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Quillen Metrics and Branched Coverings

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We give a formula to compare the Quillen metrics associated to a branched covering from holomorphic line bundles.

Introduction

The Quillen metric is a metric on the determinant of the cohomology of a holomorphic vector bundle over a complex manifold. It is the product of L^2-metric and the analytic torsion, which is the regularized determinant of the Kodaira Laplacian. By Quillen, Bismut, Gillet, and Soulé, we know that the Quillen metric has very nice proprieties.

Let $i : Y \hookrightarrow X$ be an immersion of compact complex manifolds. Let η be a holomorphic Hermitian vector bundle over Y. Let ξ be a holomorphic resolution of η over X. Bismut and Lebeau [11] have calculated the relation of the Quillen metrics associated to η and ξ.

Let $\pi : W \to S$ be a holomorphic map of compact complex manifolds. Let ξ be a holomorphic Hermitian vector bundle over W. Let $R^\bullet \pi_* \xi$ be the direct image of ξ. Let $\lambda(\xi)$ and $\lambda(R^\bullet \pi_* \xi)$ be the inverses of the determinant of the cohomology of ξ and $R^\bullet \pi_* \xi$. By [22], $\lambda(\xi) \simeq \lambda(R^\bullet \pi_* \xi)$. If π is a submersion, Berthomieu and Bismut [2] have compared the corresponding Quillen metrics on $\lambda(\xi)$ and $\lambda(R^\bullet \pi_* \xi)$.

Suppose now that W, S are arithmetic varieties over $\mathrm{Spec}(\mathbb{Z})$. Let ξ be an algebraic vector bundle on W. In [14], Gillet and Soulé conjectured that an arithmetic Riemann–Roch formula holds. In [15], by using the Bismut–Lebeau embedding formula for Quillen

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metrics [11], they proved it for the first arithmetic Chern class. By using Bismut's work [5], the family version of [11], Gillet–Roessler–Soulé [12] show that the arithmetic Riemann–Roch formula in higher degrees holds.

In [3], Bismut has conjectured an equivariant arithmetic Riemann–Roch formula. In [4], he was able to show the compatibility of his conjecture with immersions. In [23], Köhler and Roessler have obtained a version of Bismut's conjecture by using [4]. For more recent works in this direction, cf. [21, 24, 28].

In this paper, we will compare the Quillen metrics on $\lambda(\xi)$ and $\lambda(R^\bullet\pi_*\xi)$ in the case that π is a branched covering from a holomorphic line bundle. For any holomorphic line bundle over a compact complex manifold S, we give a general construction of a smooth submanifold $W \subset L$ (cf. (1.1)) from holomorphic sections of the powers of L on S such that $\pi_W : W \to S$ the projection from W on S, is a branched covering. We obtain the analogue of the result of Berthomieu–Bismut [2,Theorem 0.1] and its equivariant version [25, Theorem 3.1] in this situation. In fact, our first result, Theorem 3.4, is compatible with the arithmetic Riemann–Roch formula. Our second result, Theorem 4.1, fits perfectly well with Bismut's conjecture.

This paper is organized as follows. In Section 1, we construct a branched covering from a holomorphic line bundle. In Section 2, we describe the canonical sections of determinant lines. In Section 3, applying the Bismut–Lebeau embedding formula [11, Theorem 0.1], we calculate the Quillen norm of the canonical section. In Section 4, using the Bismut equivariant embedding formula [4, Theorem 0.1], we calculate the equivariant Quillen norm of the canonical section.

1 Branched Coverings

Let S be a compact complex manifold. Let L be a holomorphic line bundle on S.

Let $\alpha_i \in H^0(S, L^i)$ $(1 \leqslant i \leqslant d, d \geq 2, d \in \mathbb{N}^*)$. For $(x,t) \in L$, $x \in S$, set

$$F(\alpha)(x,t) = t^d + \sum_{i=1}^{d} \alpha_i(x) t^{d-i}, \qquad W = \Big\{(x,t) \in L : F(\alpha)(x,t) = 0\Big\}. \tag{1.1}$$

We suppose that W is smooth.

Let $V = \mathbb{P}(L \oplus 1)$ the projectivisation of the vector bundle $L \oplus \mathbb{C}$, here $\mathbb{C}$ is the trivial line bundle on S. We identify S with $\{(x,(0,1)) \in V : x \in S\} \subset V$. Let $\pi : V \to S$ be the natural projection with fibre Y. The complement of $\mathbb{P}(L) \simeq S$ in V is canonical isomorphic to L, so we can identify W to a sub-manifold of $V = \mathbb{P}(L \oplus 1)$. Let $\pi_W : W \to S$ be the projection induced by π. Then W is a branched covering of S of degree d.

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Let ξ be a holomorphic vector bundle on S. Let

$$\xi' = \pi_W^* \xi \tag{1.2}$$

be the pull-back of the bundle ξ on W. Let $R^\bullet \pi_{W*}\xi'$, $R^\bullet \pi_{W*}\mathcal{O}_W$ be the direct images of $\mathcal{O}_W(\xi')$, $\mathcal{O}_W$, the sheaves of holomorphic sections of ξ', and of holomorphic functions on W, respectively. By [17, Theorem 2.4.2], $R^\bullet \pi_{W*}\mathcal{O}_W = R^0 \pi_{W*}\mathcal{O}_W$ is locally free of rank d on S. By [20, Exercise 3.8.3], we have

$$R^\bullet \pi_{W*}\xi' = R^0 \pi_{W*}\mathcal{O}_W \otimes \xi. \tag{1.3}$$

Let $H^\bullet(W,\xi') = \bigoplus_{j=1}^{\dim W} H^j(W,\xi')$, $H^\bullet(S, R^0\pi_{W*}\xi')$ be the cohomology groups of $\mathcal{O}_W(\xi')$ on W, $\mathcal{O}_S(R^0\pi_{W*}\xi')$ on S, respectively.

For a complex vector space E, the determinant line of E is the complex line

$$\det E = \Lambda^{\max} E. \tag{1.4}$$

Definition 1.1. Set

$$\begin{aligned} \lambda(\xi') &= \bigotimes_i \left(\det H^i(W,\xi')\right)^{(-1)^{i+1}}, \\ \lambda(R^\bullet\pi_{W*}\xi') &= \bigotimes_i \left(\det H^i(S, R^0\pi_{W*}\xi')\right)^{(-1)^{i+1}}. \end{aligned} \tag{1.5}$$

By [22], we have the canonical isomorphism $\lambda(\xi') \simeq \lambda(R^\bullet\pi_{W*}\xi')$. Let σ be the canonical section of $\lambda(\xi') \otimes \lambda^{-1}(R^\bullet\pi_{W*}\xi')$.

Example: Let $\mathbb{CP}^n$ be the complex projective space of dimension n. Let $(z_0,\cdots,z_n) = (z_0, z)$ be the homogeneous coordinate. Let $S = \{z_0 = 0\} \hookrightarrow \mathbb{CP}^n$. Let W be a hypersurface of degree d, which doesn't contain the point $(1,0)$. Let $\pi : W \to S$ be the projection from $(1,0)$. Let $L = \mathcal{O}_S(1)$ be the hyperplane line bundle on S. By [18, page 167], we can reduce this to the situation (1.1).

Remark 1.2. Let $\pi : S_1 \to S_2$ be a finite mapping of Riemann surfaces of degree n. Let $\mathcal{M}(S_1), \mathcal{M}(S_2)$ be the meromorphic function fields on S_1, S_2. Then π is characterized by the finite field extension $\mathcal{M}(S_2) \hookrightarrow \mathcal{M}(S_1)$ [27, §2.11]. So S_1, π is constructed by an irreducible polynomial

$$P(T) = T^n + c_1 T^{n-1} + \cdots + c_n \in \mathcal{M}(S_2)[T]. \tag{1.6}$$

So our construction contains a large part of general maps of Riemann surfaces.

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Let $\iota : S \to V$, $\jmath : W \to V$ be the natural immersions. Let $\mathcal{O}_V(-1)$ be the universal line bundle over V. Let $\mathcal{O}_V(k) = \mathcal{O}_V(-1)^{\otimes -k}$. On V, we have the exact sequence of holomorphic vector bundles [6, (1.21)],

$$0 \to \mathcal{O}_V(-1) \xrightarrow{a} \pi^*L \oplus \mathbb{C} \xrightarrow{a} \frac{\pi^*L \oplus \mathbb{C}}{\mathcal{O}_V(-1)} \to 0. \tag{1.7}$$

Let $\tau_{[S]}(y) \in \left(\frac{\pi^*L\oplus\mathbb{C}}{\mathcal{O}_V(-1)}\right)_y$ be given by

$$\tau_{[S]}(y) = a_y(0,-1). \tag{1.8}$$

Then $\tau_{[S]}$ is a holomorphic section of $\frac{\pi^*L\oplus\mathbb{C}}{\mathcal{O}_V(-1)}$, which vanishes exactly on S. The map $\theta : \pi^*L \to \frac{\pi^*L\oplus\mathbb{C}}{\mathcal{O}_V(-1)}$ induced by the projection from $\pi^*L \oplus \mathbb{C}$ is an isomorphism on $L \subset \mathbb{P}(L \oplus 1)$. Under this identification, $\tau_{[S]}$ is the tautological section of π^*L on L. We have

$$\mathrm{div}(\tau_{[S]}) = S. \tag{1.9}$$

Let $\sigma_{[S]}$ be the canonical section of $[S]$ on $\mathbb{P}(L \oplus 1)$. Then $\sigma_{[S]}^{-1} \otimes \tau_{[S]}$ is a nonzero section of $[S]^{-1} \otimes \frac{\pi^*L\oplus\mathbb{C}}{\mathcal{O}_V(-1)}$. We identify the line bundle $[S]$ to $\frac{\pi^*L\oplus\mathbb{C}}{\mathcal{O}_V(-1)}$ via this section. In particular, we get

$$[S]|_S = L. \tag{1.10}$$

The exact sequence (1.7) induces also an isomorphism

$$[S] \simeq \pi^*L \otimes \mathcal{O}_V(1). \tag{1.11}$$

Remark 1.3. If the linear system $|L^d|$ hasn't any base points, then for the generic elements $\alpha_i \in H^0(S, L^i)$ $(1 \leqslant i \leqslant d)$, W is smooth.

In fact, let ν be the holomorphic section of $\mathcal{O}_V(1)$ defined by $(0,1) \in (\pi^*L \oplus \mathbb{C})^*$, then

$$\mathrm{div}[\nu] = \mathbb{P}(L). \tag{1.12}$$

By (1.8), for $c \in \mathbb{C}$, $\alpha_i \in H^0(S, L^i)$ $(1 \leqslant i \leqslant d)$, put

$$G(\alpha, c) = c\tau_{[S]}^d + \sum_{i=1}^d \alpha_i(x)\nu^i \tau_{[S]}^{d-i}, \tag{1.13}$$

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then $\{G(\alpha, c) : \alpha_i \in H^0(S, L^i), 1 \leqslant i \leqslant d, c \in \mathbb{C}\}$ is a linear system of $[dS]$ on V, and the base locus of this system is empty. By Bertini's Theorem [18, page 137], $\{G(\alpha, 1) = 0\} \subset \mathbb{P}(L \oplus 1)$ is smooth for generic elements $\alpha_i \in H^0(S, L^i)$. If we identify π^*L to $[S]$ on L as above, then $G(\alpha, 1) = F(\alpha)$, so we obtain our Remark.

2 Canonical Isomorphisms of Determinant Lines

By (1.1), we can extend $F(\alpha)$ to a meromorphic section of π^*L^d on V. Let $t : L \to \pi^*L$ be the tautological section of π^*L on $L \subset \mathbb{P}(L \oplus 1) = V$. Then t extends naturally to a meromorphic section of π^*L on V. Set

$$f(\alpha) = F(\alpha)/t^d. \tag{2.1}$$

Then $f(\alpha)$ is a meromorphic function on V, and

$$\mathrm{div}(f(\alpha)) = W - d \cdot S. \tag{2.2}$$

Let $\delta_{\{W\}}, \delta_{\{S\}}$ be the currents on V defined by the integration on W, S. By (2.2), we have

$$\frac{\overline{\partial}\partial}{2i\pi} \log |f(\alpha)|^2 = \delta_{\{W\}} - d\, \delta_{\{S\}}. \tag{2.3}$$

We will identify the line bundle $[W]$ to $[dS]$ via $f(\alpha)$. Let $\tau_{[W]}$ be the canonical section of $[W]$ on V, then

$$\tau_{[W]} = f(\alpha)\tau_{[S]}^d. \tag{2.4}$$

Let $TY = TV/S$ be the holomorphic tangent bundle to the fibre Y. By (1.6), as in [18, page 409], we have an exact sequence of holomorphic vector bundles on V,

$$0 \to \mathbb{C} \to (\pi^*L \oplus \mathbb{C}) \otimes \mathcal{O}_V(1) \to TY \to 0. \tag{2.5}$$

Let $K_Y = T^*Y$ be the relative canonical bundle on V. By (2.5),

$$K_Y \simeq \pi^*L^{-1} \otimes \mathcal{O}_V(-2). \tag{2.6}$$

Proposition 2.1. For $k > 0$, we have canonical identifications

$$\begin{aligned} &R^0\pi_*\mathcal{O}_V = \mathbb{C}, \qquad R^0\pi_*\mathcal{O}_V(-k) = 0,\\ &R^1\pi_*\mathcal{O}_V(-1) = R^1\pi_*\mathcal{O}_V = 0,\\ &R^1\pi_*\mathcal{O}_V(-(k+1)) = \bigoplus_{i=1}^{k} L^i. \end{aligned} \tag{2.7}$$

Proof. The first two equations are trivial.

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Using the Serre duality [20, page 240] and (2.6), for $m \in \mathbb{Z}$, we have

$$\begin{aligned} R^1\pi_*\mathcal{O}_V(-m) &\simeq (H^0(Y, \mathcal{O}_V(m) \otimes K_Y))^* \\ &= L \otimes (H^0(Y, \mathcal{O}_V(m-2)))^*. \end{aligned} \tag{2.8}$$

The second equation of (2.7) and (2.8) implies the third equation of (2.7).

For $k > 0$, by [18, page 165], we have

$$H^0(Y, \mathcal{O}_V(k-1)) = \mathrm{Sym}^{k-1}((L \oplus \mathbb{C})^*) = \bigoplus_{i=0}^{k-1} L^{-i}. \tag{2.9}$$

By (2.8) and (2.9), we get the last equation of (2.7). ■

Proposition 2.2. We have a canonical isomorphism,

$$R^0\pi_{W*}\mathcal{O}_W \simeq \bigoplus_{j=0}^{d-1} L^{-j}. \tag{2.10}$$

Proof. By [17, §2.4], we can identify $R^0\pi_{W*}\mathcal{O}_W$ as the sheaf of polynomial functions along the fiber L with degree $\leqslant d-1$, thus we get (2.10). ■

Using [20, Exercise 3.8.3], (1.11), (2.7), and (2.10), for $k \geq 2$, we get

$$\begin{aligned} & R^\bullet\pi_*\mathcal{O}_V([-S]) = 0, \\ & R^0\pi_*\mathcal{O}_V([-kS]) = 0, \quad R^1\pi_*\mathcal{O}_V([-kS]) = \bigoplus_{j=1}^{k-1} L^{-j}, \\ & R^0\pi_{W*}\xi' \simeq \bigoplus_{j=0}^{d-1} L^{-j} \otimes \xi. \end{aligned} \tag{2.11}$$

Note that we identify $[W]$ with $[dS]$ via (2.4), by (2.11), we have

$$R^0\pi_*\mathcal{O}_V([-W]) = 0, \quad R^1\pi_*\mathcal{O}_V([-W]) = \bigoplus_{j=1}^{d-1} L^{-j}. \tag{2.12}$$

We have the following exact sequence of sheaves over V:

$$0 \to \mathcal{O}_V([-W]) \overset{\tau_{[W]}}{\to} \mathcal{O}_V \to \jmath_*\mathcal{O}_W \to 0. \tag{2.13}$$

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By (2.7) and (2.13), we get the following exact sequence of sheaves on S:

$$0 \to R^0\pi_*\mathcal{O}_V \xrightarrow{\jmath} R^0\pi_{W*}\mathcal{O}_W \xrightarrow{\delta_1} R^1\pi_*\mathcal{O}_V([-W]) \to 0. \tag{2.14}$$

Proposition 2.3. Under the canonical identification (2.10) and (2.12), the exact sequence (2.14) is canonically split. Let $\delta : \bigoplus_{i=1}^{d-1} L^{-i} \to R^1\pi_*\mathcal{O}_V([-W])$ be the map induced by δ_1 and (2.10), then under the decomposition $L^{-1} \oplus \cdots \oplus L^{-d+1}$, we have

$$\delta^{-1} = (a_{ij}) = \begin{pmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{pmatrix}. \tag{2.15}$$

Moreover,

$$\begin{aligned} a_{ij} = \alpha_{j-i} \quad & \text{if} \quad j > i, \\ 1 \quad & \text{if} \quad i = j, \\ 0 \quad & \text{if} \quad j < i. \end{aligned} \tag{2.16}$$

Proof. Clearly, under the identification (2.7), $\jmath$ is the canonical embedding of $\mathbb{C}$ into the factor $\mathbb{C}$ in $R^0\pi_{W*}\mathcal{O}_W$, so the exact sequence (2.14) is canonical split.

To prove (2.15), we use Čech cohomology. Before prove (2.15), we explain the compatibility of (2.8), (2.9), and Čech cohomology on $\mathbb{CP}^n$.

Let $(X_0, \cdots, X_n)$ be linear coordinates on $\mathbb{C}^{n+1}$, and let $\{x_i = X_i/X_0 : i = 1, \cdots, n\}$ be the corresponding affine coordinates. Let $U_i = (X_i \neq 0) \subset \mathbb{CP}^n$. Let $K = \Lambda^n(T^*\mathbb{CP}^n)$ be the canonical line bundle on $\mathbb{CP}^n$. By [18, page 409], we have an exact sequence of holomorphic vector bundles on $\mathbb{CP}^n$

$$0 \to \mathbb{C} \to \mathcal{O}_{\mathbb{CP}^n}(1)^{n+1} \to T\mathbb{CP}^n \to 0. \tag{2.17}$$

By (2.17), we have

$$K \overset{\nu}{\simeq} \mathcal{O}_{\mathbb{CP}^n}(-(n+1)). \tag{2.18}$$

We trivialize $\mathcal{O}_{\mathbb{CP}^n}(1)$ by $(1, 0, \cdots, 0) \in \mathbb{C}^{n+1,*}$ on U_0. By [18, page 409], on U_0, we have

$$\nu(dx_1 \wedge \cdots \wedge dx_n) = 1 \in \mathcal{O}_{\mathbb{CP}^n}(-(n+1)). \tag{2.19}$$

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By [20, Remark 3.7.1.1], there exists a canonical element $a \in H^n(\mathbb{CP}^n, K)$, which defines the Serre duality μ. On $\cap_{i=0}^n U_i$, consider the cocycle, we have

$$a = \frac{1}{x_1 \cdots x_n} dx_1 \wedge \cdots \wedge dx_n. \tag{2.20}$$

By [20, Theorem 3.5.1], using Čech cohomology, on $\cap_{i=0}^n U_i$, $H^n(\mathbb{CP}^n, \mathcal{O}_{\mathbb{CP}^n}(-n-k-1))$ ($k \in \mathbb{N}$) is generated by the Čech cocycle

$$\Big\{\alpha_{l_1 \cdots l_n} = x_1^{-(l_1+1)} \cdots x_n^{-(l_n+1)} : \sum_{i=1}^n l_i \leqslant k, l_i \in \mathbb{N}\Big\}.$$

Also $H^0(\mathbb{CP}^n, \mathcal{O}_{\mathbb{CP}^n}(k))$ ($k \in \mathbb{N}$) is generated by

$$\Big\{\beta_{l_1 \cdots l_n} = x_1^{l_1} \cdots x_n^{l_n} : \sum_{i=1}^n l_i \leqslant k, l_i \in \mathbb{N}\Big\}$$

on U_0. By (2.19) and (2.20), we have the following commutative diagram:

$$\begin{array}{ccc} H^n(\mathbb{CP}^n, \mathcal{O}_{\mathbb{CP}^n}(-n-k-1)) & \xrightarrow{\text{Čech}} & \operatorname{span}\Big\{\alpha_{l_1 \cdots l_n} : \sum_{i=1}^n l_i \leqslant k, l_i \in \mathbb{N}\Big\} \\ \Big\downarrow \mu & & \Big\downarrow \mu_1 \\ H^0(\mathbb{CP}^n, \mathcal{O}_{\mathbb{CP}^n}(n+k+1) \otimes K)^* & & \\ \Big\downarrow \wr v & & \\ H^0(\mathbb{CP}^n, \mathcal{O}_{\mathbb{CP}^n}(k))^* & \xrightarrow{\text{Čech}} & \Big(\operatorname{span}\Big\{\beta_{l_1 \cdots l_n} : \sum_{i=1}^n l_i \leqslant k, l_i \in \mathbb{N}\Big\}\Big)^*. \end{array} \tag{2.21}$$

Thus, the map μ_1 is such that

$$\mu_1(\alpha_{l_1 \cdots l_n})(\beta_{l'_1 \cdots l'_n}) = \delta_{(l_1 \cdots l_n),(l'_1 \cdots l'_n)}.$$

Now we are ready to establish (2.15). Let (v, u) be the local homogeneous coordinates of $\mathbb{P}(L \oplus 1)$. Let $U_1 = \{(v, u) \in \mathbb{P}(L \oplus 1) : v \neq 0\}$, $U_2 = \{(v, u) \in \mathbb{P}(L \oplus 1) : u \neq 0\}$ with affine coordinate t as a function on U_2 with values in $\pi^* L$. We will identify U_2 with L, then $U_1 \cap U_2 = L \setminus S$, t^{-1} is a section of $\pi^* L^{-1}$ on $U_1 \cap U_2$. As explained in the proof of

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Proposition 2.2, on U_2, for $x \in S$, we have

$$(R^0\pi_{W*}\mathcal{O}_W)_x = \Big\{ \sum_{i=0}^{d-1} \gamma_i t^i : \quad \gamma_i \in \mathcal{O}_{S,x}(L^{-i}) \Big\}. \tag{2.22}$$

We recall that from (1.11) and (2.2), on V,

$$[-W] = \pi^* L^{-d} \otimes \mathcal{O}_V(-d).$$

By (2.7) and (2.21), on $U_1 \cap U_2$, for $x \in S$, we have

$$(R^1\pi_*\mathcal{O}_V([-W]))_x = \Big\{ \sum_{j=1}^{d-1} \gamma_{d-j} t^{-j} : \quad \gamma_{d-j} \in \mathcal{O}_{S,x}(L^{-d+j}) \Big\}. \tag{2.23}$$

On $U_1 \cap U_2$, we have

$$\tau_{[W]}(\gamma_{d-j}t^{-j}) = \gamma_{d-j}\Big(t^{d-j} + \sum_{i=1}^{d} \alpha_i(x) t^{d-i-j} \Big).$$

The function $\gamma_{d-j}(t^{d-j}+\sum_{i=1}^{d-j-1} \alpha_i(x)t^{d-i-j})$ is holomorphic on U_2, $\gamma_{d-j}(\sum_{i=d-j}^{d} \alpha_i(x)t^{d-i-j})$ is holomorphic on U_1. By the definition of δ_1, we have

$$\delta_1\Big(\gamma_{d-j}(t^{d-j} + \sum_{i=1}^{d-j-1} \alpha_i(x)t^{d-i-j}) \Big) = \gamma_{d-j}t^{-j}. \tag{2.24}$$

By (2.22), (2.23), and (2.24), we have (2.15) and (2.16). ■

We also have an exact sequence of sheaves over V

$$0 \to \mathcal{O}_V([-dS]) \overset{\tau^d_{[S]}}{\to} \mathcal{O}_V \to \iota_*\mathcal{O}_S\big(\bigoplus_{i=0}^{d-1} L^{-i}\big) \to 0. \tag{2.25}$$

By (2.7) and (2.25), we have the following exact sequence of sheaves over S:

$$0 \to R^0\pi_*\mathcal{O}_V \to \mathcal{O}_S\big(\bigoplus_{i=0}^{d-1} L^{-i}\big) \overset{\delta'}{\to} R^1\pi_*\mathcal{O}_V([-dS]) = \mathcal{O}_S\big(\bigoplus_{i=1}^{d-1} L^{-i}\big) \to 0. \tag{2.26}$$

Proposition 2.4. Under the identifications (2.11), the exact sequence (2.26) is naturally split, and $\delta'|_{\bigoplus_{i=1}^{d-1} L^{-i}} = \mathrm{Id}$.

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Proof. We use the notation in the proof of Proposition 2.3. On $U_1 \cap U_2$, we have

$$\tau^d_{[S]}(\gamma_{d-j} t^{-j}) = \gamma_{d-j} t^{d-j}. \tag{2.27}$$

In (2.25) as in (2.23), we have identified $\iota_* \mathcal{O}_S(L^{-i})$ to $\{\gamma_i t^i : \gamma_i \in \mathcal{O}_S(L^{-i})\}$ on U_2.

By the definition of δ' and (2.27), we have Proposition 2.4. ■

As in Definition 1.1, we define the complex lines

$$\begin{aligned} \lambda'_d(\pi^*\xi) &= \lambda(\pi^*\xi) \otimes \lambda^{-1}([-dS] \otimes \pi^*\xi), \\ \lambda_W(\xi) &= \lambda(R^0\pi_*\mathcal{O}_V \otimes \xi) \otimes \lambda(R^1\pi_*([-W]) \otimes \xi). \end{aligned} \tag{2.28}$$

By [22], (2.13), and (2.25), we have the canonical isomorphisms:

$$\lambda(\xi') \simeq \lambda'_d(\pi^*\xi), \quad \lambda(\bigoplus_{i=0}^{d-1} L^{-i} \otimes \xi) \simeq \lambda'_d(\pi^*\xi). \tag{2.29}$$

Let τ_d, σ_1 be the canonical sections of $\lambda^{-1}(\bigoplus_{i=0}^{d-1} L^{-i} \otimes \xi) \otimes \lambda'_d(\pi^*\xi)$ via (2.25), $\lambda^{-1}(\xi') \otimes \lambda'_d(\pi^*\xi)$ via (2.13). Recall that σ is the canonical section of $\lambda(\xi') \otimes \lambda^{-1}(R^\bullet \pi_{W*}\xi')$.

Proposition 2.5. Under the identifications (2.11), we have

$$\sigma = \sigma_1^{-1} \otimes \tau_d. \tag{2.30}$$

Proof. Let ν_3 be the canonical section of

$$\lambda(R^1\pi_*[-W] \otimes \xi) \otimes \lambda(R^1\pi_*[-dS] \otimes \xi)^{-1}$$

induced by δ in Proposition 2.3. Let $p_r : \bigoplus_{i=r}^{d-1} L^{-i} \to \bigoplus_{i=r+1}^{d-1} L^{-i}$ be the canonical projection. Let $\delta_r : \bigoplus_{i=r}^{d-1} L^{-i} \to \bigoplus_{i=r}^{d-1} L^{-i}$ be the map defined by the matrix (a_{ij}) as in (2.16), then we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & L^{-r} & \longrightarrow & \bigoplus_{j=r}^{d-1} L^{-j} & \xrightarrow{p_r} & \bigoplus_{j=r+1}^{d-1} L^{-j} & \longrightarrow & 0 \\ & & \downarrow \mathrm{Id} & & \downarrow \delta_r & & \downarrow \delta_{r+1} & & \\ 0 & \longrightarrow & L^{-r} & \longrightarrow & \bigoplus_{j=r}^{d-1} L^{-j} & \xrightarrow{p_r} & \bigoplus_{j=r+1}^{d-1} L^{-j} & \longrightarrow & 0. \end{array} \tag{2.31}$$

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By considering the long exact sequence from (2.31),

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(S, L^{-r}\otimes\xi) & \longrightarrow & H^0(S, \bigoplus_{j=r}^{d-1} L^{-j}\otimes\xi) & \xrightarrow{p_r} & H^0(S, \bigoplus_{j=r+1}^{d-1} L^{-j}\otimes\xi) & \longrightarrow & \cdots \\ & & \downarrow \mathrm{Id} & & \downarrow \delta_r & & \downarrow \delta_{r+1} & & \\ 0 & \longrightarrow & H^0(S, L^{-r}\otimes\xi) & \longrightarrow & H^0(S, \bigoplus_{j=r}^{d-1} L^{-j}\otimes\xi) & \xrightarrow{p_r} & H^0(S, \bigoplus_{j=r+1}^{d-1} L^{-j}\otimes\xi) & \longrightarrow & \cdots, \end{array} \tag{2.32}$$

as $\delta_{d-1} : L^{-d+1} \to L^{-d+1}$ is the identity map, by recurrence, we know the canonical section of

$$\lambda(\bigoplus_{j=r}^{d-1} L^{-j}\otimes\xi)\otimes\lambda^{-1}(\bigoplus_{j=r}^{d-1} L^{-j}\otimes\xi)$$

induced by δ_r is 1 for all $r \geq 1$. We conclude in particular that

$$\nu_3 = 1. \tag{2.33}$$

As in (1.5), we define the complex line $\lambda(\xi)$ for ξ on S. By Proposition 2.3, (2.14), and (2.27), we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^0\pi_*\mathcal{O}_V & \longrightarrow & R^0\pi_{*}J_*\mathcal{O}_W & \xrightarrow{\delta_1} & R^1\pi_*\mathcal{O}_V([-W]) & \longrightarrow & 0 \\ & & \uparrow \mathrm{Id} & & \uparrow \mathrm{Id} & & \uparrow \delta & & \\ 0 & \longrightarrow & R^0\pi_*\mathcal{O}_V & \longrightarrow & \mathcal{O}_S(\bigoplus_{j=0}^{d-1} L^{-j}) & \longrightarrow & R^1\pi_*\mathcal{O}_V([-dS]) & \longrightarrow & 0. \end{array} \tag{2.34}$$

Let ν_1, ν_2 be the canonical sections of

$$\lambda^{-1}(R^0\pi_{W*}\xi')\otimes\lambda_W(\xi), \quad \lambda^{-1}(\bigoplus_{j=0}^{d-1} L^{-j}\otimes\xi)\otimes\lambda(\xi)\otimes\lambda(R^1\pi_*\mathcal{O}_V([-dS])\otimes\xi)$$

induced by (2.14) and (2.26). By (2.15), (2.33), and (2.34), we have

$$\nu_1 = \nu_2\otimes\nu_3 = \nu_2. \tag{2.35}$$

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In our situation, the Leray spectral sequences [19, §3.7] associated to $\pi : V \to S$ and the considering vector bundles η ($\eta = [-dS] \otimes \pi^*\xi$, etc) are degenerate, as $R^0\pi_*\eta = 0$ or $R^1\pi_*\eta = 0$, so

$$H^k(V,\eta) \simeq \bigoplus_{i+j=k} H^i(S, R^j\pi_*\eta). \tag{2.36}$$

Then by (2.13), (2.14), (2.26), and (2.36), we have the following commutative diagram of long exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^0(V,\pi^*\xi) & \longrightarrow & H^0(W,\pi_W^*\xi) & \longrightarrow & H^1(V,[-W]\otimes\pi^*\xi) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & H^0(S,\xi) & \longrightarrow & H^0(S,R^0\pi_{W*}\mathcal{O}_W\otimes\xi) & \xrightarrow{\delta_1} & H^0(S,R^1\pi_*[-W]\otimes\xi) & \longrightarrow & \cdots \\
 & & \uparrow{\scriptstyle \mathrm{Id}} & & \uparrow & & \uparrow{\scriptstyle \delta} & & \\
0 & \longrightarrow & H^0(S,\xi) & \longrightarrow & H^0(S,\bigoplus_{j=0}^{d-1} L^{-j}\otimes\xi) & \xrightarrow{\delta'} & H^0(S,R^1\pi_*[-dS]\otimes\xi) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow{\scriptstyle \mathrm{Id}} & & \uparrow & & \\
0 & \longrightarrow & H^0(V,\pi^*\xi) & \longrightarrow & H^0(S,\bigoplus_{j=0}^{d-1} L^{-j}\otimes\xi) & \longrightarrow & H^1(V,[-dS]\otimes\pi^*\xi) & \longrightarrow & \cdots .
\end{array} \tag{2.37}$$

Let τ be the canonical section of ${\lambda'_d}^{-1}(\pi^*\xi) \otimes \lambda_W(\xi)$ induced by (2.36). The σ (resp. τ) is obtained from the second vertical map (resp. the rest part of the vertical maps) of the first two lines of (2.37). The σ_1 (resp. ν_1) is obtained from the first (resp. second) line of (2.37), and ν_2 and τ_d is obtained from the third and fourth line of (2.37). Finally, τ is also obtained from the first and third vertical maps of the last two lines of (2.37).

By [7, (1.3)], [22, Proposition 1], (2.13), (2.25), and (2.37), we have

$$\begin{aligned} \sigma \otimes \tau &= \sigma_1^{-1} \otimes \nu_1, \\ \tau &= \nu_2 \otimes \tau_d^{-1}. \end{aligned} \tag{2.38}$$

By (2.35) and (2.38), we have (2.30). ■

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For $0 \leqslant i \leqslant d-1$, we have an exact sequence of sheaves over V

$$0 \to \mathcal{O}_V([-(i+1)S]) \stackrel{\tau_{[S]}}{\to} \mathcal{O}_V([-iS]) \to \iota_* \mathcal{O}_S(L^{-i}) \to 0. \tag{2.39}$$

By (2.39), we have the exact sequence of sheaves over V

$$0 \to \bigoplus_{i=0}^{d-1}[-(i+1)S] \otimes \pi^*\xi \stackrel{\tau_{[S]}}{\to} \bigoplus_{i=0}^{d-1}[-iS] \otimes \pi^*\xi \to \iota_* \mathcal{O}_S(\bigoplus_{i=0}^{d-1} L^{-i} \otimes \xi) \to 0. \tag{2.40}$$

Let $\lambda_d(\pi^*\xi)$, $\lambda_V([-kS] \otimes \pi^*\xi)$ $(k \geq 1)$ be the complex lines

$$\begin{aligned} \lambda_d(\pi^*\xi) &= \lambda(\oplus_{i=0}^{d-1}[-iS] \otimes \pi^*\xi) \otimes \lambda^{-1}(\oplus_{i=1}^{d}[-iS] \otimes \pi^*\xi), \\ \lambda_V([-kS] \otimes \pi^*\xi) &= \lambda([-(k-1)S] \otimes \pi^*\xi) \otimes \lambda^{-1}([-kS] \otimes \pi^*\xi). \end{aligned} \tag{2.41}$$

By [22], (2.39), and (2.40), we have the canonical isomorphisms:

$$\begin{aligned} \lambda(\oplus_{i=0}^{d-1} L^{-i} \otimes \xi) \simeq \lambda_d(\pi^*\xi), \quad \lambda(L^{-k+1} \otimes \xi) \simeq \lambda_V([-kS] \otimes \pi^*\xi), \\ \lambda_d(\pi^*\xi) = \lambda'_d(\pi^*\xi). \end{aligned} \tag{2.42}$$

Let φ_k, ρ_d be the canonical sections of

$$\lambda^{-1}(L^{-k+1} \otimes \xi) \otimes \lambda_V([-kS] \otimes \pi^*\xi), \quad \lambda^{-1}(\oplus_{i=0}^{d-1} L^{-i} \otimes \xi) \otimes \lambda_d(\pi^*\xi).$$

Then

$$\rho_d = \bigotimes_{i=1}^{d} \varphi_i. \tag{2.43}$$

Proposition 2.6. Under the identification (2.11), we have

$$\tau_d = \rho_d. \tag{2.44}$$

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Proof. For $k \geq 1$, consider the complex of $\mathcal{O}_V$-sheaves on V

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \to & \iota_* \mathcal{O}_S(L^{-k}) & \xrightarrow{\mathrm{Id}} & \iota_* \mathcal{O}_S(\oplus_{i=0}^{k} L^{-i}) & \to & \iota_* \mathcal{O}_S(\oplus_{i=0}^{k-1} L^{-i}) & \to & 0 \to 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \to & 0 & \to & \mathcal{O}_V & \xrightarrow{\mathrm{Id}} & \mathcal{O}_V & \to & 0 \to 0 \\
 & & \uparrow & & \uparrow{\scriptstyle \tau_{[S]}^{k+1}} & & \uparrow{\scriptstyle \tau_{[S]}^{k}} & & \uparrow \\
0 & \to & 0 & \to & \mathcal{O}_V[-(k+1)S]) & \xrightarrow{\tau_{[S]}} & \mathcal{O}_V([-kS]) & \to & \iota_* \mathcal{O}_S(L^{-k}) \to 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 & & 0 & & 0 & & 0 & & 0\,.
\end{array} \tag{2.45}$$

In (2.45), the rows are exact sequences of sheaves. The second and third columns correspond to (2.25).

By [22], (2.29), (2.42), and (2.45), we have

$$\tau_k^{-1} \otimes \tau_{k+1} = \varphi_{k+1}. \tag{2.46}$$

By (2.25) and (2.39), we have also

$$\varphi_1 = \tau_1. \tag{2.47}$$

By (2.43), (2.46), and (2.47), we have (2.44). ■

3 Comparison Formula for the Quillen Metrics

Definition 3.1. Let P^V be the vector space of smooth forms on a complex manifold V, which are sums of forms of type (p, p). Let $P^{V,0}$ be the vector space of the forms $\alpha \in P^V$ such that there exist smooth forms β, γ on V for which $\alpha = \partial\beta + \overline{\partial}\gamma$.

If A is a (q, q) matrix, set

$$\mathrm{Td}(A) = \det\left(\frac{A}{1 - e^{-A}}\right), \qquad \mathrm{ch}(A) = \mathrm{Tr}[\exp(A)], \quad c_1(A) = \mathrm{Tr}[A]. \tag{3.1}$$

The genera associated to Td and ch are called the Todd genus and the Chern character.

Let P be an ad-invariant power series on square matrices. If (F, h^F) is a holomorphic Hermitian vector bundle on V, let ∇^F be the corresponding holomorphic Hermitian connection, and let R^F be its curvature. Set

$$P(F, h^F) = P\left(\frac{-R^F}{2i\pi}\right). \tag{3.2}$$

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By the Chern–Weil theory, $P(F, h^F)$ is a closed form that lies in P^V, and its cohomology class $P(F)$ does not depend on h^F.

From now on, we use the assumption and notation of Section 1 and S is a compact Kähler manifold. Then V is Kähler. Recall that we identify S with $\{(x,(0,1)) \in V : x \in S\} \subset V$.

Let $N_{S/V}, N_{W/V}$ be the normal bundles to S, W in V.

Let h^{TV} be a Kähler metric on TV. Let h^{TW}, h^{TS}, h^{TY} be the metrics on TW, TS, TY induced by h^{TV}. Let $h^{N_{S/V}}, h^{N_{W/V}}$ be the metrics on $N_{S/V}, N_{W/V}$, as the orthogonal complements of TS, TW, induced by h^{TV}.

By (1.8) and (2.4), the maps

$$\begin{aligned} N_{S/V} &\to [S]|_S, & N_{W/V} &\to [W]|_W, \\ Y &\to \partial_Y \tau_{[S]}, & Y &\to \partial_Y \tau_{[W]} \end{aligned} \tag{3.3}$$

define the canonical isomorphisms of $N_{S/V} \simeq [S]|_S, N_{W/V} \simeq [W]|_W$. Let $h^{[S]}$ (resp. $h^{[W]}$) be the Hermitian metric on $[S]$ (resp. $[W]$) on V such that the isomorphisms (3.3) are isometries.

Let $h^{[-iS]}$ be the metrics on $[-iS]$ induced by $h^{[S]}$ and let h^L be the metric on L induced by $h^{[S]}$ via (1.10). Let $h^{[-W]}$ be the dual metric on $[-W]$ induced by $h^{[W]}$.

Let h^{ξ} be a metric on ξ. Let $h^{\xi'}$ be the metric on ξ' induced by h^{ξ}. Let $h^{R\pi_{W*}\xi'}$ be the metric on $R^0\pi_{W*}\xi'$ induced by h^L, h^{ξ} under identification (2.11).

Let $\| \quad \|_{\lambda(\xi')}, \| \quad \|_{\lambda(R\pi_{W*}\xi')}$ be the Quillen metric [26], [8] on $\lambda(\xi'), \lambda(R\pi_{W*}\xi')$. Under the identification (2.11), all complex lines considered in Section 2 provide with the Quillen metrics.

Let $\zeta(s)$ be the Riemann zeta function. Let $R(x)$ be the Gillet–Soulé power series [14],

$$R(x) = \sum_{\substack{n\geq 1\\ n \text{ odd}}} \Big(\frac{2\zeta'(-n)}{\zeta(-n)} + \sum_{j=1}^{n} \frac{1}{j}\Big) \zeta(-n) \frac{x^n}{n!}. \tag{3.4}$$

We identify R to the corresponding additive genus.

Let P^V_W be the set of currents on V, which are sums of currents of type (p,p), whose wave front set is included in $N^*_{W/V,\mathbb{R}}$. Let $P^{V,0}_W$ be the set of currents $\alpha \in P^V_W$ such that there exist currents β, γ on V, whose wave front set is included in $N^*_{W/V,\mathbb{R}}$, such that $\alpha = \partial\beta + \bar{\partial}\gamma$.

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Let $(\xi_1, v), (\xi_2, v)$ be the complexes on V

$$(\xi_1, v): \quad 0 \to [-W] \otimes \pi^*\xi \xrightarrow{\tau_{[W]}} \pi^*\xi \to 0,$$
$$(\xi_2, v): \quad 0 \to \bigoplus_{i=0}^{d-1} [-(i+1)S] \otimes \pi^*\xi \xrightarrow{\tau_{[S]}} \bigoplus_{i=0}^{d-1} [-iS] \otimes \pi^*\xi \to 0. \tag{3.5}$$

Let h^{ξ_1} (resp. h^{ξ_2}) be the metrics on ξ_1 (resp. on ξ_2) induced by $h^{[W]}$ (resp. $h^{[S]}$) and h^{ξ}.

Let $T(\xi_1, h^{\xi_1}) \in P_W^V, T(\xi_2, h^{\xi_2}) \in P_S^V$ be the Bott–Chern currents constructed in [9, Theorem 2.5]. The forms $T(\xi_i, h^{\xi_i})$ verify the following equations:

$$\begin{aligned}
\frac{\overline{\partial}\partial}{2i\pi} T(\xi_1, h^{\xi_1}) &= \mathrm{Td}^{-1}(N_{W/V}, h^{N_{W/V}}) \, \mathrm{ch}(\xi', h^{\xi'}) \delta_{\{W\}} - \mathrm{ch}(\xi_1, h^{\xi_1}) \\
&= \mathrm{Td}^{-1}([W], h^{[W]}) \pi^* \mathrm{ch}(\xi, h^{\xi}) \Big(\delta_{\{W\}} - c_1([W], h^{[W]}) \Big), \\
\frac{\overline{\partial}\partial}{2i\pi} T(\xi_2, h^{\xi_2}) &= \mathrm{Td}^{-1}(N_{S/V}, h^{N_{S/V}}) \sum_{j=0}^{d-1} \mathrm{ch}([-jS], h^{[-jS]}) \, \mathrm{ch}(\xi, h^{\xi}) \delta_{\{S\}} \\
&\quad - \mathrm{ch}(\xi_2, h^{\xi_2}) \\
&= \left(\frac{1 - e^{-dx}}{x} \right) ([S], h^{[S]}) \pi^* \mathrm{ch}(\xi, h^{\xi}) \Big(\delta_{\{S\}} - c_1([S], h^{[S]}) \Big).
\end{aligned} \tag{3.6}$$

Over W, we have the exact sequence of holomorphic Hermitian vector bundles

$$0 \to TW \to TV \to N_{W/V} \to 0. \tag{3.7}$$

Let $\widetilde{\mathrm{Td}}(TW, TV|_W, h^{TV}) \in P^W / P^{W,0}$ be the Bott–Chern class constructed in [7, Theorem 1.29], such that

$$\frac{\overline{\partial}\partial}{2i\pi} \widetilde{\mathrm{Td}}(TW, TV|_W, h^{TV}) = \mathrm{Td}(TV, h^{TV}) - \mathrm{Td}(TW, h^{TW}) \, \mathrm{Td}(N_{W/V}, h^{N_{W/V}}). \tag{3.8}$$

Over S, we have the exact sequence of holomorphic Hermitian vector bundles

$$0 \to TS \to TV \to N_{S/V} \to 0. \tag{3.9}$$

Let $\widetilde{\mathrm{Td}}(TS, TV|_S, h^{TV}) \in P^S / P^{S,0}$ be the corresponding Bott–Chern class of [7]. It verifies the following equation:

$$\frac{\overline{\partial}\partial}{2i\pi} \widetilde{\mathrm{Td}}(TS, TV|_S, h^{TV}) = \mathrm{Td}(TV, h^{TV}) - \mathrm{Td}(TS, h^{TS}) \, \mathrm{Td}(N_{S/V}, h^{N_{S/V}}). \tag{3.10}$$

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The following result is a direct consequence of [13, Proposition 1.3.1] and the observation that $\mathrm{Td}^{-1}(F,h^F)\,c_1(F,h^F)=1-\mathrm{ch}(F^{-1},h^{F^{-1}})$ for any Hermitian holomorphic line bundle (F,h^F).

Lemma 3.2. For a holomorphic line bundle F on a compact complex manifold Z, and h^F, h^F_1 two metrics on F with dual metrics $h^{F^{-1}}, h_1^{F^{-1}}$ on F^{-1}, we have in $P^Z/P^{Z,0}$,

$$-\widetilde{\mathrm{ch}}(F^{-1},h^{F^{-1}},h_1^{F^{-1}})=\widetilde{\mathrm{Td}^{-1}}(F,h^F,h_1^F)c_1(F,h^F)+\mathrm{Td}^{-1}(F,h_1^F)\widetilde{c_1}(F,h^F,h_1^F), \qquad (3.11)$$

with $\widetilde{\mathrm{ch}}, \widetilde{\mathrm{Td}^{-1}}, \widetilde{c_1}$ the Bott–Chern classes such that

$$\begin{aligned}
&\frac{\overline{\partial}\partial}{2i\pi}\widetilde{\mathrm{ch}}(F^{-1},h^{F^{-1}},h_1^{F^{-1}})=\mathrm{ch}(F^{-1},h_1^{F^{-1}})-\mathrm{ch}(F^{-1},h^{F^{-1}}),\\
&\frac{\overline{\partial}\partial}{2i\pi}\widetilde{\mathrm{Td}^{-1}}(F,h^F,h_1^F)=\mathrm{Td}^{-1}(F,h_1^F)-\mathrm{Td}^{-1}(F,h^F),\\
&\frac{\overline{\partial}\partial}{2i\pi}\widetilde{c_1}(F,h^F,h_1^F)=c_1(F,h_1^F)-c_1(F,h^F).
\end{aligned} \qquad (3.12)$$

Note that by [7, Remark 1.28], we have

$$\widetilde{c_1}(\eta,h^F,h_1^F)=-\log\frac{h_1^F}{h^F}. \qquad (3.13)$$

We define

$$\begin{aligned}
\mathcal{T}(h^{[S]},h^{[W]})=&\,\mathrm{Td}^{-1}([W],h^{[W]})\log\|\tau_{[W]}\|^2_{h^{[W]}}\\
&-\mathrm{Td}^{-1}([dS],h^{[dS]})\log\|\tau^d_{[S]}\|^2_{h^{[dS]}}-\widetilde{\mathrm{ch}}([-dS],h^{[-dS]},h^{[-W]}).
\end{aligned} \qquad (3.14)$$

Lemma 3.3. In $P^V_{W\cup S}/P^{V,0}_{W\cup S}$, $\mathcal{T}(h^{[S]},h^{[W]})$ does not depend on the choice of $h^{[S]},h^{[W]}$, thus we denote it as $\mathcal{T}_{S,W}$, and we have

$$\frac{\overline{\partial}\partial}{2i\pi}\mathcal{T}_{S,W}=\mathrm{Td}^{-1}(N_{W/V},h^{N_{W/V}})\delta_{\{W\}}-\Big(\frac{1-e^{-dx}}{x}\Big)(N_{S/V},h^{N_{S/V}})\delta_{\{S\}}. \qquad (3.15)$$

Proof. By Poincaré–Lelong formula and (3.3), we get first (3.15).

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Let $h_1^{[W]}$ be another metric on $[W]$ such that (3.3) is an isometry. Then by Lemma 3.2, we have in $P^V_{W\cup S}/P^{V,0}_{W\cup S}$,

$$\begin{aligned}\mathcal{T}(h^{[S]}, h^{[W]}) - \mathcal{T}(h^{[S]}, h_1^{[W]}) &= \Big(\mathrm{Td}^{-1}([W], h^{[W]}) - \mathrm{Td}^{-1}([W], h_1^{[W]})\Big) \log \|\tau_{[W]}\|^2_{h^{[W]}} \\ &- \mathrm{Td}^{-1}([W], h_1^{[W]}) \log \frac{h_1^{[W]}}{h^{[W]}} + \widetilde{\mathrm{ch}}([-dS], h^{[-W]}, h_1^{[-W]}) \\ &= \widetilde{\mathrm{Td}^{-1}}([W], h_1^{[W]}, h^{[W]})\delta_{\{W\}} = 0, \qquad (3.16)\end{aligned}$$

as $h^{[W]} = h_1^{[W]} = h^{N_{W/V}}$ on W.

By the same argument, we know also $\mathcal{T}(h^{[S]}, h^{[W]})$ does not depend on $h^{[S]}$. ■

Theorem 3.4. The following identity holds:

$$\begin{aligned}\log(\|\sigma\|^2_{\lambda(\xi')\otimes\lambda^{-1}(R\pi_{W*}\xi')}) &= \int_V \mathrm{Td}(TV, h^{TV})\mathcal{T}_{S,W}\, \pi^*\, \mathrm{ch}(\xi, h^{\xi}) \\ &- \int_W \mathrm{Td}^{-1}(N_{W/V}, h^{N_{W/V}})\widetilde{\mathrm{Td}}(TW, TV|_W, h^{TV})\, \mathrm{ch}(\xi', h^{\xi'}) \\ &+ \int_S \Big(\frac{1-e^{-dx}}{x}\Big)(L, h^{[S]})\widetilde{\mathrm{Td}}(TS, TV|_S, h^{TV})\, \mathrm{ch}(\xi, h^{\xi}) \\ &+ \int_S \mathrm{Td}(TS)R(TS)\, \mathrm{ch}(R^\bullet\pi_{W*}\mathcal{O}_W)\, \mathrm{ch}(\xi) - \int_W \mathrm{Td}(TW)R(TW)\, \mathrm{ch}(\xi'). \qquad (3.17)\end{aligned}$$

Proof. Let $\|\ \ \|^2_{\lambda'_d(\pi^*\xi)}$ be the Quillen metric on $\lambda'_d(\pi^*\xi)$ (2.28) induced by $h^{[W]}$, h^{ξ}, and h^{TV}. Let $\|\ \ \|^2_{\lambda_d(\pi^*\xi)}$ be the Quillen metric on $\lambda_d(\pi^*\xi) \simeq \lambda'_d(\pi^*\xi)$ (2.41) induced by $h^{[S]}$, h^{ξ}, and h^{TV}. By the anomaly formula [8, Theorem 1.23], we have

$$\log \frac{\|\ \ \|^2_{\lambda'_d(\pi^*\xi)}}{\|\ \ \|^2_{\lambda_d(\pi^*\xi)}} = -\int_V \mathrm{Td}(TV, h^{TV})\widetilde{\mathrm{ch}}([-dS], h^{[-dS]}, h^{[-W]})\, \pi^*\, \mathrm{ch}(\xi, h^{\xi}). \qquad (3.18)$$

By using [11, Theorem 6.1], (2.13), (2.40), and (3.5), we have

$$\begin{aligned}\log(\|\sigma_1\|^2_{\lambda'_d(\pi^*\xi)\otimes\lambda^{-1}(\xi')}) &= -\int_V \mathrm{Td}(TV, h^{TV})T(\xi_1, h^{\xi_1}) \\ &+ \int_W \mathrm{Td}^{-1}(N_{W/V}, h^{N_{W/V}})\, \mathrm{ch}(\xi', h^{\xi'})\widetilde{\mathrm{Td}}(TW, TV|_W, h^{TV}) \\ &- \int_V \mathrm{Td}(TV)R(TV)\, \mathrm{ch}(\xi)(1-\mathrm{ch}([-W])) + \int_W \mathrm{Td}(TW)R(TW)\, \mathrm{ch}(\xi'), \qquad (3.19)\end{aligned}$$

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$$\log(\|\rho_d\|^2_{\lambda_d(\pi^*\xi)\otimes\lambda^{-1}(R^\bullet\pi_{W*}\xi')}) = -\int_V \mathrm{Td}(TV,h^{TV})T(\xi_2,h^{\xi_2})$$
$$+\int_S \mathrm{Td}^{-1}(N_{S/V},h^{N_{S/V}})\widetilde{\mathrm{Td}}(TS,TV|_S,h^{TV})\,\mathrm{ch}(R^\bullet\pi_{W*}\xi',\oplus_i h^{L^{-i}}\otimes h^{\xi})$$
$$-\int_V \mathrm{Td}(TV)R(TV)\,\mathrm{ch}(\xi)(1-\mathrm{ch}([-dS]))+\int_S \mathrm{Td}(TS)R(TS)\,\mathrm{ch}(R^\bullet\pi_{W*}\xi').$$

By [10, Remark 3.5 and Theorem 3.17],

$$\begin{aligned} T(\xi_1,h^{\xi_1}) &= \pi^*(\mathrm{ch}(\xi,h^{\xi}))\,\mathrm{Td}^{-1}([W],h^{[W]})\log\|\tau_{[W]}\|^2_{h^{[W]}} \quad \text{in} \quad P_W^V/P_W^{V,0},\\ T(\xi_2,h^{\xi_2}) &= \pi^*(\mathrm{ch}(\xi,h^{\xi}))\,\mathrm{ch}(\oplus_{i=0}^{d-1}[-iS],\oplus h^{[-iS]})\,\mathrm{Td}^{-1}([S],h^{[S]})\log\|\tau_{[S]}\|^2_{h^{[S]}}\\ &= \pi^*(\mathrm{ch}(\xi,h^{\xi}))\,\mathrm{Td}^{-1}([dS],h^{[dS]})\log\|\tau^d_{[S]}\|^2_{h^{[dS]}} \quad \text{in} \quad P_S^V/P_S^{V,0}. \end{aligned} \tag{3.20}$$

By (1.10), (2.11), and (3.3), we have

$$\mathrm{Td}^{-1}(N_{S/V},h^{N_{S/V}})\,\mathrm{ch}(R^\bullet\pi_{W*}\xi',\oplus_i h^{L^{-i}}\otimes h^{\xi}) = \Big(\frac{1-e^{-dx}}{x}\Big)(L,h^{[S]})\,\mathrm{ch}(\xi,h^{\xi}). \tag{3.21}$$

By Propositions 2.5, 2.6, and our identification of $\lambda_d(\pi^*\xi)$ to $\lambda'_d(\pi^*\xi)$ by (2.35), we have

$$\|\sigma\|^2_{\lambda(\xi')\otimes\lambda^{-1}(R\pi_{W*}\xi')} = (\|\sigma_1\|^2_{\lambda'_d(\pi^*\xi)\otimes\lambda^{-1}(\xi')})^{-1}$$
$$\cdot\|\rho_d\|^2_{\lambda_d(\pi^*\xi)\otimes\lambda^{-1}(R^\bullet\pi_{W*}\xi')}\frac{\|\ \ \|^2_{\lambda'_d(\pi^*\xi)}}{\|\ \ \|^2_{\lambda_d(\pi^*\xi)}}. \tag{3.22}$$

From Lemma 3.3, (3.18)–(3.22), we deduce (3.17). ■

Remark 3.5. From $V=\mathbb{P}(L\oplus 1)$, as holomorphic vector bundles on S, we have

$$TV|_S = TS\oplus L, \quad \text{and} \ \ TY|_S = L\simeq N_{S/V}. \tag{3.23}$$

Starting from a metric on L, by using the first Chern form of $\mathcal{O}_V(1)$ and a Kähler metric on S, we can construct a Kähler metric on V such that (3.23) is an isometry with induced metrics on TS, TY. Under this assumption, (3.9) splits with metrics as in (3.23), thus

$$\widetilde{\mathrm{Td}}(TS,TV|_S,h^{TV})=0. \tag{3.24}$$

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4 Comparison Formula for Equivariant Quillen Metrics

In the sequel, we suppose that for $1 \leqslant i \leqslant d-1$, $\alpha_i = 0$ in (1.1). So

$$W = \Big\{(x,t) \in L : t^d + \alpha_d(x) = 0\Big\}. \tag{4.1}$$

Let $G = \mathbb{Z}/d\mathbb{Z} = \{\overline{0}, \overline{1}, \cdots, \overline{d-1}\}$. In this case, the group G acts naturally on V. The action of G is defined by: for $g = \overline{1}$, $(t,u) \in L \oplus \mathbb{C}$, the homogeneous coordinate of V.

$$g \cdot (t,u) = (e^{i2\pi/d} t, u). \tag{4.2}$$

Then G preserves W, and $S = W/G$. Let G act on $\mathcal{O}_V$ by

$$g \cdot f(\cdot) = f(g^{-1} \cdot), \quad \text{for} \quad g \in G, f \in \mathcal{O}_V.$$

Let G act trivially on ξ. Then G acts also on ξ'. Let G act on L by following: for $g = \overline{1}$, $t \in L$,

$$g \cdot t = e^{i2\pi/d} t. \tag{4.3}$$

Then it induces also an action on $L^{-i}, \pi^* L$.

If given $W \in \widehat{G}$, λ_W, μ_W are complex lines, if $\lambda = \oplus_{W \in \widehat{G}} \lambda_W$, $\mu = \oplus_{W \in \widehat{G}} \mu_W$, set

$$\lambda^{-1} = \bigoplus_{W \in \widehat{G}} \lambda_W^{-1}, \quad \lambda \otimes \mu = \bigoplus_{W \in \widehat{G}} \lambda_W \otimes \mu_W. \tag{4.4}$$

Let $\lambda_G(\xi')$, $\lambda_G(R^\bullet \pi_{W*} \xi')$ be the inverse of the equivariant determinant of the cohomology of ξ' and $R^\bullet \pi_* \xi'$ on W, S [4, §2]. Then $\lambda_G(\xi')$ (resp. $\lambda_G(R^\bullet \pi_{W*} \xi')$) is a direct sum of complex lines. As in [4] and [22], we have a canonical isomorphism of direct sums of complex lines

$$\lambda_G(\xi') \simeq \lambda_G(R^\bullet \pi_{W*} \xi'). \tag{4.5}$$

Let σ_G be the canonical nonzero section of $\lambda_G(\xi') \otimes \lambda_G^{-1}(R^\bullet \pi_{W*} \xi')$.

Let h^{TV} be a G-invariant Kähler metric on V (cf. Remark 3.5 for the existence).

We provide the G-invariant Hermitian metrics $h^{[S]}, h^{[W]}, h^{\xi}$ on $[S], [W], \xi$ such that (3.3) are isometries. Then they determine the G-equivariant Quillen metrics $\|\ \|_{\lambda_G(\xi')}$, $\|\ \|_{\lambda_G(R^\bullet \pi_{W*} \xi')}$ on the equivariant determinants $\lambda_G(\xi')$, $\lambda_G(R^\bullet \pi_{W*} \xi')$ [4, §2a)].

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By our constructions, (2.13), (2.25), and (2.39) are G-equivariant exact sequences of sheaves. And the splits of (2.14) and (2.26) are also G–equivariant. Set

$$\begin{aligned}\lambda'_{d,G}(\pi^*\xi) &= \lambda_G(\pi^*\xi)\otimes\lambda_G^{-1}([-W]\otimes\pi^*\xi),\\ \lambda_{d,G}(\pi^*\xi) &= \lambda_G(\oplus_{i=1}^{d}[-iS]\otimes\pi^*\xi)\otimes\lambda_G^{-1}(\oplus_{i=0}^{d-1}[-iS]\otimes\pi^*\xi).\end{aligned} \tag{4.6}$$

As in [22], [4, §3b)], by (2.13), (2.25), and (2.39), we have the canonical isomorphisms of direct sums of complex lines

$$\lambda_G(\xi')\simeq\lambda'_{d,G}(\pi^*\xi),\quad \lambda_G(R^\bullet\pi_{W*}\xi')=\lambda_G(\oplus_{i=0}^{d-1}L^{-i}\otimes\xi)\simeq\lambda_{d,G}(\pi^*\xi). \tag{4.7}$$

Let $\sigma_{1,G},\rho_{d,G}$ be the canonical sections of $\lambda_G^{-1}(\xi')\otimes\lambda'_{d,G}(\pi^*\xi)$, $\lambda_G^{-1}(\oplus_{i=0}^{d-1}L^{-i}\otimes\xi)\otimes\lambda_{d,G}(\pi^*\xi)$.

We denote by $\Sigma=W\cap S=\{x\in S:\alpha_d(x)=0\}$. As we suppose that W is a manifold, we know that Σ is a submanifold of S and $\partial\alpha_d(x)\neq 0$ for any $x\in\Sigma$. For $g\in G$, set

$$V^g=\{x\in V:gx=x\},\qquad W^g=\{x\in W:gx=x\}. \tag{4.8}$$

If $g\neq\overline{0}$, then $V^g=S\cup\mathbb{P}(L)$, $W^g=\Sigma$.

Let $\mathrm{Td}_g(TV,g^{TV})$ be the Chern–Weil Todd form on V^g associated to the holomorphic Hermitian connection on (TV,h^{TV}) [4, §2a)], which appears in the Lefschetz formulas of Atiyah–Bott [1]. Other Chern–Weil form will be denoted in a similar way. In particular, the forms $\mathrm{ch}_g(\xi_1,h^{\xi_1})$ on V^g is the Chern–Weil representative of the g-Chern character form of (ξ_1,h^{ξ_1}). Also, we denote by $\mathrm{Td}_g(TV)$, $\mathrm{ch}_g(\xi_1)\cdots$ the cohomology classes of $\mathrm{Td}_g(TV,g^{TV})$, $\mathrm{ch}_g(\xi_1,h^{\xi_1})\cdots$ on V^g.

Let $R(\theta,x)$ be the power series in [3, (7.39)], [4, (7.43)], which verifies $R(0,x)=R(x)$. Let $R_g(TV),\cdots$ be the corresponding additive genera [3, §7c)], [4, §7g)].

Let $h^{T\Sigma}$ be the metric on $T\Sigma$ induced by h^{TS}. Let $h^{N_{\Sigma/S}}$ be the metrics on $N_{\Sigma/S}$ induced by h^{TV}. As smooth vector bundles on Σ, we have the following G-equivariant orthogonal splitting:

$$TV|_\Sigma=T\Sigma\oplus N_{\Sigma/S}\oplus N_{S/V}=T\Sigma\oplus N_{\Sigma/W}\oplus N_{W/V}, \tag{4.9}$$

as G acts trivially on $T\Sigma$, $N_{\Sigma/S}$, and nontrivially on $N_{S/V},N_{\Sigma/W}$, we conclude that

$$N_{\Sigma/S}=N_{W/V},\quad h^{N_{\Sigma/S}}=h^{N_{W/V}},\quad N_{\Sigma/W}=N_{S/V},\quad h^{N_{\Sigma/W}}=h^{N_{S/V}}\quad\text{on }\Sigma. \tag{4.10}$$

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Theorem 4.1. For $g=\bar{j}\,(0<j\leqslant d-1)$, the following identity holds:

$$\begin{aligned}
&\log(\|\sigma_G\|^2_{\lambda_G(\xi')\otimes\lambda_G^{-1}(R\pi_{W*}\xi')})(g)\\
&\quad=\int_S \mathrm{Td}(TS,h^{TS})\,\mathrm{Td}_g(N_{S/V},h^{N_{S/V}})\,\mathrm{Td}^{-1}([W],h^{[dS]})\,\mathrm{ch}(\xi,h^{\xi})\log\|\alpha_d\|^2_{h^{[dS]}}\\
&\qquad-\int_\Sigma \mathrm{Td}^{-1}(N_{W/V},h^{N_{W/V}})\,\mathrm{Td}_g(N_{S/V},h^{N_{S/V}})\widetilde{\mathrm{Td}}(T\Sigma,TS|_\Sigma,h^{TS})\,\mathrm{ch}(\xi,h^{\xi})\\
&\qquad+\int_\Sigma \mathrm{Td}(TS,h^{TS})\,\mathrm{Td}_g(N_{S/V},h^{N_{S/V}})\widetilde{\mathrm{Td}^{-1}}(N_{W/V},h^{[dS]},h^{N_{W/V}})\,\mathrm{ch}(\xi,h^{\xi})\\
&\qquad+\int_S \mathrm{Td}(TS)R(TS)\,\mathrm{ch}(\xi)\,\mathrm{ch}_g(R^\bullet\pi_{W*}\mathcal{O}_W)-\int_\Sigma \mathrm{Td}_g(TW)R_g(TW)\,\mathrm{ch}(\xi).
\end{aligned}\tag{4.11}$$

Proof. By the anomaly formula [4, Theorem 2.5], we have

$$\log\left(\frac{\|\ \ \|^2_{\lambda'_d(\pi^*\xi)}}{\|\ \ \|^2_{\lambda_d(\pi^*\xi)}}\right)(g)=-\int_{S\cup\mathbb{P}(L)}\mathrm{Td}_g(TV,h^{TV})\widetilde{\mathrm{ch}}_g([-dS],h^{[-dS]},h^{[-W]})\,\mathrm{ch}(\xi,h^{\xi}).\tag{4.12}$$

By applying [4, Theorem 0.1] to (2.13) and (2.40), we have

$$\begin{aligned}
&\log(\|\sigma_{1,G}\|^2_{\lambda'_{d,G}(\pi^*\xi)\otimes\lambda_G^{-1}(\xi')})(g)=-\int_{S\cup\mathbb{P}(L)}\mathrm{Td}_g(TV,h^{TV})T_g(\xi_1,h^{\xi_1})\\
&\qquad+\int_\Sigma \mathrm{Td}_g^{-1}(N_{W/V},h^{N_{W/V}})\,\mathrm{ch}_g(\xi,h^{\xi})\widetilde{\mathrm{Td}}_g(TW|_\Sigma,TV|_\Sigma,h^{TV})\\
&-\int_{S\cup\mathbb{P}(L)}\mathrm{Td}_g(TV)R_g(TV)\pi^*\,\mathrm{ch}(\xi)(1-\mathrm{ch}_g([-W]))+\int_\Sigma \mathrm{Td}_g(TW)R_g(TW)\,\mathrm{ch}_g(\xi).
\end{aligned}\tag{4.13}$$

$$\begin{aligned}
&\log(\|\rho_{d,G}\|^2_{\lambda_{d,G}(\pi^*\xi)\otimes\lambda_G^{-1}(R^\bullet\pi_{W*}\xi)})(g)=-\int_{S\cup\mathbb{P}(L)}\mathrm{Td}_g(TV,h^{TV})T_g(\xi_2,h^{\xi_2})\\
&\quad+\int_S \mathrm{Td}_g^{-1}(N_{S/V},h^{N_{S/V}})\widetilde{\mathrm{Td}}_g(TS,TV|_S,h^{TV})\,\mathrm{ch}(\xi,h^{\xi})\,\mathrm{ch}_g(R^\bullet\pi_{W*}\mathcal{O}_W,\oplus h^{[-iS]})\\
&\qquad-\int_{S\cup\mathbb{P}(L)}\mathrm{Td}_g(TV)R_g(TV)\pi^*\,\mathrm{ch}(\xi)(1-\mathrm{ch}_g([-dS]))\\
&\qquad\qquad+\int_S \mathrm{Td}_g(TS)R_g(TS)\pi^*\,\mathrm{ch}(\xi)\,\mathrm{ch}_g(R^\bullet\pi_{W*}\mathcal{O}_W).
\end{aligned}$$

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In this case, since the identifications in Section 2 is G-equivariant, as in (3.22), we have

$$\|\sigma_G\|^2_{\lambda_G(\xi')\otimes\lambda_G^{-1}(R\pi_{W*}\xi')}(g) = \Big\{(\|\sigma_{1,G}\|^2_{\lambda'_{d,G}(\pi^*\xi)\otimes\lambda_G^{-1}(\xi')})^{-1} \cdot \|\rho_{d,G}\|^2_{\lambda_{d,G}(\pi^*\xi)\otimes\lambda_G^{-1}(R^\bullet\pi_{W*}\xi')} \frac{\|\ \|^2_{\lambda'_d(\pi^*\xi)}}{\|\ \|^2_{\lambda_d(\pi^*\xi)}}\Big\}(g). \quad (4.14)$$

Note that by (1.11), $g=\bar{j}$ acts on $[S]$ as multiplication by $e^{i2\pi j/d}$ on $S=V^g\cap S$, and g acts on $[W]|_S$ as $g^d=\mathrm{Id}$. By [4, §6b)], [10, Theorem 3.17], (3.5), and (4.8), on $S=V^g\cap S$, we calculate easily

$$\begin{aligned} T_g(\xi_1,h^{\xi_1}) &= \mathrm{ch}(\xi,h^\xi)\,\mathrm{Td}^{-1}([W],h^{[W]})\log\|\alpha_d\|^2_{h^{[W]}} \quad \text{in} \quad P^S_\Sigma/P^{S,0}_\Sigma, \\ T_g(\xi_2,h^{\xi_2}) &= 0 \quad \text{in} \quad P^S/P^{S,0}. \end{aligned} \quad (4.15)$$

In the second equation of (4.15), we use $\tau_{[S]}=0$ on $S^g=V^g\cap S$, thus the form $\mathrm{Tr}_s[gN_H\exp(-C_u^2)]$ in the definition of $T_g(\xi_2,h^{\xi_2})$ does not depend on u, and automatically $T_g(\xi_2,h^{\xi_2})$ vanishes.

As explain above, on $S\cup\mathbb{P}(L)$, g acts as identity on $[dS]=[W]$, and by the argument in (3.16), and (4.10), we know in $P^S_\Sigma/P^{S,0}_\Sigma$,

$$\begin{aligned} \mathrm{Td}^{-1}([W],h^{[W]})\log\|\alpha_d\|^2_{h^{[W]}} &- \widetilde{\mathrm{ch}}_g([-dS],h^{[-dS]},h^{[-W]}) \\ &- \mathrm{Td}^{-1}([W],h^{[dS]})\log\|\alpha_d\|^2_{h^{[dS]}} \\ &= \widetilde{\mathrm{Td}^{-1}}([W],h^{[dS]},h^{[W]})\delta_{\{\Sigma\}} = \widetilde{\mathrm{Td}^{-1}}([W],h^{[dS]},h^{N_{W/V}})\delta_{\{\Sigma\}}. \end{aligned} \quad (4.16)$$

On $\mathbb{P}(L)$, by (1.8), g acts on $[S]$ as identity, and by (1.7), we have

$$[-W]=[-S]=\mathcal{O}_{\mathbb{P}(L)} \quad \text{on } \mathbb{P}(L). \quad (4.17)$$

By using [4, §6b)], we have also

$$\begin{aligned} T_g(\xi_1,h^{\xi_1}) &= \widetilde{\mathrm{ch}}\Big([-W]|_{\mathbb{P}(L)},h^{\mathcal{O}_V},h^{[-W]}\Big)\,\mathrm{ch}(\xi,h^\xi) \quad \text{in } P^{\mathbb{P}(L)}/P^{\mathbb{P}(L),0}, \\ T_g(\xi_2,h^{\xi_2}) &= \sum_{i=1}^d \widetilde{\mathrm{ch}}\Big([-iS]|_{\mathbb{P}(L)},h^{[-(i-1)S]},h^{[-iS]}\Big)\,\mathrm{ch}(\xi,h^\xi) \quad \text{in } P^{\mathbb{P}(L)}/P^{\mathbb{P}(L),0}. \end{aligned} \quad (4.18)$$

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From (4.17), we have

$$\sum_{i=1}^{d} \widetilde{\mathrm{ch}}\Big([-iS]|_{\mathbb{P}(L)}, h^{[-(i-1)S]}, h^{[-iS]}\Big) + \widetilde{\mathrm{ch}}_g([-dS], h^{[-dS]}, h^{[-W]}) = \widetilde{\mathrm{ch}}\Big([-W]|_{\mathbb{P}(L)}, h^{\mathcal{O}_V}, h^{[-W]}\Big) \quad \text{in } P^{\mathbb{P}(L)}/P^{\mathbb{P}(L),0}. \tag{4.19}$$

From (4.17)–(4.19), we see that the contribution from $\mathbb{P}(L)$ in (4.14) via (4.12) and (4.13) is zero.

Since g acts on $N_{W/V} = [W]$ on Σ as Id, we have

$$\mathrm{Td}_g^{-1}(N_{W/V}, h^{N_{W/V}}) = \mathrm{Td}^{-1}(N_{W/V}, h^{N_{W/V}}) \quad \text{on} \quad \Sigma. \tag{4.20}$$

The restriction of the exact sequence (3.7) on Σ is split as in [4, (6.8)] to two following exact sequences:

$$0 \to T\Sigma \to TS \to N_{\Sigma/S} \to 0, \quad 0 \to N_{\Sigma/W} \to N_{S/V} \to 0 \to 0. \tag{4.21}$$

By (3.6), (4.10), and (4.21), we have

$$\begin{aligned} \widetilde{\mathrm{Td}}_g(TW|_\Sigma, TV|_\Sigma, h^{TV|_\Sigma}) &= \mathrm{Td}_g(N_{S/V}, h^{N_{S/V}})\widetilde{\mathrm{Td}}(T\Sigma, TS|_\Sigma, h^{TS}) \quad \text{in } P^\Sigma/P^{\Sigma,0}, \\ \mathrm{Td}_g(TV, h^{TV}) &= \mathrm{Td}_g(N_{S/V}, h^{N_{S/V}})\,\mathrm{Td}(TS, h^{TS}) \quad \text{on} \quad S. \end{aligned} \tag{4.22}$$

As (3.9) splits G-equivariantly and isometrically, as in (3.23), we get

$$\widetilde{\mathrm{Td}}_g(TS, TV|_S, h^{TV}) = 0 \quad \text{in} \quad P^S/P^{S,0}. \tag{4.23}$$

By (4.12)–(4.23), we have (4.11). ■

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