T_{\mathbb{R}}^HW\subset T_{\mathbb{R}}W$ be the sub-bundle defined by (0.13). Then the triple $(\pi, h^{T^{(1,0)}X}, T_{\mathbb{R}}^HW)$ defines a Kähler fibration.

We will add a subscript p to denote the corresponding objects in Sect. 2.1 associated to $L^p\otimes E$.

By (0.3), for $p>p_0$,

$$H^0(X_s, L^p\otimes E) = H^\bullet(X_s, L^p\otimes E) \tag{2.10}$$

forms a smooth vector bundle $H^\bullet(X,L^p\otimes E) = H^0(X,L^p\otimes E)$ on S. Thus $H^\bullet(X,L^p\otimes E)$ is canonically a holomorphic vector bundle on S. The L^2-product on $\Omega^{0,\bullet}(X,L^p\otimes E)$ induces naturally a metric $h^{H^\bullet(X,L^p\otimes E)}$ on $H^\bullet(X,L^p\otimes E)$ by (2.8).

In this case, by Theorem 2.2, (0.9), (0.11), (2.4) and (2.10) for any $p>p_0$,

$$\nabla^{\mathrm{Ker}(D_p)} = \nabla^{H^0(X,L^p\otimes E)} \tag{2.11}$$

is the Chern connection on $(H^0(X,L^p\otimes E), h^{H^0(X,L^p\otimes E)})$.

By Theorem 1.8, (1.114), (2.5), (2.6), (2.10) and $a_j=2\pi$ in (1.89), we get

Theorem 2.3 *Under the assumptions of this Section, for the asymptotic expansion of $R^{H^0(X,L^p\otimes E)}(x,x')$ in Theorem 1.8, the polynomials $\mathscr{J}_r(Z,Z')\in\Lambda^2(T_{\mathbb{R}}^*S)\otimes \mathrm{End}(E)_{x_0}$ ($x_0\in X_s, s\in S$), in Z,Z' is of the same parity as r and $\deg\mathscr{J}_r\le 3r$, whose coefficients are polynomials in R^{TX}, R^E (and T, R^L) and their derivatives of order $\le r-2$ (and $\le r-1$, $\le r$).*

Now we will compute $b_{2,1}$ in (0.24) by using (1.123).

We fix $x_0 \in W$ and we use the notation in Section 1.5. Especially, $\{w_i\}$ (resp. $\{e_i\}$) is an orthonormal basis of $(T^{(1,0)}_{x_0}X, g^{TX})$ (resp. $(T_{\mathbb{R},x_0}X, g^{TX})$), and we will also use the complex coordinates here.

We will evaluate our tensors at x_0, and most of time, we will omit the subscript x_0. Set

$$\begin{aligned}\widetilde{\mathcal{O}}_2 =& \frac{1}{3}\left\langle R^{TX}_{x_0}(\mathcal{R}, e_i)\mathcal{R}, e_j\right\rangle \nabla_{0,e_i}\nabla_{0,e_j} - R^E_{x_0}(w_j, \overline{w}_j) - \frac{r^X_{x_0}}{6}\\ &+ \left(\left\langle \frac{1}{3}R^{TX}_{x_0}(\mathcal{R}, e_k)e_k + \frac{\pi}{3}R^{TX}_{x_0}(z, \overline{z})\mathcal{R}, e_j\right\rangle - R^E_{x_0}(\mathcal{R}, e_j)\right)\nabla_{0,e_j}.\end{aligned} \tag{2.12}$$

Lemma 2.4 *Under the assumptions of this Section, for $\mathcal{O}_1, \mathcal{O}_2$ in (1.83), we have*

$$\begin{aligned}\mathcal{O}^{(0)}_1 =& 0,\\ \mathcal{O}^{(0)}_2 =& \widetilde{\mathcal{O}}_2 - \langle R^{TX}(\mathcal{R}, e_l)w_i, \overline{w}_j\rangle_{x_0}\overline{w}^j \wedge i_{\overline{w}_i}\nabla_{0,e_l}\\ &+ 2\Big(R^E_{x_0} + \tfrac{1}{2}\operatorname{Tr}\Big[R^{T^{(1,0)}X}_{x_0}\Big]\Big)(w_i, \overline{w}_j)\overline{w}^j \wedge i_{\overline{w}_i},\\ \mathcal{O}^{(1)}_1 =& - f^\alpha \wedge c(e_i)\nabla_{0,T(f^H_\alpha, e_i)},\\ \mathcal{O}^{(2)}_1 =& \frac{1}{2}f^\alpha \wedge f^\beta\Big[-\nabla_{0,T(f^H_\alpha, f^H_\beta)} + \nabla_Z(R^L(f^H_\alpha, f^H_\beta))\Big],\\ \mathcal{O}^{(2)}_2 =& \frac{1}{2}f^\alpha \wedge f^\beta\Big\{\langle R^{TX}(f^H_\alpha, f^H_\beta)w_i, \overline{w}_j\rangle_{x_0}\overline{w}^j \wedge i_{\overline{w}_i} + R^E_{x_0}(f^H_\alpha, f^H_\beta)\\ &+ \frac{1}{2}(\nabla\nabla(R^L(f^H_\alpha, f^H_\beta)))_{x_0,(Z,Z)} - \langle \nabla^{TX}_Z(T(f^H_\alpha, f^H_\beta)), e_i\rangle_{x_0}\nabla_{0,e_i}\Big\}.\end{aligned} \tag{2.13}$$

Proof By (1.33), (1.34), (1.89) and (2.1), we have in our situation

$$\mathbf{J} = J^{T_\mathbb{R}X}, \quad a_j = 2\pi, \quad \tau = 2\pi n. \tag{2.14}$$

At first, as $J^{T_\mathbb{R}X}$ is integrable along the fiber X, we know that $J^{T_\mathbb{R}X}$ is parallel with respect to ∇^{TX} along the fiber, thus as in [30, (4.1.103)], in our normal coordinates,

$$\nabla^{TX}_{e_j}e_i = 0, \quad (\partial_k R^L)_{x_0}(e_j, e_i) = 0 \quad \text{at } x_0. \tag{2.15}$$

Note that for a $(1,1)$-form R, by (1.8) as in [30, (1.3.3)], we have

$$\frac{1}{2}R(e_i, e_j)c(e_i)c(e_j) = 2R(w_i, \overline{w}_j)\overline{w}^j \wedge i_{\overline{w}_i} - R(w_i, \overline{w}_i). \tag{2.16}$$

The first two equations of (2.13) follow from [30, Theorem 4.1.25] (or [22, Theorem 5.1]) where the restriction of the operators on $\mathscr{C}^\infty(\mathbb{R}^{2n}, E_{x_0})$ are obtained, and also from [29, Theorem 2.2], (2.5), (2.14), (2.16) as well as the fact that R^{TX}, R^E are $(1,1)$-forms.

Note that by (0.1) and (0.13), we have $R^L(f_\alpha^H, e_j) = 0$. Now the last three equations of (2.13) follow from (1.81), (2.4), (2.5) and (2.15). □

From (0.13), (1.123) and (2.13), we get

$$\begin{aligned}\mathscr{Q}_2 = &-P^N\mathscr{O}_1^{(1)}(\mathscr{L}_0^{(0)})^{-1}P^{N^\perp}\mathscr{O}_1^{(1)}P^N + P^N\mathscr{O}_2^{(2)}P^N\\ &-(\mathscr{L}_0^{(0)})^{-1}P^{N^\perp}\mathscr{O}_2^{(0)}P^N\mathscr{O}_0^{(2)} - \mathscr{O}_0^{(2)}P^N\mathscr{O}_2^{(0)}(\mathscr{L}_0^{(0)})^{-1}P^{N^\perp}. \quad (2.17)\end{aligned}$$

Note that for the Riemannian curvature R^{TX}, for $U, V, W, Y \in T_\mathbb{R}X$, we have

$$\begin{aligned}&\left\langle R^{TX}(U,V)W, Y\right\rangle = \left\langle R^{TX}(W,Y)U, V\right\rangle,\\ &R^{TX}(U,V)W + R^{TX}(V,W)U + R^{TX}(W,U)V = 0.\end{aligned} \quad (2.18)$$

For $\phi \in T^*_\mathbb{R}X$, by (1.91), we have

$$\phi(e_i)e_i = 2\phi(\tfrac{\partial}{\partial z_j})\tfrac{\partial}{\partial \overline{z}_j} + 2\phi(\tfrac{\partial}{\partial \overline{z}_j})\tfrac{\partial}{\partial z_j}, \quad \phi(e_i)\nabla_{0,e_i} = \phi(\tfrac{\partial}{\partial z_j})b_j^+ - \phi(\tfrac{\partial}{\partial \overline{z}_j})b_j. \quad (2.19)$$

By (1.92), (1.98) and (2.14), we have

$$\begin{aligned}&(b_i^+P^N)(Z,Z') = 0, \quad (b_iP^N)(Z,Z') = 2\pi(\overline{z}_i - \overline{z}'_i)P^N(Z,Z'),\\ &P^N(0,0) = I_{\mathbb{C}\otimes E}.\end{aligned} \quad (2.20)$$

From (1.93), (2.12), (2.14), (2.19), (2.20) and the fact that R^{TX}, R^E are (1, 1)-forms, we get (cf. [30, (4.1.109)])

$$\begin{aligned}(P^{N^\perp}\widetilde{\mathscr{O}}_2P^N)(\cdot,0) = &\Big\{P^{N^\perp}\Big[\frac{1}{3}\left\langle R^{TX}(\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_i})\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_j}\right\rangle b_ib_j\\ &-\frac{4\pi}{3}\left\langle R^{TX}(\mathcal{R}, \tfrac{\partial}{\partial z_k})\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_k}\right\rangle - \frac{2}{3}\left\langle R^{TX}(\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_k})\tfrac{\partial}{\partial z_k}, \tfrac{\partial}{\partial \overline{z}_j}\right\rangle b_j\\ &-\frac{\pi}{3}\left\langle R^{TX}(z,\overline{z})z, \tfrac{\partial}{\partial \overline{z}_j}\right\rangle b_j + R^E(\mathcal{R}, \tfrac{\partial}{\partial \overline{z}_j})b_j\Big]P^N\Big\}(\cdot,0)\\ = &\Big\{P^{N^\perp}\Big[\frac{1}{6}\left\langle R^{TX}(z, \tfrac{\partial}{\partial \overline{z}_i})z, \tfrac{\partial}{\partial \overline{z}_j}\right\rangle b_ib_j + R^E(z, \tfrac{\partial}{\partial \overline{z}_j})b_j\Big]P^N\Big\}(\cdot,0).\end{aligned} \quad (2.21)$$

By Theorem 1.12, (1.93), (2.18) and (2.21), we get (cf. [30, (4.1.109)])

$$\begin{aligned}(P^{N^\perp}\widetilde{\mathscr{O}}_2P^N)(\cdot,0) = &\Big\{P^{N^\perp}\Big[\frac{b_ib_j}{6}\left\langle R^{TX}(z, \tfrac{\partial}{\partial \overline{z}_i})z, \tfrac{\partial}{\partial \overline{z}_j}\right\rangle\\ &+\frac{4b_i}{3}\left\langle R^{TX}(z, \tfrac{\partial}{\partial \overline{z}_i})\tfrac{\partial}{\partial z_j}, \tfrac{\partial}{\partial \overline{z}_j}\right\rangle + b_jR^E(z, \tfrac{\partial}{\partial \overline{z}_j})\Big]P^N\Big\}(\cdot,0).\end{aligned} \quad (2.22)$$

From (1.98) and (2.13), we have

$$P^{N^\perp}(\mathscr{O}_2^{(0)} - \widetilde{\mathscr{O}}_2)P^N = 0. \quad (2.23)$$

From Theorem 1.12, (2.22) and (2.23), we get

$$\Big((\mathscr{L}_0^{(0)})^{-1}P^{N^\perp}\mathscr{O}_2^{(0)}P^N\Big)(\cdot,0)=\bigg\{\bigg[\frac{b_ib_j}{48\pi}\Big\langle R^{TX}(z,\tfrac{\partial}{\partial\overline{z}_i})z,\tfrac{\partial}{\partial\overline{z}_j}\Big\rangle + \frac{b_i}{3\pi}\Big\langle R^{TX}(z,\tfrac{\partial}{\partial\overline{z}_i})\tfrac{\partial}{\partial z_j},\tfrac{\partial}{\partial\overline{z}_j}\Big\rangle+\frac{b_j}{4\pi}R^E(z,\tfrac{\partial}{\partial\overline{z}_j})\bigg]P^N\bigg\}(\cdot,0). \tag{2.24}$$

Let $h_i(Z)$ (resp. $F(Z)$) be polynomials in Z with degree 1 (resp. 2). By Theorem 1.12, (1.93) and (2.20), we have

$$\begin{aligned}&(b_jh_j\mathscr{P})(0,0)=-2\frac{\partial h_j}{\partial z_j},\quad (b_ib_jF(Z)\mathscr{P})(0,0)=4\frac{\partial^2F}{\partial z_i\partial z_j},\\&(\mathscr{P}h_jb_j\mathscr{P})(0,0)=2\frac{\partial h_j}{\partial z_j}.\end{aligned}\tag{2.25}$$

Note that $\mathscr{L}_t^{(0)}$ is a formally self-adjoint elliptic operator with respect to $\|\ \ \|_0$, thus $\mathscr{L}_0^{(0)}$, $\mathscr{O}_r^{(0)}$ are also formally self-adjoint with respect to $\|\ \ \|_0$. Thus from (2.5), (2.20), (2.24) and (2.25) (cf. [31, (2.39)] or [30, (4.1.110)]), we get

$$\begin{aligned}-(P^N\mathscr{O}_2^{(0)}(\mathscr{L}_0^{(0)})^{-1}P^{N^\perp})(0,0)&=-((\mathscr{L}_0^{(0)})^{-1}P^{N^\perp}\mathscr{O}_2^{(0)}P^N)(0,0)\\&=\frac{1}{2\pi}\Big\{\Big\langle R^{TX}(\tfrac{\partial}{\partial z_i},\tfrac{\partial}{\partial\overline{z}_j})\tfrac{\partial}{\partial z_j},\tfrac{\partial}{\partial\overline{z}_i}\Big\rangle+R^E(\tfrac{\partial}{\partial z_j},\tfrac{\partial}{\partial\overline{z}_j})\Big\}I_{\mathbb{C}\otimes E}\\&=\frac{1}{2\pi}\Big\{\frac{1}{8}r^X_{x_0}+R^E_{x_0}(\tfrac{\partial}{\partial z_j},\tfrac{\partial}{\partial\overline{z}_j})\Big\}I_{\mathbb{C}\otimes E}.\end{aligned}\tag{2.26}$$

By (0.13), (1.80), (1.98) and (2.2), we have

$$\mathscr{O}_0^{(2)}=-2\pi\sqrt{-1}\omega^H_{x_0},\quad \mathscr{O}_0^{(2)}P^N=P^N\mathscr{O}_0^{(2)}. \tag{2.27}$$

From (2.27) and (2.26), we get

$$\begin{aligned}&\Big(-(\mathscr{L}_0^{(0)})^{-1}P^{N^\perp}\mathscr{O}_2^{(0)}P^N\mathscr{O}_0^{(2)}-\mathscr{O}_0^{(2)}P^N\mathscr{O}_2^{(0)}(\mathscr{L}_0^{(0)})^{-1}P^{N^\perp}\Big)(0,0)\\&=-2\sqrt{-1}\omega^H_{x_0}\Big\{\frac{1}{8}r^X_{x_0}+R^E_{x_0}(\tfrac{\partial}{\partial z_j},\tfrac{\partial}{\partial\overline{z}_j})\Big\}I_{\mathbb{C}\otimes E}.\end{aligned}\tag{2.28}$$

Let $\{g_\alpha\}$ be a basis of the holomorphic tangent bundle $T^{(1,0)}S$ with dual basis $\{g^\alpha\}$. From [10, Theorem 1.7] (or [9, Theorem 2.5]) and (1.6),

$$\begin{aligned}&\text{the tensor } T \text{ is a real } (1,1)- \text{ form with values in } T_{\mathbb{R}}X \text{ and}\\&T(g^H_\alpha,\overline{w}_i)\in T^{(1,0)}X,\quad T(\overline{g}^H_\alpha,w_i)\in T^{(0,1)}X.\end{aligned}\tag{2.29}$$

By (1.90) and (2.29), we get

$$T(g_\alpha^H, \overline{w}_i) = 2\langle T(g_\alpha^H, \overline{w}_j), \tfrac{\partial}{\partial \overline{z}_k}\rangle \tfrac{\partial}{\partial z_k}. \tag{2.30}$$

By (1.8), (1.91), (2.13) and (2.30), we get

$$\begin{aligned}\mathscr{O}_1^{(1)} &= \sqrt{2}\left(\overline{g}^\alpha \wedge i_{\overline{w}_j} \nabla_{0,T(\overline{g}_\alpha^H, w_j)} - g^\alpha \wedge \overline{w}^j \nabla_{0,T(g_\alpha^H, \overline{w}_j)}\right)\\ &= \sqrt{2}\overline{g}^\alpha \wedge i_{\overline{w}_j} \langle T(\overline{g}_\alpha^H, w_j), \tfrac{\partial}{\partial z_k}\rangle b_k^+\\ &\quad +\sqrt{2} g^\alpha \wedge \overline{w}^j \langle T(g_\alpha^H, \overline{w}_j), \tfrac{\partial}{\partial \overline{z}_k}\rangle b_k.\end{aligned} \tag{2.31}$$

Thus by (1.98), (2.20) and (2.31), we have

$$\mathscr{O}_1^{(1)} P^N = \sqrt{2} g^\alpha \wedge \overline{w}^j \langle T(g_\alpha^H, \overline{w}_j), \tfrac{\partial}{\partial \overline{z}_k}\rangle b_k P^N. \tag{2.32}$$

By Theorem 1.12, (1.94), (2.14) and (2.32), we get $\mathscr{L}_0^{(0)} \mathscr{O}_1^{(1)} P^N = 8\pi \mathscr{O}_1^{(1)} P^N$. Now from (1.90), (1.93), (2.20), (2.31) and (2.32), we get

$$\begin{aligned}&-P^N \mathscr{O}_1^{(1)} P^{N\perp} (\mathscr{L}_0^{(0)})^{-1} \mathscr{O}_1^{(1)} P^N\\ &= -\frac{1}{4\pi} P^N \overline{g}^\beta \wedge i_{\overline{w}_i} \langle T(\overline{g}_\beta^H, w_i), \tfrac{\partial}{\partial z_l}\rangle b_l^+ g^\alpha \wedge \overline{w}^j \langle T(g_\alpha^H, \overline{w}_j), \tfrac{\partial}{\partial \overline{z}_k}\rangle b_k P^N\\ &= \frac{1}{2} \overline{g}^\beta \wedge g^\alpha \langle T(\overline{g}_\beta^H, w_j), T(g_\alpha^H, \overline{w}_j)\rangle_{x_0} P^N.\end{aligned} \tag{2.33}$$

Let $F(Z)$ be a polynomial in Z with degree 2. Then by (2.20), we have

$$(F(Z)\mathscr{P})(Z, 0) = \left(\frac{1}{2}\frac{\partial^2 F}{\partial z_i \partial z_j} z_i z_j + \frac{\partial^2 F}{\partial z_i \partial \overline{z}_j} z_i \frac{b_j}{2\pi} + \frac{1}{2}\frac{\partial^2 F}{\partial \overline{z}_i \partial \overline{z}_j}\frac{b_i b_j}{4\pi^2}\right)\mathscr{P}(Z, 0). \tag{2.34}$$

By Theorem 1.12, (1.93) and (2.34),

$$(\mathscr{P} F(Z)\mathscr{P})(Z, 0) = \left(\frac{1}{2}\frac{\partial^2 F}{\partial z_i \partial z_j} z_i z_j + \frac{1}{\pi}\frac{\partial^2 F}{\partial z_j \partial \overline{z}_j}\right)\mathscr{P}(Z, 0). \tag{2.35}$$

From (1.98), (2.13), (2.19) and (2.20), we have

$$\begin{aligned}P^N \mathscr{O}_2^{(2)} P^N = \frac{1}{2} f^\alpha \wedge f^\beta P^N \Big[&R^E(f_\alpha^H, f_\beta^H) + \frac{1}{2}(\nabla\nabla(R^L(f_\alpha^H, f_\beta^H)))_{(Z,Z)}\\ &+\langle \nabla_Z^{TX}(T(f_\alpha^H, f_\beta^H)), \tfrac{\partial}{\partial \overline{z}_j}\rangle b_j\Big] P^N.\end{aligned} \tag{2.36}$$

From (0.1), (2.25), (2.35) and (2.36), we get

$$(P^N \mathscr{O}_2^{(2)} P^N)(0, 0) = \frac{1}{2} f^\alpha \wedge f^\beta \Big[R^E(f_\alpha^H, f_\beta^H)$$

$$-2\sqrt{-1}(\nabla\nabla(\omega(f_\alpha^H, f_\beta^H)))_{(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \overline{z}_j})}$$
$$+2\langle \nabla^{TX}_{\frac{\partial}{\partial z_j}}(T(f_\alpha^H, f_\beta^H)), \tfrac{\partial}{\partial \overline{z}_j}\rangle\Big] I_{\mathbb{C}\otimes E}. \tag{2.37}$$

From (2.4), for $U, V \in T_{\mathbb{R}}S$, we get

$$\begin{aligned}\nabla_{e_j}\nabla_{e_i}(\omega(U^H, V^H)) &= -\nabla_{e_j}(\omega^X(T(U^H, V^H), e_i)) \\ &= -\omega^X(\nabla^{TX}_{e_j}T(U^H, V^H), e_i) \\ &\quad -\omega^X(T(U^H, V^H), \nabla^{TX}_{e_j}e_i).\end{aligned} \tag{2.38}$$

Recall that we are using the normal coordinates, from (0.2), (2.15) and (2.38), at x_0, we have

$$\begin{aligned}(\nabla\nabla(\omega^H(U^H, V^H)))_{(e_i,e_j)} &= (\nabla\nabla(\omega^H(U^H, V^H)))_{(e_j,e_i)} \\ &= \nabla_{e_j}\nabla_{e_i}(\omega^H(U^H, V^H)) \\ &= \left\langle \nabla^{TX}_{e_j}T(U^H, V^H), J^{T_{\mathbb{R}}X}e_i\right\rangle.\end{aligned} \tag{2.39}$$

From (2.37) and (2.39), we get

$$(P^N \mathcal{O}_2^{(2)} P^N)(0,0) = \frac{1}{2} f^\alpha \wedge f^\beta R^E_{x_0}(f_\alpha^H, f_\beta^H) I_{\mathbb{C}\otimes E}. \tag{2.40}$$

Now by [6, Theorem 4.14] (cf. [5, Proposition 10.9], [8, (11.61)]), for the tensor S in (1.4), we have for $X, Y \in T_{\mathbb{R}}X$, $Z, W \in T_{\mathbb{R}}W$,

$$\begin{aligned}&\left\langle R^{TX}(X,Y)P^{TX}Z, P^{TX}W\right\rangle + \left\langle (SP^{TX}S)(X,Y)Z, W\right\rangle \\ &\quad + \left\langle (\nabla^{TX}S)(X,Y)Z, W\right\rangle = \left\langle R^{TX}(Z,W)X, Y\right\rangle.\end{aligned} \tag{2.41}$$

By (1.6), if $U, V \in T_{\mathbb{R}}S$, $X, Y \in T_{\mathbb{R}}X$, we have

$$\left\langle (\nabla^{TX}S)(X,Y)U^H, V^H\right\rangle = \frac{1}{2}\left\langle \nabla^{TX}_X T(U^H, V^H), Y\right\rangle - \frac{1}{2}\left\langle \nabla^{TX}_Y T(U^H, V^H), X\right\rangle. \tag{2.42}$$

Note that we are using the normal coordinates, thus as in (2.15), for a function h along the fiber X, the positive Laplacian Δ_X acts on h as

$$\Delta_X h = -4\frac{\partial^2 h}{\partial z_j \partial \overline{z}_j} \quad \text{at } x_0. \tag{2.43}$$

From (2.39), (2.42) and (2.43), we get

$$\begin{aligned}\Big\langle(\nabla^{TX}S)(\tfrac{\partial}{\partial z_j},\tfrac{\partial}{\partial \overline{z}_j})f_\alpha^H, f_\beta^H\Big\rangle_{x_0} &= \frac{1}{2}\langle\nabla^{TX}_{\frac{\partial}{\partial z_j}}(T(f_\alpha^H, f_\beta^H)), \tfrac{\partial}{\partial \overline{z}_j}\rangle\\ &\quad -\frac{1}{2}\Big\langle\nabla^{TX}_{\frac{\partial}{\partial \overline{z}_j}}(T(f_\alpha^H, f_\beta^H)), \tfrac{\partial}{\partial z_j}\Big\rangle\\ &=\sqrt{-1}\frac{\partial^2}{\partial z_j\partial\overline{z}_j}\omega^H(f_\alpha^H, f_\beta^H)\\ &= -\frac{\sqrt{-1}}{4}\Delta_X\omega^H(f_\alpha^H, f_\beta^H).\end{aligned}\tag{2.44}$$

By (1.6) and the fact that $S(\cdot)$ takes values in anti-symmetric elements of $\mathrm{End}(T_{\mathbb{R}}W)$, we find that for $U, V\in T_{\mathbb{R}}S$, $X, Y\in T_{\mathbb{R}}X$,

$$\begin{aligned}&\Big\langle(SP^{TX}S)(X,Y)U^H, V^H\Big\rangle\\ &= \Big\langle S(X)P^{TX}S(Y)U^H, V^H\Big\rangle - \Big\langle S(Y)P^{TX}S(X)U^H, V^H\Big\rangle\\ &= \Big\langle P^{TX}S(X)U^H, P^{TX}S(Y)V^H\Big\rangle - \Big\langle P^{TX}S(Y)U^H, P^{TX}S(X)V^H\Big\rangle\\ &= \Big\langle T(U^H,X), T(V^H,Y)\Big\rangle - \Big\langle T(U^H,Y), T(V^H,X)\Big\rangle.\end{aligned}\tag{2.45}$$

From (2.20), (2.29), (2.33), (2.41), (2.44) and (2.45), we get

$$\begin{aligned}&-(P^N\mathscr{O}_1^{(1)}P^{N^\perp}(\mathscr{L}_0^{(0)})^{-1}\mathscr{O}_1^{(1)}P^N)(0,0)\\ &=\frac{1}{2}\Big\langle(SP^{TX}S)(w_j,\overline{w}_j)\overline{g}_\beta^H, g_\alpha^H\Big\rangle\overline{g}^\beta\wedge g^\alpha P^N(0,0)\\ &=\frac{1}{2}\Big\langle(SP^{TX}S)(\tfrac{\partial}{\partial z_j},\tfrac{\partial}{\partial \overline{z}_j})f_\alpha^H, f_\beta^H\Big\rangle f^\alpha\wedge f^\beta I_{\mathbb{C}\otimes E}\\ &=\frac{1}{2}f^\alpha\wedge f^\beta\Big[\Big\langle R^{TX}(f_\alpha^H, f_\beta^H)\tfrac{\partial}{\partial z_j},\tfrac{\partial}{\partial \overline{z}_j}\Big\rangle+\frac{\sqrt{-1}}{4}\Delta_X(\omega(f_\alpha^H, f_\beta^H))\Big]I_{\mathbb{C}\otimes E}\\ &=\Big[\Big(\frac{1}{2}\mathrm{Tr}[R_{x_0}^{T^{(1,0)}X}]\Big)^H+\frac{\sqrt{-1}}{4}\Delta_{X,x_0}\omega^H\Big]I_{\mathbb{C}\otimes E}.\end{aligned}\tag{2.46}$$

As we work on E, $I_{\mathbb{C}\otimes E}=\mathrm{Id}_E$. From (2.17), (2.28), (2.40) and (2.46), we get

$$\begin{aligned}b_{2,1}(x_0)=\mathscr{Q}_2(0,0) &= -2\sqrt{-1}\omega_{x_0}^H\Big\{\frac{1}{8}r_{x_0}^X+R_{x_0}^E(\tfrac{\partial}{\partial z_j},\tfrac{\partial}{\partial \overline{z}_j})\Big\}\\ &\quad+\Big[R_{x_0}^E+\frac{1}{2}\mathrm{Tr}[R_{x_0}^{T^{(1,0)}X}]\Big]^H+\frac{\sqrt{-1}}{4}\Delta_{X,x_0}\omega^H.\end{aligned}\tag{2.47}$$

Note that for any 2-form ϑ on W, by (1.19) and (2.2), we have

$$(\vartheta\wedge\omega^n)^{(2)}=(n\vartheta\wedge(\omega^X)^{n-1})^{(0)}\wedge\omega^H+\vartheta^H\wedge(\omega^X)^n$$

$$= (-\sqrt{-1}\vartheta(w_j, \overline{w}_j)\omega^H + \vartheta^H) \wedge (\omega^X)^n. \tag{2.48}$$

From (2.2), (2.5), (2.47) and (2.48), we get the first two equations of (0.24).

By [30, Lemma 7.2.4, p. 314], for any $h \in \mathscr{C}^\infty(X_s)$ and $f \in \mathscr{C}^\infty(X_s, \mathrm{End}(E))$, we have

$$\begin{aligned} T_{f,p}(x,x) =& f(x)p^n + \mathscr{O}(p^{n-1}), \\ T_{h,p}(x,x) =& h(x)p^n + \Big(b_1(x)h(x) - \frac{1}{4\pi}(\Delta_X h)(x)\Big)p^{n-1} + \mathscr{O}(p^{n-2}), \\ & \text{with } b_1(x) = \frac{1}{8\pi}r^X + \frac{1}{2\pi}R^E(w_j, \overline{w}_j). \end{aligned} \tag{2.49}$$

Theorem 0.4, (2.47) and (2.49) imply that in (0.20), we have

$$\begin{aligned} R_1 &= b_{2,1} - \Big(b_1(x)b_{2,0}(x) - \frac{1}{4\pi}(\Delta_X b_{2,0})(x)\Big) \\ &= \Big(R^E + \frac{1}{2}\,\mathrm{Tr}[R^{T^{(1,0)}X}]\Big)^H - \frac{\sqrt{-1}}{4}\Delta_X \omega^H. \end{aligned} \tag{2.50}$$

The proof of Theorem 0.6 is completed.

Proof of Corollary 0.5 Let $h^{T^{(1,0)}S}$ be a Hermitian metric on $T^{(1,0)}S$, and $h^{TS\otimes H^0}$ be the Hermitian metric on $T^{(1,0)}S \otimes H^0(X, L^p \otimes E)$ induced by $h^{T^{(1,0)}S}$ and $h^{H^0(X,L^p\otimes E)}$.

We define $\dot{T}_{R_0,p} \in \mathrm{End}(T^{(1,0)}S \otimes H^0(X, L^p \otimes E))$ such that for $u, v \in T^{(1,0)}S$, $\sigma_1, \sigma_2 \in H^0(X, L^p \otimes E)$,

$$h^{TS\otimes H^0}(\dot{T}_{R_0,p}(u \otimes \sigma_1), v \otimes \sigma_2) := \langle T_{R_0(u^H,\overline{v}^H),p}\sigma_1, \sigma_2\rangle. \tag{2.51}$$

We define for $u, v \in T^{(1,0)}S$, $\xi, \eta \in L^p \otimes E$,

$$h_p(u \otimes \xi, v \otimes \eta) = -2\pi\sqrt{-1}\omega^H(u^H, \overline{v}^H)h^{L^p\otimes E}(\xi, \eta). \tag{2.52}$$

As ω is a Kähler form on W, h_p is in fact a Hermitian metric on $\pi^*(T^{(1,0)}S) \otimes L^p \otimes E$. But as $R_0 = -2\pi\sqrt{-1}\omega^H$, we know at $s \in S$,

$$h^{T^{(1,0)}S\otimes H^0}(\dot{T}_{R_0,p}(u \otimes \sigma_1), v \otimes \sigma_2) = \int_{X_s} h_p(u \otimes \sigma_1(x), v \otimes \sigma_2(x))(\omega^X(x))^n/n! \tag{2.53}$$

Thus $\dot{T}_{R_0,p}$ is positive definite. Combining with (0.20), we get Corollary 0.5. □

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