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Orbifold Submersion and Analytic Torsions

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Abstract. In this paper, we establish the curvature theorem of determinant line bundles for an orbifold Kähler fibration as an extension of Bismut–Gillet–Soulé's curvature theorem. Then we introduce Bismut–Köhler analytic torsion form for an orbifold Kähler fibration. Finally we calculate the behaviour of the Quillen metric by orbifold submersions as an extension of Berthomieu–Bismut's result.

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0. Introduction

Let ξ be a Hermitian vector bundle on a compact Hermitian complex manifold X. Let $\lambda(\xi)$ be the inverse of the determinant of the cohomology of ξ. Quillen defined first a metric on $\lambda(\xi)$ in the case that X is a Riemann surface. Quillen metric is the product of the L^2 metric on $\lambda(\xi)$ by the Ray–Singer analytic torsion of the Dolbeault complex. The logarithm of the Ray–Singer analytic torsion [39] is a linear combination of derivatives at zero of the zeta function of the Hodge Laplacians acting on smooth forms of various degrees. In [12], Bismut, Gillet, and Soulé have established a general theory on Quillen metric for any dimensional compact Kähler manifolds, in particular their anomaly formulas for Quillen metrics computes the variation of Quillen metric on the metrics on ξ and TX by using some Bott–Chern classes; for a holomorphic submersion, they proved their determinant line bundle from spectral theory has canonically a holomorphic structure, and is isomorphic canonically to the Knudsen–Mumford line bundle from sheaf theory, as holomorphic line bundles. They have shown that the Quillen metric is a smooth metric on the determinant line bundle $\lambda(\xi)$ of the cohomology groups of the fibers, even both L^2-metric and the analytic torsion could be discontinuous, their curvature formula calculates the curvature of $\lambda(\xi)$ with Quillen metric which refines the degree two part of the Riemann–Roch–Grothendieck theorem at the differential form level.

Later, Bismut and Köhler [13] (refer also [11], [22] in the special case) have extended the analytic torsion of Ray–Singer to the analytic torsion forms T for a holomorphic submersion. In particular, the equation on $\frac{\overline{\partial}\partial}{2i\pi}T$ gives a refinement of the Riemann–Roch–Grothendieck theorem at the level of differential forms. They have also established the corresponding anomaly formulas.

In [22], Gillet and Soulé had conjectured an arithmetic Riemann–Roch theorem in Arakelov geometry. The analytic torsion form is contained in their definition of direct image. In [23], they have established it for the first arithmetic Chern class and Bismut–Lebeau's embedding formula [15] for Quillen metric plays an important role in their proof. In [24], they have established the high degree version by using Bismut's embedding formula [6] for torsion forms. For the various equivariant extensions cf. [29], [5], [16], and the recent works [17, 18].

Note also that for a submersion $\pi : M \to B$ of compact Kähler manifolds and a holomorphic vector bundle ξ on M, by [28], there exists a canonical isomorphism σ from $\lambda_M(\xi)$, the determinant of the cohomology of ξ over M, to $\lambda(R^\bullet\pi_*\xi)$, here $R^\bullet\pi_*\xi$ is the direct image of ξ. In [2], Berthomieu and Bismut have obtained a formula for the Quillen norm of σ in terms of Bott–Chern classes on M and the analytic torsion forms of the fibration π. In our thesis [31, 32], we establish the family version of [2].

In [34], we define the analytic torsion for orbifolds and established the corresponding anomaly formula and embedding formula. This paper is a continuation of [34]. For an orbifold submersion, we will study the curvature formula for the Quillen metric and define the analytic torsion form, then extend Berthomieu–Bismut's result [2] for an orbifold submersion.

An complex orbifold can be always represented locally by $\mathbb{C}^n/G$ where the finite group G acts $\mathbb{C}$-linearly on $\mathbb{C}^n$. The simplest complex orbifold is a global orbifold M/G where G is a finite group acting holomorphically on a complex manifold M.

We will use the heat kernel method to solve our problem. Thanks to finite propagation speed of the solution of the hyperbolic equation [20], [35, Appendix D], we can use the local family index theory of Bismut [3]. Since, locally, we have to meet G-manifold, to generalize the results to the orbifold case, we must understand very well the situation of G-equivariant complex manifolds. After localized, we will apply the results of [5] and [33] to our situation.

Orbifold appears naturally in many important cases, for example: the symplectic reduction, the problem on moduli spaces. In [27], Kawasaki has extended the Riemann–Roch–Hirzebruch theorem to the orbifold case. Bismut and Labourie [14] also proved the Verlinde formula by using Kawasaki's theorem.

For applications of the analytic torsion in Arakelov geometry, cf. the book [42], in particular the recent works [17], [36], [37]. We also hope our results have corresponding versions in Arakelov geometry. For applications of analytic torsion on the moduli space of $K3$ surfaces, cf. Yoshikawa's works [43], [44], in particular, in [45], for general abelian Calabi–Yau orbifolds of dimension three, BCOV invariant was defined and the curvature theorem was proved for global orbifolds there.

Let us explain the contain of this paper in detail now. For a complex vector space F, we denote $\det F = \Lambda^{\max} F$ and denote by $(\det F)^{-1} := \det F^*$ its dual line.

Let ξ be a holomorphic orbifold vector bundle on an n-dimensional complex orbifold X. Let $H^\bullet(X,\xi)$ be the cohomology of sheaf of holomorphic sections of ξ over X.

The determinant of the cohomology of ξ over X is defined as

$$\lambda(\xi) := (\det H^\bullet(X,\xi))^{-1} = \otimes_{j=0}^n (\det H^j(X,\xi))^{(-1)^{j+1}}. \tag{0.1}$$

Let ΣX be the strata of X which has a natural orbifold structure. Let m_i be the multiplicity of the connected component X_i of $X \cup \Sigma X$ (cf. (1.2)). For α a differential form on $X \cup \Sigma X$, we denote simply

$$\int_{X\cup\Sigma X} \alpha = \sum_i \frac{1}{m_i} \int_{X_i} \alpha. \tag{0.2}$$

Let h^{TX}, h^ξ be Hermitian metrics on TX, ξ. Then as in the smooth case, in [34], we defined the analytic torsion and the Quillen metric on the complex line $\lambda(\xi)$ (cf. (3.4)) and established the anomaly formula in [34, Theorem 4.2], and the local term are certain integral of differential forms on $X \cup \Sigma X$, not on X. For example, $\mathrm{Td}^\Sigma(TX, h^{TX})$ is the Todd form on $X \cup \Sigma X$ associated with the holomorphic Hermitian connection on (TX, h^{TX}), which appears in Kawasaki's formulas [27]. Other Chern–Weil forms will be denoted in a similar way. In particular, the form $\mathrm{ch}^\Sigma(\xi, h^\xi)$ (cf. (2.8)) on $X \cup \Sigma X$ is the Chern–Weil representative of the Chern character of $(\xi^{\mathrm{pr}}, h^\xi)$, with ξ^{pr} the maximal proper orbifold subbundle of ξ.

As the space of $\mathscr{C}^\infty$ sections of an orbifold vector bundle is identified as the space of $\mathscr{C}^\infty$ sections of its maximal proper orbifold subbundle. In the whole paper, we can assume that ξ is a proper orbifold vector bundle.

Let $\pi : M \to B$ be a proper orbifold submersion of complex orbifolds. Then by Proposition 1.4, locally π is a quotient of a fibration with fiber of a compact orbifold X, by a finite group.

We assume that π is a Kähler fibration in the sense of Bismut–Gillet–Soulé, i.e., there is a smooth closed real $(1,1)$-form on M such that it induces a Kähler form along the fiber, cf. Definition 1.7. Let ξ be a holomorphic orbifold vector bundle on M. Let h^ξ be a Hermitian metric on ξ.

When the base B is a complex manifold, then the direct image $R^\bullet\pi_*\xi$ is well defined as an element in K-group of B. In this case, we establish in Theorem 2.3 the family local index theorem as an extension of Bismut's family local index theorem.

When B is a complex orbifold, as one of our main results, in Section 3.3, we define the determinant line bundle as a proper orbifold holomorphic line bundle on B by using the spectral analysis, also Knudsen–Mumford orbifold line bundle from sheaf theory, then Theorem 3.5 as an extension of [12, Theorem 3.14], shows the canonical isomorphism of these orbifold line bundles is holomorphic. In Theorem 3.6, we compute the curvature of the associated Chern connection as a consequence of the family local index theorem. Thus we extend Bismut–Gillet–Soulé's classical curvature theorem [12, Theorem 0.3] to the orbifold case.

We assume now that the direct image $R^k\pi_*\xi(0 \le k \le \dim X)$ are orbifold vector bundle on B. Then in Section 4, we introduce the analytic torsion form which is a differential form on $B \cup \Sigma B$, and we establish its anomaly formula.

Now we assume further that M, B are compact Kähler orbifolds. Let σ be the canonical section of $\lambda_M(\xi) \otimes \lambda^{-1}(R^\bullet\pi_*\xi)$.

Let h^{TM}, h^{TB} be Kähler metrics on TM and TB. Let h^{TX} be the metric on TX induced by h^{TM}. Let ω^M be the Kähler form of h^{TM}.

Let $H^\bullet(X, \xi|_X)$ be the cohomology of $\xi|_X$. Let $h^{H(X,\xi|_X)}$ be the L^2-metric on $H^\bullet(X, \xi|_X)$ constructed in Section 4 associated to h^{TX}, h^ξ. Let $T(\omega^M, h^\xi)$ be the analytic torsion forms on $B \cup \Sigma B$ constructed in Section 4, which extend the analytic torsion forms of Bismut–Köhler to the orbifold case. Let $\widetilde{\mathrm{Td}}^\Sigma(TM, TB, h^{TM}, h^{TB})$ be the Bott–Chern class on $M \cup \Sigma M$ constructed as in [10] such that

$$\frac{\overline{\partial}\partial}{2i\pi}\widetilde{\mathrm{Td}}^\Sigma(TM, TB, h^{TM}, h^{TB}) = \mathrm{Td}^\Sigma(TM, h^{TM}) - \pi^*(\mathrm{Td}^\Sigma(TB, h^{TB}))\,\mathrm{Td}^\Sigma(TX, h^{TX}). \tag{0.3}$$

Let $|| \quad ||_{\lambda_M(\xi)\otimes\lambda^{-1}(R^\bullet\pi_*\xi)}$ be the Quillen metric on the complex line $\lambda_M(\xi) \otimes \lambda^{-1}(R^\bullet\pi_*\xi)$ attached to the metrics $h^{TM}, h^\xi, h^{TB}, h^{H(X,\xi|_X)}$ on $TM, \xi, TB, R^\bullet\pi_*\xi$. The last purpose of this paper is to calculate the Quillen metric

$$||\sigma||_{\lambda_M(\xi)\otimes\lambda^{-1}(R^\bullet\pi_*\xi)}$$

as an extension of [2, Theorem 3.1]

Theorem 5.1. *The following identity holds,*

$$\log(||\sigma||^2_{\lambda_M(\xi)\otimes\lambda^{-1}(R^\bullet\pi_*\xi)}) = -\int_{B\cup\Sigma B} \mathrm{Td}^\Sigma(TB, h^{TB})T(\omega^M, h^\xi) + \int_{M\cup\Sigma M} \widetilde{\mathrm{Td}}^\Sigma(TM, TB, h^{TM}, h^{TB})\,\mathrm{ch}^\Sigma(\xi, h^\xi). \tag{0.4}$$

Let $m_{i,B}, m_{i,M}$ be the multiplicities of the connected components B_i, M_i of $B \cup \Sigma B, M \cup \Sigma M$. Then we can reformulate (0.4) as

$$\log(||\sigma||^2_{\lambda_M(\xi)\otimes\lambda^{-1}(R^\bullet\pi_*\xi)}) = -\sum_i \frac{1}{m_{i,B}} \int_{B_i} \mathrm{Td}^\Sigma(TB, h^{TB})T(\omega^M, h^\xi) + \sum_i \frac{1}{m_{i,M}} \int_{M_i} \widetilde{\mathrm{Td}}^\Sigma(TM, TB, h^{TM}, h^{TB})\,\mathrm{ch}^\Sigma(\xi, h^\xi). \tag{0.5}$$

This paper is organized as follows. The first four sections are concerned with some generalities of orbifolds and of analytic torsions. In Section 1, we recall the definition of orbifold, and construct the Bismut superconnection for a submersion of orbifolds. In Section 2, We extend Kawasaki's theorem to a relative situation. In Section 3, we construct the Quillen metrics for an orbifold, and prove their anomaly formulas. In Section 4, we construct the analytic torsion forms for a submersion of orbifolds. In Section 5, we extend the result of [2] to the orbifold case.

The first version of this paper was written in 1998 when I was visiting at ICTP. The first part was published in [34]. For the recent works on the analytic torsion for orbifold flat vector bundles, cf. recent works [21], [41].

In the whole paper, we use the superconnection formalism of Quillen [38]. If $E = E^+ \oplus E^-$ is a $\mathbb{Z}_2$-graded vector space, and $\tau = \pm 1$ defines the $\mathbb{Z}_2$-grading, for $A \in \mathrm{End}(E)$, we denote $\mathrm{Tr}_s[A]$ the supertrace of A, i.e.,

$$\mathrm{Tr}_s[A] = \mathrm{Tr}[\tau A]. \tag{0.6}$$

The reader is referred for more details to [4], [10], [2].

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1. Orbifolds and superconnections

In this section, we extend the Bismut superconnection of [3] to a Kähler fibration of orbifolds.

This section is organized as follows. In Section 1.1, we recall the definition of an orbifold following [34, §1.1]. In Section 1.2, we describe the Kähler fibration. In Section 1.3, we explain the construction of the Bismut superconnection $B_u (u > 0)$ [3] for a submersion of orbifolds.

1.1. Definition of an orbifold

We define at first a category $\mathcal{M}_s$ as follows: The objects of $\mathcal{M}_s$ are the class of pairs (G, M) where M is a connected smooth manifold and G is a finite group acting effectively on M. Let (G, M) and (G', M') be two objects, then a morphism $\Phi : (G, M) \to (G', M')$ is a family of open embedding $\varphi : M \to M'$ satisfying:

i) For each $\varphi \in \Phi$, there is an injective group homomorphism $\lambda_\varphi : G \to G'$ that makes φ be λ_φ-equivariant.
ii) For $g \in G', \varphi \in \Phi$, we define $g\varphi : M \to M'$ by $(g\varphi)(x) = g\varphi(x)$ for $x \in M$. If $(g\varphi)(M) \cap \varphi(M) \neq \phi$, then $g \in \lambda_\varphi(G)$.
iii) For $\varphi \in \Phi$, we have $\Phi = \{g\varphi : g \in G'\}$.

Definition 1.1. Let X be a paracompact Hausdorff space and let $\mathcal{U}$ be a cover of X consisting of connected open subsets. We assume $\mathcal{U}$ satisfies the condition:

$$\begin{aligned}&\text{For any } x \in U \cap U',\ U, U' \in \mathcal{U}, \text{ there}\\ &\text{is } U'' \in \mathcal{U} \text{ such that } x \in U'' \subset U \cap U'.\end{aligned} \tag{1.1}$$

Then an orbifold structure $\mathcal{V}$ on X is the following:

i) For $U \in \mathcal{U}$, $\mathcal{V}(U) = ((G_U, \widetilde{U}) \xrightarrow{\tau} U)$ is a ramified covering $\widetilde{U} \to U$ giving an identification $U \simeq \widetilde{U}/G_U$.
ii) For $U, V \in \mathcal{U}, U \subset V$, there is a morphism $\varphi_{VU} : (G_U, \widetilde{U}) \to (G_V, \widetilde{V})$ that covers the inclusion $U \subset V$.
iii) For $U, V, W \in \mathcal{U}, U \subset V \subset W$, we have $\varphi_{WU} = \varphi_{WV} \circ \varphi_{VU}$.

If $\mathcal{U}'$ is a refinement of $\mathcal{U}$ satisfying (1.1), then there is an orbifold structure $\mathcal{V}'$ such that $\mathcal{V}\cup\mathcal{V}'$ is an orbifold structure. We consider $\mathcal{V}$ and $\mathcal{V}'$ to be equivalent. Such an equivalence class is called an orbifold structure over X. So we may choose $\mathcal{U}$ arbitrarily fine.

In the above definition, we can replace $\mathcal{M}_s$ by a category of manifolds with an additional structure such as orientation, Riemannian metric or complex structure. We understand that the morphisms (and the groups) preserve the specified structure. So we can define oriented, Riemannian or complex orbifolds.

Let $(X,\mathcal{V})$ be an orbifold. For each $x \in X$, we can choose a small neighbourhood $(G_x, \widetilde{U}_x) \to U_x$ such that $\widetilde{x} \in \widetilde{U}_x$, the unique inverse image of x, is a fixed point of G_x. (Such G_x is unique up to isomorphisms for each $x \in X$, [40, p. 468].) Let (1), $(h_x^1), \dots, (h_x^{\rho_x})$ be the conjugacy classes in G_x. Let $Z_{G_x}(h_x^j)$ be the centralizer of h_x^j in G_x. One also notes $\widetilde{U}_x^{h_x^j}$ the fixed points of h_x^j over $\widetilde{U}_x$. Then we have a natural bijection

$$\Big\{(y,(h_y^j)) : y \in U_x, j = 1,\dots,\rho_y\Big\} \simeq \coprod_{j=1}^{\rho_x} \widetilde{U}_x^{h_x^j}/Z_{G_x}(h_x^j). \tag{1.2}$$

So we can define globally

$$\Sigma X = \{(x,(h_x^j)) : x \in X, G_x \neq \{1\}, j = 1,\dots,\rho_x\}. \tag{1.3}$$

Then ΣX has a natural orbifold structure defined by

$$\Big\{(Z_{G_x}(h_x^j)/K_x^j, \widetilde{U}_x^{h_x^j}) \to \widetilde{U}_x^{h_x^j}/Z_{G_x}(h_x^j)\Big\}_{(x,U_x,j)}. \tag{1.4}$$

Here K_x^j is the kernel of the representation $Z_{G_x}(h_x^j) \to \mathrm{Diffeo}\,(\widetilde{U}_x^{h_x^j})$, the diffeomorphism group of $\widetilde{U}_x^{h_x^j}$. The number $m = |K_x^j|$ is called the multiplicity of ΣX in X at $(x,(h_x^j))$. Since the multiplicity is locally constant on ΣX, we may assign the multiplicity m_i to each connected component ΣX_i of ΣX.

Definition 1.2. An orbifold vector bundle ξ over an orbifold $(X,\mathcal{V})$ is defined as follows: ξ is an orbifold and for $U \in \mathcal{U}$, $(G_U^\xi, \widetilde{p}_U : \widetilde{\xi}_U \to \widetilde{U})$ is a G_U^ξ-equivariant vector bundle such that the morphism $\varphi_{\xi_U \xi_V}$ is a morphism of equivariant vector bundles, and $(G_U^\xi, \widetilde{\xi}_U)$ (resp. $(G_U^\xi/K_U, \widetilde{U}), K_U = \mathrm{Ker}(G_U^\xi \to \mathrm{Diffeo}(\widetilde{U}))$ (In general, G_U^ξ does not act effectively on $\widetilde{U}$, i.e., $K_U \neq \{1\}$) is the orbifold structure of ξ (resp. X). For $x \in X$, we denote the fiber of the vector bundle $\widetilde{\xi}_U$ at an inverse image of x in $\widetilde{U}$, as the vector space $\widetilde{\xi}_x$.

If G_U^ξ acts effectively on $\widetilde{U}$ for $U \in \mathcal{U}$, we call that ξ is a *proper* orbifold vector bundle.

For an orbifold vector bundle ξ, let $\widetilde{\xi_U^{\mathrm{pr}}}$ be the maximal K_U-invariant subbundle of $\widetilde{\xi}_U \to \widetilde{U}$, then $(G_U, \widetilde{\xi_U^{\mathrm{pr}}})$ defines a proper orbifold vector bundle ξ^{pr}.

A natural example is the (proper) orbifold tangent bundle TX which is defined by:

$$(G_U, T\widetilde{U} \to \widetilde{U}), \quad \text{for} \quad U \in \mathcal{U}$$

Let $\xi \to X$ be an orbifold vector bundle. A section $s : X \to \xi$ is called $\mathscr{C}^\infty$ (or $\mathscr{C}^k$) if for each $U \in \mathcal{U}$, $s_{|U}$ is covered by a G_U^ξ-invariant smooth (or $\mathscr{C}^k$) section $\widetilde{s}_U : \widetilde{U} \to \widetilde{\xi}_U$.

If X is oriented, we define the integral $\int_X \omega$ for a form over X (i.e., a section of $\Lambda(T^*X)$ over X): if $\mathrm{supp}(\omega) \subset U \in \mathcal{U}$, then

$$\int_X \omega = \frac{1}{|G_U|} \int_{\widetilde{U}} \widetilde{\omega}_U. \tag{1.5}$$

In the sequel, if G does not act effectively on the connected manifold M, we will identify the couple (G, M) as an element $(G/K, M)$ in $\mathcal{M}_s$, with $K = \mathrm{Ker}(G \to \mathrm{Diffeo}(M))$.

Definition 1.3. Let M, B be two orbifolds, a map $\pi : M \to B$ is said to define an orbifold submersion if there exist $\mathcal{U}, \mathcal{U}'$ open covers of M, B, such that $\pi(\mathcal{U}) \subset \mathcal{U}'$, and $(G_U, \widetilde{U})_{U\in\mathcal{U}}$, $(G_V, \widetilde{V})_{V\in\mathcal{V}}$ are the orbifold structures of M, B; for $U \in \mathcal{U}$, there is $\widetilde{\pi} : \widetilde{U} \to \widetilde{V}$ a G_U-equivariant submersion of $\widetilde{U}$ onto $\widetilde{V}$ that covers $\pi : U \to V = \pi(U)$, and $(G_U, \widetilde{V}) = (G_V, \widetilde{V})$ in $\mathcal{M}_s$; if $U_1 \subset U_2, U_1, U_2 \in \mathcal{U}$, then $\Phi_{\pi(U_2)\pi(U_1)}$ is induced by $\Phi_{U_2U_1}$.

Let $\pi : M \to B$ be an orbifold submersion of M onto B, then the related tangent bundle TM/B is defined by: over $\widetilde{U}$, $((G_U, T\widetilde{U}/\widetilde{V}) \to \widetilde{U})$.

Proposition 1.4. *If $\pi : M \to B$ is a proper orbifold submersion of M onto B, then for each $b \in B$, there exists a small neighborhood $(G_b, \widetilde{V}_b) \to V_b$, $\widetilde{M}_b$ an orbifold, such that π is induced by a G_b-equivariant orbifold submersion $\widetilde{\pi}_b : \widetilde{M}_b \to \widetilde{V}_b$ with compact fiber $\overline{X}$.*

Proof. Let $\mathcal{U}$ be a cover of M in Definition 1.3. For $U \in \mathcal{U}$, set

$$K_U = \mathrm{Ker}\{G_U \to \mathrm{Diffeo}(\widetilde{\pi}(\widetilde{U}))\}. \tag{1.6}$$

As π is proper, for $b \in B$, we can find $V \subset B$ open, $b \in V$, $(G_b, \widetilde{V}) \xrightarrow{\gamma} V$ be a ramified covering of V, and $\gamma^{-1}(b) = \{b_0\}$, such that there is $((G_{U_i}, \widetilde{U}_i) \to U_i)_{i\in I}$ ($I = \{1, \cdots, q\}$) induced by the orbifold structure of M, the map $\widetilde{\pi} : (G_{U_i}, \widetilde{U}_i) \to (G_{U_i}, \widetilde{V}) = (G_b, \widetilde{V})$ is a G_{U_i}-equivariant submersion of $\widetilde{U}_i$ onto $\widetilde{V}$, and $\pi^{-1}(V) = \cup_{i\in I} U_i$.

For $W_1 \subset W_2, W_1, W_2 \in \mathcal{U}$, by definition, there exist morphisms

$$\begin{aligned} \Phi_{W_2W_1} &: (G_{W_1}, \widetilde{W_1}) \to (G_{W_2}, \widetilde{W_2}), \\ \Phi_{\pi(W_2)\pi(W_1)} &: (G_{\pi(W_1)}, \widetilde{\pi}(\widetilde{W_1})) \to (G_{\pi(W_2)}, \widetilde{\pi}(\widetilde{W_2})) \text{ in } \mathcal{M}_s, \end{aligned} \tag{1.7}$$

such that $\Phi_{\pi(W_2)\pi(W_1)}$ is induced by $\Phi_{W_2W_1}$. We note that $\widetilde{\pi}(\widetilde{W_j})$ is a ramified covering of $\pi(W_j)$ for $j = 1, 2$.

Let $\widetilde{\mathcal{U}} = \{(\widetilde{W}, \varphi)$: there exist $i \in I$, such that $(G_W, \widetilde{W}) \to W \subset U_i$, $W \in \mathcal{U}$, and $\varphi \in \Phi_{V\pi(W)}\}$. Let $a_1 = (\widetilde{W}_1, \varphi_1)$, $a_2 = (\widetilde{W}_2, \varphi_2) \in \widetilde{\mathcal{U}}$, $W_1 \subset W_2$, for each

$\psi \in \Phi_{W_2W_1}$, we also denote $\psi \in \Phi_{\pi(W_2)\pi(W_1)}$ the associated open embedding. Thus for $\psi \in \Phi_{W_2W_1}$, we have the commutative diagram

$$\begin{array}{ccccccccc} \widetilde{W}_1 & \xrightarrow{g} & \widetilde{W}_1 & \xrightarrow{\widetilde{\pi}} & \widetilde{\pi}(\widetilde{W}_1) & \xrightarrow{g} & \widetilde{\pi}(\widetilde{W}_1) & \xrightarrow{\varphi_1} & \widetilde{V} \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\psi} & & \\ \widetilde{W}_2 & \xrightarrow{\lambda_\psi(g)} & \widetilde{W}_2 & \xrightarrow{\widetilde{\pi}} & \widetilde{\pi}(\widetilde{W}_2) & \xrightarrow{\lambda_\psi(g)} & \widetilde{\pi}(\widetilde{W}_2) & \xrightarrow{\varphi_2} & \widetilde{V} \end{array} \tag{1.8}$$

Put

$$\Phi_{a_2a_1} = \{\psi \in \Phi_{W_2W_1} : \varphi_2\psi = \varphi_1 \text{ as a map from } \widetilde{\pi}(\widetilde{W}_1) \text{ to } \widetilde{V}\}. \tag{1.9}$$

I.e., for $\psi \in \Phi_{a_2a_1}$, the commutative diagram (1.8) is completed by the identity map $\mathrm{Id} : \widetilde{V} \to \widetilde{V}$.

Claim: *$\Phi_{a_2a_1} : (K_{W_1}, (\widetilde{W}_1, \varphi_1)) \to (K_{W_2}, (\widetilde{W}_2, \varphi_2))$ is a morphism in $\mathcal{M}_s$.*

Proof of the claim. The K_{W_1}-action on $(\widetilde{W}_1, \varphi_1)$ is defines by its action on $\widetilde{W}_1$. i) For $\psi \in \Phi_{a_2a_1} \subset \Phi_{W_2W_1}$, the injective group homomorphism $\lambda_\psi : G_{W_1} \to G_{W_2}$ makes that ψ is λ_ψ-equivariant. Note that for $g \in G_{W_1}$, $\widetilde{x} \in \widetilde{W}_1$, by (1.8), we have

$$\psi(g\widetilde{x}) = \lambda_\psi(g)\psi(\widetilde{x}), \quad \psi\widetilde{\pi}(g\widetilde{x}) = \lambda_\psi(g)\widetilde{\pi}\psi(\widetilde{x}) = \lambda_\psi(g)\psi\widetilde{\pi}(\widetilde{x}). \tag{1.10}$$

Thus if $g \in K_{W_1}$, $\lambda_\psi(g) \in G_{W_2}$ fixes $\widetilde{\pi}(\psi(\widetilde{W}_1)) = \psi(\widetilde{\pi}(\widetilde{W}_1))$, an open set of $\widetilde{\pi}(\widetilde{W}_2)$. But G_{W_2} is compact and acts on $\widetilde{W}_2$ which is connected, thus we conclude that $\lambda_\psi(g)$ acts as identity on $\widetilde{\pi}(\widetilde{W}_2)$, i.e., $\lambda_\psi(g) \in K_{W_2}$. Thus λ_ψ induces an injective group homomorphism $\lambda_\psi : K_{W_1} \to K_{W_2}$.

ii) Assume now $(h\psi)(\widetilde{W}_1) \cap \psi(\widetilde{W}_1) \neq \phi$, and $h \in K_{W_2}$. The first condition implies $h \in \lambda_\psi(G_{W_1})$, i.e., there exists $g \in G_{W_1}$ such that $h = \lambda_\psi(g)$. But $h \in K_{W_2}$ means that $\lambda_\psi(g)$ acts as identity on $\widetilde{\pi}(\widetilde{W}_2)$, this implies that g acts as identity $\widetilde{\pi}(\widetilde{W}_1)$ by (1.8), i.e., $g \in K_{W_1}$. We conclude that $h \in \lambda_\psi(K_{W_1})$.

iii) For any $\psi', \psi \in \Phi_{a_2a_1}$, there exists $g \in G_{W_2}$ such that $g\psi = \psi'$. By (1.9),

$$\lambda_{\varphi_2}(g)\varphi_2\psi = \varphi_2 g\psi = \varphi_2\psi' = \varphi_1 = \varphi_2\psi. \tag{1.11}$$

Thus $\lambda_{\varphi_2}(g)$ acts as identity on an open set $\varphi_2\psi(\widetilde{\pi}(\widetilde{W}_1))$ of $\widetilde{V}$, thus as identity on $\widetilde{V}$, this implies that g acts as identity on $\widetilde{\pi}(\widetilde{W}_2)$, i.e., $g \in K_{W_2}$, thus $\Phi_{a_2a_1} = \{g\psi : g \in K_{W_2}\}$.

The proof of the claim is completed. □

For $i \in I$, we denote $\widetilde{U}_i = (\widetilde{U}_i, 1) \in \widetilde{\mathcal{U}}$. We define an equivalence relation $\sim$ on $\overline{M} = \cup_{i\in I}\widetilde{U}_i/K_{U_i}$: For $\widetilde{x} \in \widetilde{U}_i, \widetilde{y} \in \widetilde{U}_j$, $\widetilde{x} \sim \widetilde{y}$ if and only if there exist $(G_W, \widetilde{W}) \to W \subset U_i \cap U_j$, $\varphi_1, \varphi_2 \in \Phi_{V\pi(W)}$, $\widetilde{z} \in \widetilde{W}$ such that

$$\widetilde{x} \in \Phi_{\widetilde{U}_ia_1}(\{\widetilde{z}\}), \widetilde{y} \in \Phi_{\widetilde{U}_ja_2}(\{\widetilde{z}\}), \text{ for } a_1 = (\widetilde{W}, \varphi_1), a_2 = (\widetilde{W}, \varphi_2) \in \widetilde{\mathcal{U}}. \tag{1.12}$$

We can interpret (1.12) for $\widetilde{x}, \widetilde{z}$ by the following commutative diagram:

$$\begin{tikzcd}
\widetilde{z} \in & \widetilde{W} \arrow[r, "\widetilde{\pi}"] \arrow[d, "\Phi_{\widetilde{U}_i a_1}"] & \widetilde{\pi}(\widetilde{W}) \arrow[d] \arrow[dr, "\varphi_1"] & \\
\widetilde{x} \in & \widetilde{U}_i \arrow[r, "\widetilde{\pi}"] & \widetilde{\pi}(\widetilde{U}_i) = \widetilde{V} \arrow[r, "\mathrm{Id}"] & \widetilde{V}
\end{tikzcd} \tag{1.13}$$

Let $\widetilde{M}_b = \overline{M}/\sim$. Let $\mathcal{U}' = \{(\widetilde{W}, \varphi)/K_W : (\widetilde{W}, \varphi) \in \widetilde{\mathcal{U}}\}$, then $\mathcal{U}'$ is a covering of $\widetilde{M}_b$ which satisfies the condition (1.1). We get the conditions i), ii), iii) of Definition 1.1 from the claim. So $\mathcal{U}'$ defines an orbifold structure on $\widetilde{M}_b$.

Note that K_{U_i} is a normal subgroup of G_{U_i} and $G_b = G_{U_i}/K_{U_i}$, thus G_b acts naturally on $\widetilde{U}_i/K_{U_i}$, so G_b acts on $\widetilde{M}_b$. Let $\widetilde{\pi} : \widetilde{M}_b \to \widetilde{V}$ be induced by $\widetilde{\pi} : \widetilde{U}_i \to \widetilde{V}$, then $\widetilde{\pi}$ is an orbifold submersion, and $\widetilde{\pi}$ is G_b -equivariant.

Now the procedure is standard. Note that the kernel of $d\widetilde{\pi} : T\widetilde{M}_b \to T\widetilde{V}$ is an orbifold vector bundle. By choosing a horizontal subbundle $T^H\widetilde{M}_b$ of $T\widetilde{M}_b$ (for example, by taking the orthogonal complement of $\mathrm{Ker}(d\widetilde{\pi})$ with respect to a metric on $T\widetilde{M}_b$), such that

$$T\widetilde{M}_b = \mathrm{Ker}(d\widetilde{\pi}) \oplus T^H\widetilde{M}_b. \tag{1.14}$$

As $\widetilde{V}$ is a manifold, we know that $T^H\widetilde{M}_b$ is a usual vector bundle. Now the horizontal lift of any ball $B(p,r)$, with the center p and radius r, in $\widetilde{V}$ along the radius direction gives a trivialization

$$\widetilde{\pi}^{-1}(B(p,r)) = B(p,r) \times \overline{X}_p. \tag{1.15}$$

Note that for any point in V such that $G_p = \{1\}$, $\overline{X}_p = \pi^{-1}(\{p\})$, thus as a real orbifold, the fiber $\overline{X}$ has a canonical model.

The proof of Proposition 1.4 is completed. □

Let $(X, \mathcal{V})$ be a compact connected Riemannian orbifold. For $x, y \in X$, put

$$\begin{aligned} d(x,y) = \mathrm{Inf}\Big\{ \textstyle\sum_i \int_{t_{i-1}}^{t_i} |\frac{\partial}{\partial t}\widetilde{\gamma}_i(t)|dt \Big| & \gamma : [0,1] \to X, \gamma(0) = x, \gamma(1) = y, \text{such that} \\ & \text{there exist } t_0 = 0 < t_1 < \cdots < t_k = 1, U_i \in \mathcal{U}, \gamma([t_{i-1}, t_i]) \subset U_i, \\ & \widetilde{\gamma}_i : [t_{i-1}, t_i] \to \widetilde{U}_i \quad \mathscr{C}^\infty, \text{ that covers } \gamma_{|[t_{i-1},t_i]}. \Big\} \end{aligned}$$

Then (X, d) is a metric space.

1.2. Kähler fibrations

In the rest of this paper, we always work on complex orbifolds, especially, all morphisms considered in Section 1.1 are holomorphic. For an orbifold complex vector bundle, we denote the underlying real orbifold vector bundle by adding a subscript $\mathbb{R}$.

Definition 1.5. A Kähler form on a complex orbifold X is a real closed $(1,1)$-form ω on X such that ω induces a (orbifold) metric on TX.

Let $\pi : M \to B$ be a proper holomorphic orbifold submersion of M onto B. Let TM, TB be the holomorphic tangent bundles to M, B. From Proposition 1.4, the holomorphic relative tangent bundle TX of the fibration π is well defined as an orbifold vector bundle over M. Let J^{TX} be the complex structure on the real relative tangent bundle $T_{\mathbb{R}}X$.

Lemma 1.6. *For $\pi : M \to B$ a proper holomorphic orbifold submersion, for any $b \in B$, we can choose $\widetilde{M}_b$ in Proposition 1.4 such that π is induced by a G_b-equivariant holomorphic orbifold submersion $\widetilde{\pi}_b : \widetilde{M}_b \to \widetilde{V}_b$.*

Proof. As all morphisms in the proof of Proposition 1.4 are holomorphic, we get Lemma 1.6 from the proof of Proposition 1.4. □

Let h^{TX} be a Hermitian metric on TX. Let T^HM be an orbifold vector subbundle of TM, such that

$$TM = T^HM \oplus TX. \tag{1.16}$$

We now define the Kähler fibration as in [11, Definition 1.4].

Definition 1.7. The triple (π, h^{TX}, T^HM) is said to define a Kähler fibration if there exists a smooth real 2-form ω of complex type (1,1), which has the following properties:

a) ω is closed.
b) $T^H_{\mathbb{R}}M$ and $T_{\mathbb{R}}X$ are orthogonal with respect to ω,
c) If $X, Y \in T_{\mathbb{R}}X$, then $\omega(X,Y) = \left\langle X, J^{TX}Y \right\rangle_{g^{T_{\mathbb{R}}X}}$ with $g^{T_{\mathbb{R}}X}$ the metric on $T_{\mathbb{R}}X$ induced by h^{TX}.

Now we have an analogue of [11, Theorems 1.5 and 1.7].

Theorem 1.8. *Let ω be a real smooth 2-form on M of complex type $(1,1)$, which has the following two properties:*

a) *ω is closed.*
b) *The bilinear map $X, Y \in T_{\mathbb{R}}X \to \omega(J^{TX}X, Y)$ defines a Hermitian product h^{TX} on TX.*

For $x \in M$, set

$$T^H_xM = \{Y \in T_xM : \text{ for any } X \in T_xX, \omega(X, \overline{Y}) = 0\}.$$

Then T^HM is an orbifold subbundle of TM such that $TM = T^HM \oplus TX$. Also (π, h^{TX}, T^HM) is a Kähler fibration, and ω is an associated $(1,1)$-form.

A smooth real $(1,1)$-form ω' on M is associated with the Kähler fibration (π, h^{TX}, T^HM) if and only if there is a real smooth closed $(1,1)$-form η on B such that

$$\omega' - \omega = \pi^*\eta.$$

Proof. The proof is as same as in [11, Theorems 1.5 and 1.7]. □

1.3. The Bismut superconnection of a Kähler fibration

In this part, we will define the Bismut superconnection by proceeding as in [13, §1], [2, §2].

Let $\pi : M \to B$ be a proper holomorphic orbifold submersion of M onto B with fibre X. Let ω^M be a real closed (1,1) form on M taken as in Theorem 1.8. Let ξ be a complex orbifold vector bundle on M. Let h^ξ be a Hermitian metric on ξ. Let ∇^{TX}, ∇^ξ be the holomorphic Hermitian connections on $(TX, h^{TX}), (\xi, h^\xi)$.

We will temporarily assume that B is a complex manifold. Then π is a fibration of M on B which is modelled on orbifold X: There is an open covering $\mathcal{U}$ of B such that if $U \in \mathcal{U}$, $\pi^{-1}(U)$ is diffeomorphic to $U \times X$.

Definition 1.9. For $0 \leq k \leq \dim X$, $b \in B$, let E_b^k be the vector space of $\mathscr{C}^\infty$ sections of $(\Lambda^k(T^{*(0,1)}X) \otimes \xi)|_{X_b}$ over X_b. Set

$$E_b = \oplus_{k=0}^{\dim X} E_b^k, \quad E_b^+ = \oplus_{k \,\mathrm{even}} E_b^k, \quad E_b^- = \oplus_{k \,\mathrm{odd}} E_b^k. \tag{1.17}$$

As in [3, §1f)], [11, §1d)], we can regard the E_b's as the fibers of a smooth $\mathbb{Z}$-graded infinite-dimensional vector bundle over the base B. Smooth sections of E over B will be identified with smooth sections of $\Lambda(T^{*(0,1)}X) \otimes \xi$ over M.

Let dv_X be the Riemannian volume form on X associated with h^{TX}. Let $\langle\ \ \rangle_{\Lambda(T^{*(0,1)}X)\otimes\xi}$ be the Hermitian product induced by h^{TX}, h^ξ on $\Lambda(T^{*(0,1)}X)\otimes\xi$. The Hermitian product $\langle\ \ \rangle$ on E is defined by: If $s, s' \in E$, set

$$\langle s, s'\rangle = \left(\frac{1}{2\pi}\right)^{\dim X} \int_X \langle s, s'\rangle_{\Lambda(T^{*(0,1)}X)\otimes\xi}\, dv_X. \tag{1.18}$$

For $b \in B$, let $\overline{\partial}^{X_b}$ be the Dolbeault operator acting on E_b, and let $\overline{\partial}^{X_b *}$ be its formal adjoint with respect to the Hermitian product (1.18). Set

$$D^X = \overline{\partial}^{X_b} + \overline{\partial}^{X_b *}. \tag{1.19}$$

If $U \in T_\mathbb{R}B$, let U^H be the lift of U in $T_\mathbb{R}^H M$, so that $\pi_* U^H = U$.

Definition 1.10. If $U \in T_\mathbb{R}B$, if s is a smooth section of E over B, set

$$\nabla_U^E s = \nabla_{U^H}^{\Lambda(T^{*(0,1)}X)\otimes\xi} s. \tag{1.20}$$

Let $c(T_\mathbb{R}X)$ be the Clifford algebra of $(T_\mathbb{R}X, h^{TX})$. The bundle $\Lambda(T^{*(0,1)}X)\otimes\xi$ is a $c(T_\mathbb{R}X)$-Clifford module. In fact, if $U \in TX$, let $U' \in T^{*(0,1)}X$ correspond to U by the metric h^{TX}. If $U, V \in TX$, set

$$c(U) = \sqrt{2}U'\wedge, \quad c(\overline{V}) = -\sqrt{2}i_{\overline{V}}. \tag{1.21}$$

Let P^{TX} be the projection $TM \simeq T^H M \oplus TX \to TX$.

If U, V are smooth vector fields on B, set

$$T(U^H, V^H) = -P^{TX}[U^H, V^H]. \tag{1.22}$$

Then T is a tensor. By [11, Theorem 1.7], we know that as a 2-form, T is of complex type (1,1).

Let $f_1, \dots, f_{2m}$ be a base of $T_{\mathbb{R}}B$, and let $f^1, \dots, f^{2m}$ be the dual base of $T^*_{\mathbb{R}}B$.

Definition 1.11. Set

$$c(T) = \frac{1}{2} \sum_{1 \le \alpha, \beta \le 2m} f^\alpha f^\beta c\Big(T(f^H_\alpha, f^H_\beta)\Big). \tag{1.23}$$

Then $c(T)$ is a section of $(\Lambda^2(T^*_{\mathbb{R}}B) \widehat{\otimes} \operatorname{End}(\Lambda(T^{*(0,1)}X) \otimes \xi))^{\text{odd}}$.

Definition 1.12. For $u > 0$, let B_u be the Bismut superconnection constructed in [3, §3], [11, §2a)],

$$B_u = \sqrt{u} D^X + \nabla^E - \frac{c(T)}{2\sqrt{2u}}. \tag{1.24}$$

Let N_V be the number operator defining the $\mathbb{Z}$-grading on $\Lambda(T^{*(0,1)}X) \otimes \xi$ and on E. N_V acts by multiplication by k on $\Lambda^k(T^{*(0,1)}X) \otimes \xi$. If $U, V \in T_{\mathbb{R}}B$, set

$$\omega^{H\overline{H}}(U, V) = \omega^M(U^H, V^H). \tag{1.25}$$

Definition 1.13. For $u > 0$, set

$$N_u = N_V + \frac{i\omega^{H\overline{H}}}{u}. \tag{1.26}$$

In general, B is not a complex manifold. By Proposition 1.4, we verify easily that the above objects go down to B (Ex, E is an orbifold bundle over B), so we can define the Bismut superconnection B_u $(u > 0)$ over B as locally over $\widetilde{V}_b$.

2. Family index theorem

In this section, we describe basic properties of the operator $\overline{\partial}^X$ on a complex orbifold, and we extend Kawasaki's theorem to a relative situation.

This section is organized as follows. In Section 2.1, we give the Hodge decomposition for $\overline{\partial}^X$ operator over a complex orbifold. In Section 2.2, we state the family version of Kawasaki's theorem.

We use the notation of Section 1.

2.1. $\overline{\partial}$-operator on a complex orbifold

Let X be a compact complex orbifold of complex dimension l. Let ξ be a holomorphic orbifold vector bundle on X.

Let $\mathcal{O}_X$ be the sheaf over X of local G_U-invariant holomorphic functions over $\widetilde{U}$, for $U \in \mathcal{U}$. Then by [19], $(X, \mathcal{O}_X)$ is an analytic space. The local G^ξ_U -invariant holomorphic sections of $\widetilde{\xi} \to \widetilde{U}$ define also a coherent analytic sheaf $\mathcal{O}_X(\xi)$ over X.

Let $\mathcal{D}^k(\xi)$ be the sheaf of $\mathscr{C}^\infty$ sections of $\Lambda^k(T^{*(0,1)}X) \otimes \xi$ over X. Then we have the operator $\overline{\partial}^X : \mathcal{D}^k(\xi) \to \mathcal{D}^{k+1}(\xi)$ and an exact sequence of $\mathcal{O}_X$-sheaves

$$0 \to \mathcal{O}_X(\xi) \to \mathcal{D}^1(\xi) \overset{\overline{\partial}^X}{\to} \cdots \overset{\overline{\partial}^X}{\to} \mathcal{D}^l(\xi) \to 0. \tag{2.1}$$

Put $\Omega^k(X,\xi)=\Gamma(X,\mathcal{D}^k(\xi))$, $\Omega^\bullet(X,\xi)=\oplus_k\Omega^k(X,\xi)$, then we have $(\Omega^\bullet(X,\xi),\overline{\partial}^X)$ the Dolbeault complex of $\mathscr{C}^\infty$ sections of $\Lambda(T^{*(0,1)}X)\otimes\xi$ over X:

$$0\to\Omega^0(X,\xi)\overset{\overline{\partial}^X}{\to}\cdots\overset{\overline{\partial}^X}{\to}\Omega^l(X,\xi)\to 0. \tag{2.2}$$

The sheaves $\mathcal{D}^k(\xi)$ are fine [27], so their higher cohomology groups vanish. So

$$H^\bullet(\Omega^\bullet(X,\xi),\overline{\partial}^X)\simeq H^\bullet(X,\mathcal{O}_X(\xi)). \tag{2.3}$$

In the sequel, we also note $H^\bullet(X,\mathcal{O}_X(\xi))$ by $H^\bullet(X,\xi)$.

Let h^{TX},h^ξ be Hermitian metrics on TX,ξ. Then D^X in (1.19) induced by h^{TX},h^ξ is an elliptic operator and

$$D^{X,2}=\overline{\partial}^X\overline{\partial}^{X*}+\overline{\partial}^{X*}\overline{\partial}^X \tag{2.4}$$

preserves the $\mathbb{Z}$-grading on $\Omega^\bullet(X,\xi)$.

The following proposition is [34, Proposition 2.2].

Proposition 2.1 (The Hodge Decomposition Theorem). *There is a L^2-orthogonal direct sum decomposition of the ξ-value $(0,k)$-forms*

$$\Omega^k(X,\xi)=\mathrm{Ker}(D^X)\oplus\mathrm{Im}(\overline{\partial}^X)\oplus\mathrm{Im}(\overline{\partial}^{X*}). \tag{2.5}$$

From (2.3), (2.5), there is a canonical identification

$$\mathrm{Ker}(D^X)\simeq H^\bullet(X,\xi). \tag{2.6}$$

Definition 2.2. Let P^X be the vector space of smooth forms on X, which are sums of forms of type (k,k). Let $P^{X,0}$ be the vector space of the forms $\alpha\in P^X$ such that there exist smooth forms β,γ on X for which $\alpha=\partial\beta+\overline{\partial}\gamma$.

We define $P^{X\cup\Sigma X}$, $P^{X\cup\Sigma X,0}$ in the same way.

2.2. Family index theorem

We use the notation of Section 1.3.

Let M be a complex orbifold. Let ΣM be the strata of M defined by (1.3). Let B be a complex manifold. Let $\pi:M\to B$ be a proper orbifold holomorphic submersion of M onto B with compact fibre X. Then $\pi':M\cup\Sigma M\to B$ is also an orbifold submersion with compact fibre $X\cup\Sigma X$. Let m_i be the multiplicity of each connected component M_i ($m_i=1$, if $M_i=M$) of $M\cup\Sigma M$. Let ξ be an orbifold vector bundle on M. Let ξ^{pr} be the maximal proper orbifold subbundle of ξ.

We assume that π is a Kähler fibration with respect to a real closed $(1,1)$-form ω^M on M. Let D^X_+,D^X_- be the restrictions of D^X to E^+,E^-.

Let B_u $(u>0)$ be the Bismut superconnection on E constructed in Section 1.3 which is attached to the (1,1) form ω^M on M and to the metric h^ξ on ξ.

If A is a (q,q) matrix, set

$$\begin{aligned}&\mathrm{Td}(A)=\det\Big(\frac{A}{1-e^{-A}}\Big),\quad \mathrm{Td}'(A)=\frac{\partial}{\partial u}\mathrm{Td}(A+u)|_{u=0},\\&\mathrm{ch}(A)=\mathrm{Tr}[\exp(A)].\end{aligned}\tag{2.7}$$

The genera associated with Td and ch are called the Todd genus and the Chern character.

Let $\mathcal{U}$ be a cover of $(M,\mathcal{V})$ which defines the submersion π as in Definition 1.3. Recall that for $U\in\mathcal{U}$, we denote $\mathcal{V}(U)=((G_U,\widetilde{U})\to U)$. By [5, (2.20)], [33, (1.15)], [34, (1.6), (1.7)], the forms $\mathrm{Td}^\Sigma(TX,h^{TX})$, $\mathrm{ch}^\Sigma(\xi,h^\xi)$ over $M\cup\Sigma M$ are defined by: on $\widetilde{U}^g/Z_{G_U}(g)$ $(g\in G_U)$, as $\mathrm{Td}_g(\widetilde{TX},h^{TX})$ and

$$\mathrm{ch}_g(\widetilde{\xi^{\mathrm{pr}}},h^\xi)=\mathrm{Tr}\Big[g\exp(\frac{i}{2\pi}R^{\widetilde{\xi^{\mathrm{pr}}}})\Big],\tag{2.8}$$

where $R^{\widetilde{\xi^{\mathrm{pr}}}}$ is the curvature of the holomorphic Hermitian connection on $(\widetilde{\xi^{\mathrm{pr}}},h^\xi)$. Then $\mathrm{Td}^\Sigma(TX,h^{TX}),\mathrm{ch}^\Sigma(\xi,h^\xi)$ are closed on $M\cup\Sigma M$, and their cohomology classes don't depend on the metrics h^{TX},h^ξ.

Let Φ be the homomorphism of $\Lambda^{\mathrm{even}}(T^*_{\mathbb{R}}B)$ into itself: $\alpha\to(2i\pi)^{-\deg\alpha/2}\alpha$.

The following result extends [11, Theorem 2.2].

Theorem 2.3. *For any $u>0$, the differential forms on B, $\mathrm{Tr}_s[\exp(-B_u^2)]$ are elements of P^B. They are closed and they are in the same cohomology class, which does not depend on $u>0$. Also uniformly on compact sets in B,*

$$\lim_{u\to0}\Phi\,\mathrm{Tr}_s[\exp(-B_u^2)]=\sum_i\frac{1}{m_i}\int_{M_i/B}\mathrm{Td}^\Sigma(TX,h^{TX})\,\mathrm{ch}^\Sigma(\xi,h^\xi),\tag{2.9}$$

and the differential form in the right-hand side of (2.9) is also in the same cohomology class as $\Phi\,\mathrm{Tr}_s[\exp(-B_u^2)]$.

If B is compact, then the index bundle as an element in the K-group $K(B)$ is well defined:

$$\mathrm{Ind}(D_+^X)=\mathrm{Ker}(D_+^X)-\mathrm{Ker}(D_-^X)\in K(B).\tag{2.10}$$

The differential forms considered above represent in cohomology $\mathrm{ch}(\mathrm{Ker}(D_+^X)-\mathrm{Ker}(D_-^X))$.

Proof. Let $P_u(x,y,b)$ $(x,y\in\pi^{-1}(b),b\in B)$ be the kernel of the heat operator $\exp(-B_u^2)$ with respect to the Riemannian volume form $dv_X(y)$ on (TX,h^{TX}). By the method of [1, Theorem 9.50], we know $P_u(x,y,b)$ defines a smooth family of smoothing operators along the fibers X.

Proceeding as in [11, Theorem 2.2], $\mathrm{Tr}_s[\exp(-B_u^2)]\in P^B$. They are closed and they are in the same cohomology class.

In [34, §6.6], we observe that the finite propagation speed for hyperbolic equations [20, §7.8], [35, Appendix D.2] holds for orbifolds. By (1.24), and using finite propagation speed as in [6, §11b)], [7], one shows that the problem of calculating the limit of $\mathrm{Tr}_s[\exp(-B_u^2)]$ as $u\to0$ is local on $X_b(b\in B)$.

By Definition 1.3 and the discussion between (1.2)–(1.4), for each $x \in M$, we can choose a chart $\tau : (G_x, \widetilde{U}_x) \to U_x$, such that $\tau^{-1}(x)$ is a point $\widetilde{x}$ and $\widetilde{\pi} : \widetilde{U}_x \to \pi(U_x)$ is a G_x-equivariant submersion. For $\epsilon > 0$, let $B(\widetilde{x}, \epsilon) \subset \widetilde{U}_x$ be the ball with the center $\widetilde{x}$ and radius ϵ. If ϵ is small enough, there exist $x_i \in \pi^{-1}(b)(i \in I = \{1, \dots, k\})$, such that $\{(G_{x_i}, B(\widetilde{x}_i, \frac{\epsilon}{2})) \to B(\widetilde{x}_i, \frac{\epsilon}{2})/G_{x_i}\}_{i \in I}$ is a cover of $\pi^{-1}(b)$. Let $\{\rho_{x_i}\}$ be a partition of unity subordinate to this cover. Then we can replace X by $(\widetilde{TX})_{x_i}/G_{x_i} = \mathbb{C}^l/G_{x_i}$ $(l = \dim X)$, with $0 \in (\widetilde{TX})_{x_i}$ representing x_i.

Note that if Q_U has a $\mathscr{C}^k$-kernel $\widetilde{Q}_U(\widetilde{y}_1, \widetilde{y}_2)$ over $\widetilde{U} \times \widetilde{U}$, then for $y_1, y_2 \in U$,

$$Q_U(y_1, y_2) = \frac{1}{|K_U^\xi|} \sum_{g \in G_U^\xi} (g, 1) \widetilde{Q}_U(g^{-1}\widetilde{y}_1, \widetilde{y}_2), \tag{2.11}$$

is the kernel of the operator

$$Q_U : \mathscr{C}^\infty(U, \xi|_U) \to \mathscr{C}^\infty(U, \xi|_U),$$

with $\tau(\widetilde{y}_i) = y_i (i = 1, 2)$.

Let ${}'\nabla^{\Lambda(T^*_\mathbb{R} B) \otimes \Lambda(T^{*(0,1)}X)}$ be the connection on $\Lambda(T^*_\mathbb{R} B) \otimes \Lambda(T^{*(0,1)}X)$ along the fibre X given as in [6, Definition 11.7].

For $u > 0$, let $\psi_u : \Lambda(T^*_\mathbb{R} B) \to \Lambda(T^*_\mathbb{R} B)$ be the map

$$\alpha \in \Lambda(T^*_\mathbb{R} B) \to u^{-\frac{\deg \alpha}{2}} \alpha \in \Lambda(T^*_\mathbb{R} B).$$

Taken $y \in \mathbb{C}^l$, set $Y = y + \overline{y}$. We identify

$$(\Lambda(T^*_\mathbb{R} B) \otimes \Lambda(T^{*(0,1)}X))_Y, \ \xi_Y \quad \text{with} \quad (\Lambda(T^*_\mathbb{R} B) \otimes \Lambda(T^{*(0,1)}X))_0, \ \xi_0$$

by parallel transport along the curve $t \in [0, 1] \to tY$ with respect to the connection

$$\psi_u {}'\nabla^{\Lambda(T^*_\mathbb{R} B) \otimes \Lambda(T^{*(0,1)}X)} \psi_u^{-1}, \ \nabla^\xi.$$

Let $dv_{T_{x_i}X}(y)$ be the Riemannian volume form on $((\widetilde{TX})_{x_i}, h^{TX}_{x_i}) \simeq \mathbb{R}^{2l}$. For $y \in \mathbb{C}^l, |y| < \epsilon/2$, set

$$dv_X(y) = k(y) dv_{T_{x_i}X}(y). \tag{2.12}$$

Let $\widetilde{P}_u(x, y, b)(x, y \in (\widetilde{TX})_{x_i})$ be the kernel of $\exp(-B_u^2)$ associated to $dv_{T_{x_i}X}(y)$. Then by (2.11), and using finite propagation speed as in [6, §11b)], we get

$$\begin{aligned} &\lim_{u \to 0} \int_{M/B} \rho_{x_i} \Phi \operatorname{Tr}_s[P_u(y, y, b)] dv_X(y) \\ &\quad = \lim_{u \to 0} \int_{\widetilde{U}_{x_i}/V_{x_i}} \rho_{x_i} \frac{1}{|G_{x_i}|} \sum_{g \in G_{x_i}} \Phi \operatorname{Tr}_s[g \widetilde{P}_u(g^{-1}y, y, b)] k(y) dv_{T_{x_i}X}(y). \end{aligned} \tag{2.13}$$

By [3, Theorems 4.11–4.15] and [33, Proof of Theorem 2.12], (1.4), one finds that

$$\begin{aligned}
&\lim_{u\to 0}\int_{\widetilde{U}_{x_i}/V_{x_i}} \frac{1}{|G_{x_i}|}\sum_{g\in G_{x_i}} \rho_{x_i}\Phi\,\mathrm{Tr}_s\Big[g\exp(-B_u^2)(g^{-1}y,y,b)\Big]k(y)dv_{T_{x_i}X}(y)\\
&\quad = \frac{1}{|G_{x_i}|}\sum_{g\in G_{x_i}}\int_{\widetilde{U}^g_{x_i}/V_{x_i}} \rho_{x_i}\,\mathrm{Td}_g(TX,g^{TX})\,\mathrm{ch}_g(\xi,h^\xi) \qquad (2.14)\\
&\quad = \sum_j \frac{1}{m_j}\int_{X_j} \rho_{x_i}\,\mathrm{Td}^\Sigma(TX,g^{TX})\,\mathrm{ch}^\Sigma(\xi,h^\xi).
\end{aligned}$$

By (2.13), (2.14), we get (2.9).

Using the same argument of [3, Theorem 3.4] (also. [1, Chap. 9]), we get the last part of Theorem 2.3. □

3. Quillen metrics and curvature theorem

In this section, we construct the Quillen metrics on the inverse of the determinant of the cohomology of a holomorphic orbifold vector bundle, and establish the curvature formula. We extend the results of [12] to complex orbifolds.

This section is organized as follows. In Section 3.1, by [12], we construct the Quillen metrics. In Section 3.2, we recall our anomaly formulas. In Section 3.3, we establish the curvature formula.

In this section, we use the notation of Section 1.1. We remark that all the morphisms considered in Section 1.1 are holomorphic in the rest of the paper.

3.1. Quillen metrics

Let X be a compact complex orbifold of complex dimension l. Let ξ be a holomorphic orbifold vector bundle on X. Let h^{TX}, h^ξ be smooth Hermitian metrics on TX, ξ. Let $h^{H(X,\xi)}$ be the corresponding metric on $H^\bullet(X,\xi)$ induced by the restriction of the L_2-metric (1.18) to $\mathrm{Ker}(D^X)$ via the canonical isomorphism (2.6).

Let $\lambda(\xi)$ be the inverse of the determinant of the cohomology of ξ on X.

$$\det H^\bullet(X,\xi) = \otimes_{i=0}^{\dim X}(\det H^i(X,\xi))^{(-1)^i}, \quad \lambda(\xi) = (\det H^\bullet(X,\xi))^{-1}. \tag{3.1}$$

Let $|\quad|_{\lambda(\xi)}$ be the metric on $\lambda(\xi)$ induced by $h^{H(X,\xi)}$. The metric $|\quad|_{\lambda(\xi)}$ will be called the L_2-metric on $\lambda(\xi)$.

Let P be the orthogonal projection operator from $\Omega^\bullet(X,\xi)$ on $\mathrm{Ker}(D^X)$ with respect to the Hermitian product (1.18). Set $P^\perp = 1 - P$. Let N be the number operator defining the $\mathbb{Z}$-grading of $\Omega^\bullet(X,\xi)$, i.e., N acts by multiplication by k on $\Omega^k(X,\xi)$. For $s\in\mathbb{C}, \mathrm{Re}(s) > \dim X$, set

$$\theta^\xi(s) = -\mathrm{Tr}_s[N(D^{X,2})^{-s}P^\perp]. \tag{3.2}$$

Then

$$\theta^\xi(s) = \frac{-1}{\Gamma(s)}\int_0^{+\infty} t^{s-1}\,\mathrm{Tr}_s\Big[N\exp(-tD^{X,2})P^\perp\Big]dt. \tag{3.3}$$

From the small time asymptotic expansion of the heat kernel (cf. [34, Proposition 2.1]), (3.3), $\theta^\xi(s)$ extends to a meromorphic function of $s \in \mathbb{C}$ which is holomorphic at $s = 0$.

Following [38], [12], now we define the Quillen metric on the line $\lambda(\xi)$.

Definition 3.1. Let $\| \ \|_{\lambda(\xi)}$ be the Quillen metric on the line $\lambda(\xi)$,

$$\| \ \|_{\lambda(\xi)} = | \ |_{\lambda(\xi)} \exp\Big(-\frac{1}{2}\frac{\partial \theta^\xi}{\partial s}(0)\Big). \tag{3.4}$$

3.2. Anomaly formulas for Quillen metrics

Let $h^{'TX}, h'^\xi$ be another couple of metrics on TX, ξ. We denote with a $'$ the objects attached to $h^{'TX}, h'^\xi$.

As in [10, §1f)], in [34, (1.8)], we constructed classes $\widetilde{\mathrm{Td}}^\Sigma(TX, h^{TX}, h'^{TX})$ and $\widetilde{\mathrm{ch}}^\Sigma(\xi, h^\xi, h'^\xi)$ in $P^{X\cup\Sigma X}/P^{X\cup\Sigma X,0}$ such that

$$\begin{aligned} \frac{\overline{\partial}\partial}{2i\pi}\widetilde{\mathrm{Td}}^\Sigma(TX, h^{TX}, h'^{TX}) &= \mathrm{Td}^\Sigma(TX, h'^{TX}) - \mathrm{Td}^\Sigma(TX, h^{TX}),\\ \frac{\overline{\partial}\partial}{2i\pi}\widetilde{\mathrm{ch}}^\Sigma(\xi, h^\xi, h'^\xi) &= \mathrm{ch}^\Sigma(\xi, h'^\xi) - \mathrm{ch}^\Sigma(\xi, h^\xi). \end{aligned} \tag{3.5}$$

Let m_i be the multiplicity of each connected component X_i of $X \cup \Sigma X$.

The following result is [34, Theorem 0.1] which extends the anomaly formulas of [12, Theorem 1.23], to orbifolds.

Theorem 3.2. *Assume that the metrics h^{TX} and h'^{TX} are Kähler. Then*

$$\begin{aligned} \log\left(\frac{\| \ \|'^2_{\lambda(\xi)}}{\| \ \|^2_{\lambda(\xi)}}\right) = \sum_i \Big(&\frac{1}{m_i}\int_{X_i} \widetilde{\mathrm{Td}}^\Sigma(TX, h^{TX}, h'^{TX})\,\mathrm{ch}^\Sigma(\xi, h^\xi)\\ &+\frac{1}{m_i}\int_{X_i} \mathrm{Td}^\Sigma(TX, h'^{TX})\widetilde{\mathrm{ch}}^\Sigma(\xi, h^\xi, h'^\xi)\Big). \end{aligned} \tag{3.6}$$

3.3. The curvature of the determinant line bundle for a Kähler fibration

We now do the same assumption as in Section 1.3 and we use the same notations.

Let $\pi : M \to B$ be a proper holomorphic orbifold submersion of M onto B with compact fibre X. Let ξ be a holomorphic orbifold vector bundle on M. Let ω^M be a real, closed (1,1) form on M taken as in Theorem 1.8. Let h^{TX} be the metric on TX induced by ω^M. Let h^ξ be a Hermitian metric on ξ.

We will temporarily assume that B is a complex manifold. Let λ be the $\mathscr{C}^\infty$ determinant line bundle on B constructed as in [12, §1b)]. By proceeding as in [12, §1c)], we can define a holomorphic structure on the line bundle λ.

We explain the construction in detail here. Let $\nabla^{E''}$ be the anti-holomorphic part of the connection ∇^E in (1.20) on the infinite-dimensional vector bundle E on B. For $a > 0$, set

$$U^a = \{y \in B : a \notin \mathrm{Spec}(D_y^2)\}, \tag{3.7}$$

where $\mathrm{Spec}(D_y^2)$ is the spectrum of the operator D_y^2. Then on U^a, the sum of the eigenspaces of the operator D_y^2 acting on E_y^j of eigenvalues $< a$, $K_y^{a,j}$ forms a smooth finite-dimensional vector bundle. On U^a, λ coincides with the line bundle λ^a

$$\lambda^a = \otimes_{j=0}^{\dim X} (\det K^{b,j})^{(-1)^{j+1}}, \tag{3.8}$$

and for $0 < a < c$, over $U^a \cap U^c$, we identify λ^a and λ^c by

$$s \in \lambda^a \to s \otimes T(\overline{\partial}^{(a,c)}) \in \lambda^c, \tag{3.9}$$

with $\overline{\partial}^{(a,c)}$ the restriction of $\overline{\partial}^X$ to $K^{c,j}/K^{a,j}$ and the torsion $T(\overline{\partial}^{(a,c)})$ for the complex $(K^{c,j}/K^{a,j}, \overline{\partial}^{(a,c)})$ is defined in [10, Definition 1.1]. Let P^a be the orthogonal projection operator from E onto K^a. By [12, Theorem 1.3], the holomorphic structure on λ^a is defined by

$$\overline{\partial}^{\lambda^a} = \mathrm{Tr}_s[P^a \nabla^{E''} P^a] \tag{3.10}$$

and the identification λ^a and λ^c in (3.9) is holomorphic.

The sheaf $\mathcal{O}_M$ is coherent as explained in Section 2.1. By [35, Theorem 5.4.16], $(M, \mathcal{O}_M)$ is a normal complex space and $\mathcal{O}_M(\xi)$ is a $\mathcal{O}_M$-coherent analytic sheaf, thus by a theorem of Grauert [25], for all $i \geq 0$, the $\mathcal{O}_B$-module $R^i\pi_*\xi$ is coherent. If $i > \dim M$, then $R^i\pi_*\xi = 0$. The functor $R^\bullet\pi_*$ maps the derived category of $\mathcal{O}_M$-module to the derived category of $\mathcal{O}_B$-modules and sends coherent sheaves to complexes with coherent cohomology. As B is a complex manifold, for any $y \in B$, the local ring $\mathcal{O}_{B,y}$ is regular, hence all coherent analytic sheaves on B is perfect and more generally any complex with bounded coherent cohomology is perfect. Thus as in [12, Theorem 3.4], we can associate a (graded) invertible holomorphic sheaf $\det(R^\bullet\pi_*\xi)$ on B, and the associated Knudsen–Mumford determinant line bundle is

$$\lambda^{KM} = (\det(R^\bullet\pi_*\xi))^{-1}. \tag{3.11}$$

In particular, if $R^i\pi_*\xi$ is locally free for all i, we get

$$\lambda^{KM}(\xi) = \otimes_{i\geq 0} (\det(R^i\pi_*\xi))^{(-1)^{i+1}}. \tag{3.12}$$

Let $\mathcal{O}_B^\infty$ be the sheaf of $\mathscr{C}^\infty$ functions on B. Let $\mathscr{H}^j_{\overline{\partial}}(\xi)$ be the cohomology sheaves of the relative Dolbeault complex $(\mathcal{D}_X^\bullet(\xi), \overline{\partial}^X)$ in (2.1) as $\mathcal{O}_B^\infty$-modules. Let $\mathcal{D}_M^\bullet$ be the sheaf of Dolbeault complexes on M, then we can use the partition of unity argument for $\mathcal{D}_M^\bullet$, thus $\mathcal{D}_M^\bullet$ is fine, from the argument of [12, p. 342],

$$R^j\pi_*\xi = \mathscr{H}^j(\pi_*(\mathcal{D}_M^\bullet(\xi))). \tag{3.13}$$

The natural map $T^*M \to T^*X$ induces a map of complexes $\mathcal{D}_M^\bullet(\xi) \to \mathcal{D}_X^\bullet(\xi)$, thus a canonical map on cohomology sheaves

$$\varrho_j : (R^j\pi_*\xi) \otimes_{\mathcal{O}_B} \mathcal{O}_B^\infty \to \mathscr{H}^j_{\overline{\partial}}(\xi). \tag{3.14}$$

Again as B is a manifold, the algebraic argument in the proof of [12, Theorem 3.5] holds, thus we get the analogue of [12, Theorem 3.5]:

Theorem 3.3. *For all $j \geq 0$, the map ϱ_j is an isomorphism.*

Under the assumption of the Kähler fibration, as in [11, Theorem 2.8], we have

$$\overline{\partial}^M = \nabla^{E''} + \overline{\partial}^X. \tag{3.15}$$

From the arguments of the proof of [12, Corollary 3.9, Theorem 3.14], by Theorem 3.3 and (3.15), we get the analogue of [12, Theorem 3.14]:

Theorem 3.4. *The smooth isomorphism λ^{KM} and λ via (3.14) is an isomorphism of holomorphic line bundles.*

If B is not a complex manifold, then for each $b \in B$, we consider over $\widetilde{V}_b$ as in Lemma 1.6. By proceeding as in the proof of Proposition 1.4, we construct $\widetilde{\xi}$ a holomorphic orbifold vector bundle on $\widetilde{M}_b$ induced by ξ. Then the above construction gives a G_b-equivariant holomorphic line bundle $\widetilde{\lambda}$ on $\widetilde{V}_b$ and natural compatibilities for different local charts $(G_b, \widetilde{V}_b)$ in Lemma 1.6. Thus we get the determinant line bundle λ as a holomorphic orbifold line bundle on B.

From the algebraic side, the Knudsen–Mumford line bundle $\widetilde{\lambda}^{KM}$ on $\widetilde{V}_b$ is also well defined and G_b-action lifts naturally on it. Thus we get the Knudsen–Mumford line bundle λ^{KM} as a holomorphic orbifold line bundle over B. Moreover, the isomorphism ϱ_j in (3.14) over $\widetilde{V}_b$ is G_b-equivariant via the argument from [12, §3]. Thus we get

Theorem 3.5. *The smooth isomorphism λ^{KM} and λ via (3.14) is an isomorphism of holomorphic orbifold line bundles.*

For $\alpha \in \Lambda(T^*_{\mathbb{R}}B)$, $\alpha^{(j)}$ denotes the component of α in $\Lambda^j(T^*_{\mathbb{R}}B)$.

Let m_j be the multiplicity of the component X_j of $X \cup \Sigma X$ in Proposition 1.4. The following result extends the curvature theorem [10, Theorem 0.3], [12, Theorem 1.27] to orbifolds.

Theorem 3.6. *The Quillen metric $\| \quad \|_\lambda$ on λ is a smooth metric on B. Let ∇^λ be the holomorphic Hermitian connection on the Hermitian orbifold line bundle $(\lambda, \| \quad \|_\lambda)$, then*

$$(\nabla^\lambda)^2 = 2i\pi\left[\sum_j \frac{1}{m_j}\int_{X_j} \mathrm{Td}^\Sigma(TX, g^{TX})\,\mathrm{ch}^\Sigma(\xi, h^\xi)\right]^{(2)}. \tag{3.16}$$

Proof. Note that for $b \in B$, the Quillen metric $\| \quad \|_{\widetilde{\lambda}}$ on the G_b-equivariant holomorphic line bundle $\widetilde{\lambda}(\xi)$ over $\widetilde{V}_b$, is smooth and G_b-invariant. Thus $\| \quad \|_\lambda$ on the orbifold line bundle $\lambda(\xi)$ is smooth over B.

We still need to compute the curvature of $(\widetilde{\lambda}, \| \quad \|_{\widetilde{\lambda}})$ on $\widetilde{V}_b$, for $b \in B$.

As the argument of [12, Theorem 1.8] is purely functional analysis, by [8, Theorem 1.18] and [9, Theorem 1.19], we know $(\nabla^{\lambda})^2$ over $\widetilde{V}_b$ is the constant term in the asymptotic of

$$\mathrm{Tr}_s[\exp(-B_u^2)]^{(2)} \text{ as } u \to 0. \tag{3.17}$$

Now by combining with Theorem 2.3 for the fibration $\widetilde{\pi}_b : \widetilde{M}_b \to \widetilde{V}_b$, we get (3.16). □

4. Analytic torsion forms and anomaly formulas

In this section, we construct the analytic torsion forms associated with an orbifold submersion, and we explain the anomaly formulas. This extends the results of [11], [13] to the orbifold case.

This section is organized as follows. In Section 4.1, we describe the transgression formulas of the superconnection forms, which depend on $u \in]0, +\infty[$. In Section 4.2, proceeding as in [3], [11], [1], we obtain the results on the asymptotics of these forms as $u \to 0$ and $u \to +\infty$. In Section 4.3, we construct the analytic torsion forms, which extend [13]. In Section 4.4, we give the anomaly formulas of the analytic torsion forms, which extend [13] to the orbifold case.

We use here the same notation as in Sections 1, 2.1.

4.1. Superconnection forms and double transgression formulas

Let $\pi : M \to B$ be a proper holomorphic orbifold submersion of M onto B with compact fibre X. Let $n = \dim M$. Let ξ be a holomorphic orbifold vector bundle on M.

By Lemma 1.6, for each $b \in B$, there exists a neighbourhood $(G_b, \widetilde{V}_b) \to V_b$, an orbifold $\widetilde{M}_b$, such that π is induced by a G_b-equivariant orbifold submersion $\widetilde{\pi}_b : \widetilde{M}_b \to \widetilde{V}_b$ with compact fibre X. By proceeding as in the proof of Proposition 1.4, we construct $\widetilde{\xi}$ a holomorphic orbifold vector bundle on $\widetilde{M}_b$ induced by ξ.

The direct image $R^\bullet \pi_* \xi$ is well defined as a $\mathcal{O}_B$-sheaf. Let $\mathcal{D}^j_M(\xi)$ be the sheaf of $\mathscr{C}^\infty$ sections of $\Lambda^j(T^{*(0,1)}M) \otimes \xi$ over M. We have an exact sequence of $\mathcal{O}_M$-sheaves:

$$0 \to \mathcal{O}_M(\xi) \to \mathcal{D}^1_M(\xi) \overset{\overline{\partial}^M}{\to} \cdots \overset{\overline{\partial}^M}{\to} \mathcal{D}^n_M(\xi) \to 0. \tag{4.1}$$

The sheaves $\mathcal{D}^j_M(\xi)$ are fine, as we can apply the partition of unity argument for $\mathcal{D}^\bullet_M(\xi)$, so $(\mathcal{D}^\bullet_M(\xi), \overline{\partial}^M)$ is a π_*-acyclic resolution of $\mathcal{O}_M(\xi)$. So the direct image $R^\bullet \pi_* \xi$ is defined by the presheaf, cf. (3.13):

$$V \to H^\bullet(\Gamma(\pi^{-1}(V), \mathcal{D}^\bullet_M(\xi)), \overline{\partial}^M)\,.$$

But for $b \in B$, on V_b, the presheaf $V \to H^\bullet(\Gamma(\pi^{-1}(V), \mathcal{D}^\bullet_M(\xi)), \overline{\partial}^M)$ is exactly the G_b-invariant sections of $R^\bullet \widetilde{\pi}_{b*} \widetilde{\xi}$ over $\widetilde{V}_b$.

If on each $\widetilde{V}_b$, we define a G_b-equivariant coherent sheaf $R^\bullet\widetilde{\pi}_{b*}\widetilde{\xi}$, then by construction, we verify that this defines a proper coherent sheaf on B.

By the above discussion, the direct image $R^\bullet\pi_*\xi$ is an orbifold $\mathcal{O}_B$-coherent sheaf: over $\widetilde{V}_b$, it is defined by $R^\bullet\widetilde{\pi}_{b*}\widetilde{\xi}$.

We make the *basic assumption* that for $0 \leq k \leq \dim X$, $b \in B$, the sheaves $R^k\widetilde{\pi}_{b*}\widetilde{\xi}$ is locally free. Then $R^\bullet\pi_*\xi$ is a proper orbifold vector bundle over B. For $p \in \widetilde{V}_b$, let $H^\bullet(X_p, \widetilde{\xi}|_{X_p}) = \oplus_{k=0}^{\dim X} H^k(X_p, \widetilde{\xi}|_{X_p})$ be the cohomology of the sheaf of holomorphic sections of $\widetilde{\xi}$ restricted to X_p. Then the $H^\bullet(X_p, \widetilde{\xi}|_{X_p})$ are the fibres of a G_b-equivariant holomorphic $\mathbb{Z}$-graded vector bundle $H^\bullet(X_p, \widetilde{\xi}|_{X_p})$ on $\widetilde{V}_b$, and $H^\bullet(X_p, \widetilde{\xi}|_{X_p}) = R^\bullet\widetilde{\pi}_{b*}\widetilde{\xi}$. So the $H^\bullet(X_p, \widetilde{\xi}|_{X_p})$ defined an orbifold vector bundle $H^\bullet(X, \xi|_X)$.

Let ω^M be a real closed (1,1) form on M such that ω^M induces a Kähler metric on TX (cf. Theorem 1.8). Let h^ξ be a Hermitian metric on ξ.

We verify easily that the objects on M (for example: ω^M, ξ, h^ξ) lift on $\widetilde{M}_b$. We denote with a $\widetilde{\ }$ the objects we considered in Section 1.1 which are attached to $\widetilde{\pi}_b : \widetilde{M}_b \to \widetilde{V}_b$.

For $p \in \widetilde{V}_b$, set

$$K_p = \left\{ f \in \widetilde{E}_p : \overline{\partial}^{X_p} f = 0, \overline{\partial}^{X_p *} f = 0 \right\}. \tag{4.2}$$

By the Hodge theory (2.6),

$$K_p \simeq H^\bullet(X_p, \widetilde{\xi}|_{X_p}). \tag{4.3}$$

The identification (4.3) induces an identification of the corresponding smooth vector bundles on $\widetilde{V}_b$. Also K inherits a G_b-invariant Hermitian product from the L_2-Hermitian product on $\widetilde{E}$. Let $h^{H(X,\xi|X)}$ be the corresponding smooth metric on $H^\bullet(X, \xi|_X)$.

Recall that $\widetilde{E}$ is a G_b-equivariant bundle over $\widetilde{V}_b$ and the contribution of ξ is only from its maximal proper orbifold subbundle ξ^{pr} of ξ.

Let B_u be the Bismut superconnection on E constructed in Section 1.3.

Let $\widetilde{P}_u(x, y, p)$ $(x, y \in \widetilde{\pi}^{-1}(p), p \in \widetilde{V}_b)$ be the kernel associated to the operator $\exp(-B_u^2)$ with respect to $dv_X(y)/(2\pi)^{\dim X}$, then we know $\widetilde{P}_u(x, y, p)$ defines a smooth family of smoothing operators.

We define $\mathrm{Tr}_s^\Sigma[\exp(-B_u^2)]$, $\mathrm{Tr}_s^\Sigma[N_u\exp(-B_u^2)]$ as forms over $B \cup \Sigma B$ by: If a connected component B_i of $B \cup \Sigma B$, is locally defined by $\Big((Z_{G_b}(h), \widetilde{V}_b^h) \to \widetilde{V}_b^h/Z_{G_b}(h)\Big)$ $(h \in G_b$, $\widetilde{V}_b^h$ is the fixed point of h over $\widetilde{V}_b)$, then over $\widetilde{V}_b^h/Z_{G_b}(h)$,

$$\begin{aligned} \mathrm{Tr}_s^\Sigma[\exp(-B_u^2)] &= \mathrm{Tr}_s[h\exp(-B_u^2)], \\ \mathrm{Tr}_s^\Sigma[N_u\exp(-B_u^2)] &= \mathrm{Tr}_s[hN_u\exp(-B_u^2)]. \end{aligned} \tag{4.4}$$

As in [11, Theorems 2.2 and 2.9], the forms

$$\Phi\,\mathrm{Tr}_s^\Sigma[\exp(-B_u^2)] \quad \text{and} \quad \Phi\,\mathrm{Tr}_s^\Sigma[N_u\exp(-B_u^2)]$$

lie in $P^{B\cup\Sigma B}$. By Theorem 2.3, we know that the forms $\Phi \operatorname{Tr}_s^{\Sigma}[\exp(-B_u^2)]$ are closed and that their cohomology class is constant and equal to $\operatorname{ch}^{\Sigma}(H^{\bullet}(X, \xi|_X))$.

Theorem 4.1. *For $u > 0$, the following identity holds*

$$\frac{\partial}{\partial u}\Phi \operatorname{Tr}_s^{\Sigma}[\exp(-B_u^2)] = -\frac{1}{u}\frac{\overline{\partial}\partial}{2i\pi}\Phi \operatorname{Tr}_s^{\Sigma}[N_u \exp(-B_u^2)]. \tag{4.5}$$

Proof. In (4.4), the action h commutes with B_u, N_u. Now, by proceeding as in [11, Theorem 2.9], we get (4.5). □

If $(\alpha_u)_{u>0}$ is a family of smooth forms over $B \cup \Sigma B$, we write that as $u \to 0$ (resp. $u \to +\infty$), $\alpha_u = O(u^k)$, if for any compact subset $K \subset B \cup \Sigma B$, and any $j \in \mathbb{N}$, there is $C > 0$ such that the sup of α_u and its derivative of order $\le j$ on K are dominated by Cu^k.

4.2. The asymptotics of the superconnection forms

Clearly, for $b \in B$, in Proposition 1.4, we can choose the ramified covering $(G_{U_i}, \widetilde{U}_i)$ of $\pi^{-1}(V_b)$ as the type $(G_x, \widetilde{U}_x)$ such that $\widetilde{U}_x$ is a neighbourhood of $0 \in \mathbb{C}^n (n = \dim M)$ and such that G_x acts linearly on $\mathbb{C}^n$. Now, we fixe a choice of

$$(G_{U_i}, \widetilde{U}_i) = (G_{x_i}, \widetilde{U}_{x_i})_{i\in I}(I = \{1, \dots, k\}), (G_b, \widetilde{V}_b) \to V_b. \tag{4.6}$$

Let $\widetilde{\pi} : (G_{U_i}, \widetilde{U}_i) \to (G_b, \widetilde{V}_b)$ be the G_{U_i}-equivariant holomorphic submersion of $\widetilde{U}_i$ onto $\widetilde{V}_b$, and $\pi^{-1}(V_b) = \cup_{i\in I} U_i$. The map $\widetilde{\pi}$ induces naturally a morphism $\pi_i : G_{U_i} \to G_b$. Let $K_{x_i} = K_{U_i} = \operatorname{Ker}\{\pi_i : G_{U_i} \to G_b\}$. Then for $h \in G_b, g \in \pi_i^{-1}(h)$, $\widetilde{\pi} : \widetilde{U}_i^g \to \widetilde{V}_b^h$ is also a submersion. Let ρ_i be a partition of unity of $\pi^{-1}(V_1)$ subordinate to $\{U_i\}_{i\in I}$, for $b \in V_1 \subset V_b$ compact.

Let $\beta = \inf_{i\in I}\{\text{injectivity radius of } x_i \text{ on } \widetilde{U}_{x_i}\}$. Take $\alpha \in]0, \beta/4]$.

Let f be a smooth even function defined on $\mathbb{R}$ with values in $[0, 1]$, such that

$$f(t) = \begin{cases} 1 & \text{for} \quad |t| \le \alpha/2 \\ 0 & \text{for} \quad |t| \ge \alpha. \end{cases} \tag{4.7}$$

Set

$$g(t) = 1 - f(t). \tag{4.8}$$

Definition 4.2. For $u \in]0, 1], a \in \mathbb{C}$, set

$$F_u(a) = \int_{-\infty}^{+\infty} \exp(ita\sqrt{2}) \exp\left(\frac{-t^2}{2}\right) f(ut)\frac{dt}{\sqrt{2\pi}}, \tag{4.9}$$
$$G_u(a) = \int_{-\infty}^{+\infty} \exp(ita\sqrt{2}) \exp\left(\frac{-t^2}{2}\right) g(ut)\frac{dt}{\sqrt{2\pi}}.$$

Clearly

$$F_u(a) + G_u(a) = \exp(-a^2). \tag{4.10}$$

The functions $F_u(a), G_u(a)$ are even holomorphic functions. So there exist holomorphic functions $\widetilde{F}_u(a)$, $\widetilde{G}_u(a)$ such that

$$F_u(a) = \widetilde{F}_u(a^2), \quad G_u(a) = \widetilde{G}_u(a^2). \tag{4.11}$$

Let μ be a form on $M \cup \Sigma M$, we define $\int_{X\cup\Sigma X} \mu$ as a form over $B \cup \Sigma B$: locally over $\widetilde{V}_b^h / Z_{G_b}(h) \subset B \cup \Sigma B$, we denote

$$\int_{X\cup\Sigma X} \mu = \sum_{i\in I} \frac{1}{|K_{U_i}|} \sum_{g\in\tau_{U_i}^{-1}(h)} \int_{\widetilde{U}_i^g/\widetilde{V}^h} \rho_i \mu. \tag{4.12}$$

Put

$$\begin{aligned} C_{-1} &= \int_{X\cup\Sigma X} \frac{\omega^M}{2\pi} \operatorname{Td}^\Sigma(TX, h^{TX}) \operatorname{ch}^\Sigma(\xi, h^\xi), \\ C_0 &= \int_{X\cup\Sigma X} \Big(-(\operatorname{Td}')^\Sigma(TX, h^{TX}) + \dim X \operatorname{Td}^\Sigma(TX, h^{TX}) \Big) \operatorname{ch}^\Sigma(\xi, h^\xi). \end{aligned} \tag{4.13}$$

Set

$$\begin{aligned} \operatorname{ch}^\Sigma(H^\bullet(X, \xi|_X), h^{H(X,\xi|_X)}) &= \sum_{k=0}^{\dim X} (-1)^k \operatorname{ch}^\Sigma(H^k(X, \xi|_X), h^{H(X,\xi|_X)}), \\ \operatorname{ch}'^\Sigma(H^\bullet(X, \xi|_X), h^{H(X,\xi|_X)}) &= \sum_{k=0}^{\dim X} (-1)^k k \operatorname{ch}^\Sigma(H^k(X, \xi|_X), h^{H(X,\xi|_X)}). \end{aligned} \tag{4.14}$$

Theorem 4.3. *As $u \to 0$*

$$\Phi \operatorname{Tr}_s^\Sigma[\exp(-B_u^2)] = \int_{X\cup\Sigma X} \operatorname{Td}^\Sigma(TX, h^{TX}) \operatorname{ch}^\Sigma(\xi, h^\xi) + O(u). \tag{4.15}$$

There are forms $C'_j \in P^{B\cup\Sigma B} (j \geq -1)$ such that for $k \in \mathbb{N}$, as $u \to 0$

$$\Phi \operatorname{Tr}_s^\Sigma[N_u \exp(-B_u^2)] = \sum_{j=-1}^{k} C'_j u^j + O(u^{k+1}). \tag{4.16}$$

Also

$$\begin{aligned} C'_{-1} &= C_{-1}, \\ C'_0 &= C_0 \quad \text{in} \quad P^{B\cup\Sigma B}/P^{B\cup\Sigma B,0}. \end{aligned} \tag{4.17}$$

Proof. Recall that in the construction of the orbifold $\widetilde{M}_b$, we use the local coordinate system $(K_{U_i}, \widetilde{U}_i) \to \widetilde{U}_i/K_{U_i}$.

By (2.8) and the definition of smooth sections for an orbifold vector bundle, only the maximal proper orbifold subbundle ξ^{pr} of ξ makes contributions in various steps, thus we will assume simply that ξ is a proper orbifold vector bundle on M.

Following (4.4), we will calculate the following limit as $u \to 0$,

$$I_i(h, u) = \int_X \rho_i(p, x) \Phi \operatorname{Tr}_s[h\widetilde{P}_u(x, x, p)] dv_X. \tag{4.18}$$

Lemma 4.4. *There exist $c > 0, C > 0$ such that for $u \in]0,1]$*

$$\left|\operatorname{Tr}_s[\rho_i h \widetilde{G}_u(B_u^2)]\right| \leq c \exp\left(\frac{-C}{u^2}\right). \tag{4.19}$$

Proof. By proceeding as in the proof of [2, Proposition 8.3], we have (4.19). □

Let $\widetilde{F}_u(B_u^2)(x_1, x_2)((x_1, x_2) \in X_p \times X_p)$ be the smooth kernel associated with $\widetilde{F}_u(B_u^2)$ with respect to $dv_X(x_2)/(2\pi)^{\dim X}$. Using (4.5), (4.9), and finite propagation speed [20, §7.8], [35, Appendix D. 2], it is clear that $\widetilde{F}_u(B_u^2)(x, x') = 0$ if $d(x, x') > \alpha$, and $\widetilde{F}_u(B_u^2)(x, x')$ depends only on the restriction of B_u^2 to $B^X(x, \alpha)$.

We replace X by $(\widetilde{TX})_{x_i}/K_{x_i} = \mathbb{C}^l/K_{x_i}$ ($l = \dim X$), with $0 \in (\widetilde{TX})_{x_i}$ representing x_i, and that the extended fibration over $\mathbb{C}^l$ coincides with the given fibration over $B(0, 2\alpha) \subset \mathbb{C}^l$.

Let Δ^{TX} be the standard Laplacian on $((\widetilde{TX})_{x_i}, h^{TX}_{x_i})$. Let $\rho(Y)$ be a $\mathscr{C}^\infty$ function over $\mathbb{C}^l$ which is equal 1 if $|Y| \leq \alpha$, equal 0 if $|Y| \geq 2\alpha$. Let

$$L_u^1 = (1 - \rho^2(Y))\left(-\frac{1}{2}u\Delta^{TX}\right) + \rho^2(Y)B_u^2. \tag{4.20}$$

Let $\widetilde{F}_u(L_u^1)(x, x')(x, x' \in \mathbb{C}^l)$ be the smooth kernel of $\widetilde{F}_u(L_u^1)$ with respect to $dv_{T_{x_i}X}(x')/(2\pi)^{\dim X}$. For $y \in \mathbb{C}^l$, $|y| < 2\alpha$, as in (2.12), set

$$dv_X(y) = k(y) dv_{T_{x_i}X}(y). \tag{4.21}$$

Then, for $|y| < 2\alpha$, $y \in (\widetilde{TX})^g_{x_i}$, we get

$$dv_{X^g}(y) = k(y) dv_{T_{x_i}X^g}(y). \tag{4.22}$$

By (2.11) and the above discussion, if α is enough small, for

$$(x, x') \in \operatorname{supp}(\rho_i) \times \operatorname{supp}(\rho_i),$$

we get

$$\widetilde{F}_u(B_u^2)(x, x') = k(x') \sum_{g \in K_{x_i}} (g, 1)\widetilde{F}_u(L_u^1)(g^{-1}\widetilde{x}, \widetilde{x}'). \tag{4.23}$$

Note that K_{x_i} acts on $\widetilde{\xi}$ as we explained above (4.18) that ξ is proper.

By (4.18), (4.19), (4.23), we get

$$\begin{aligned}\lim_{u\to 0} I_i(h, u) &= \lim_{u\to 0} \int_X \rho_i(p, x)\Phi \operatorname{Tr}_s[h\widetilde{F}_u(B_u^2)(x, x)] dv_X/(2\pi)^{\dim X} \qquad (4.24)\\ &= \lim_{u\to 0} \int_{\mathbb{C}^l} \frac{1}{|K_{x_i}|} \sum_{g \in K_{x_i}} \rho_i(p, x)\Phi \operatorname{Tr}_s[h(g\widetilde{F}_u(L_u^1))(\widetilde{x}, \widetilde{x})] k(\widetilde{x}) dv_{T_{x_i}X}/(2\pi)^{\dim X}\\ &= \lim_{u\to 0} \int_{\mathbb{C}^l} \frac{1}{|K_{x_i}|} \sum_{g \in \tau_{U_i}^{-1}(h)} \rho_i(p, x)\Phi \operatorname{Tr}_s[g\widetilde{F}_u(L_u^1)(g^{-1}\widetilde{x}, \widetilde{x})] k(\widetilde{x}) dv_{T_{x_i}X}/(2\pi)^{\dim X}.\end{aligned}$$

We observe that for any $k \in \mathbb{N}$, $c > 0$, there is $C > 0, C' > 0$ such that for $u \in]0,1]$,

$$\sup_{|\mathrm{Im}(a)| \leq c} |a|^k \left| \widetilde{F}_u(a^2) - \exp(-a^2) \right| \leq C' \exp(\frac{-C}{u^2}). \tag{4.25}$$

For each $g \in \tau_{U_i}^{-1}(h)$, by using (4.22), (4.25), and by proceeding as in [33, (2.42)–(2.51)], we get

$$\begin{aligned}
&\lim_{u\to 0} \int_{\mathbb{C}^l} \rho_i(p, \widetilde{x}) \Phi \operatorname{Tr}_s[g\widetilde{F}_u(L_u^1)(g^{-1}\widetilde{x}, \widetilde{x})] k(\widetilde{x}) dv_{T_{x_i}X}/(2\pi)^l \\
&\quad = \lim_{u\to 0} \int_{(\widetilde{TX})^g_{x_i}} \int_{z\in \widetilde{N}_{X^g/X}} \rho_i(p, (\widetilde{x}, \widetilde{z})) \Phi \operatorname{Tr}_s[g\widetilde{F}_u(L_u^1)(g^{-1}(\widetilde{x}, \widetilde{z}), (\widetilde{x}, \widetilde{z}))] \\
&\qquad k(\widetilde{x}, \widetilde{z}) dv_{T_{x_i}X^g}(\widetilde{x}) dv_{N_{X^g/X}, x_i}(\widetilde{z})/(2\pi)^l \\
&\qquad\quad = \int_{(\widetilde{TX})^g_{x_i}} \rho_i(p, \widetilde{x}) \operatorname{Td}_g(\widetilde{TX}, h^{TX}) \operatorname{ch}_g(\widetilde{\xi}, h^{\xi}).
\end{aligned} \tag{4.26}$$

By (4.12), (4.18), (4.19), (4.24), (4.26), we get (4.15).

By combining the techniques of proof of [11, Theorems 2.2, 2.3, 2.9 and 2.16] and the proof of (4.15), we get (4.16) and (4.17). □

Theorem 4.5. *As $u \to +\infty$*

$$\begin{aligned}
\Phi \operatorname{Tr}_s^{\Sigma}[\exp(-B_u^2)] &= \operatorname{ch}^{\Sigma}(H^\bullet(X, \xi|_X), h^{H(X,\xi|_X)}) + O\left(\frac{1}{\sqrt{u}}\right), \\
\Phi \operatorname{Tr}_s^{\Sigma}[N_u \exp(-B_u^2)] &= \operatorname{ch}'^{\Sigma}(H^\bullet(X, \xi|_X), h^{H(X,\xi|_X)}) + O\left(\frac{1}{\sqrt{u}}\right).
\end{aligned} \tag{4.27}$$

Proof. Equation (4.27) was stated in [13, Theorem 3.4], if M, B are complex manifolds. By proceeding as in [1, Theorem 9.23], we get also (4.27) in our situation. □

4.3. Analytic torsion forms

For $s \in \mathbb{C}, \mathrm{Re}(s) > 1$, set

$$\zeta_1(s) = -\frac{1}{\Gamma(s)} \int_0^1 u^{s-1} \Big(\Phi \operatorname{Tr}_s^{\Sigma}[N_u \exp(-B_u^2)] - \operatorname{ch}'^{\Sigma}(H^\bullet(X, \xi|_X), h^{H(X,\xi|_X)}) \Big) du.$$

Using (4.16), we see that $\zeta_1(s)$ extends to a holomorphic function of $s \in \mathbb{C}$ near $s = 0$.

For $s \in \mathbb{C}, \mathrm{Re}(s) < \frac{1}{2}$, set

$$\zeta_2(s) = -\frac{1}{\Gamma(s)} \int_1^{+\infty} u^{s-1} \Big(\Phi \operatorname{Tr}_s^{\Sigma}[N_u \exp(-B_u^2)] - \operatorname{ch}'^{\Sigma}(H^\bullet(X, \xi|_X), h^{H(X,\xi|_X)}) \Big) du.$$

Then $\zeta_2(s)$ is a holomorphic function of s.

Definition 4.6. Set

$$T(\omega^M, h^{\xi}) = \frac{\partial}{\partial s}(\zeta_1 + \zeta_2)(0). \tag{4.28}$$

Then $T(\omega^M, h^\xi)$ is a smooth form on $B \cup \Sigma B$. Using (4.16), (4.27), we get

$$\begin{aligned} T(\omega^M, h^\xi) = & - \int_0^1 \Big(\Phi \operatorname{Tr}_s^\Sigma[N_u \exp(-B_u^2)] - \frac{C'_{-1}}{u} - C'_0\Big)\frac{du}{u} \\ & - \int_1^{+\infty} \Big(\Phi \operatorname{Tr}_s^\Sigma[N_u \exp(-B_u^2)] - \operatorname{ch}'^\Sigma(H^\bullet(X, \xi|_X), h^{H(X,\xi|_X)})\Big)\frac{du}{u} \\ & + C'_{-1} + \Gamma'(1)\Big(C'_0 - \operatorname{ch}'^\Sigma(H^\bullet(X, \xi|_X), h^{H(X,\xi|_X)})\Big). \end{aligned} \tag{4.29}$$

Theorem 4.7. *The form $T(\omega^M, h^\xi)$ lies in $P^{B\cup\Sigma B}$. Moreover*

$$\begin{aligned} \frac{\overline{\partial}\partial}{2i\pi} T(\omega^M, h^\xi) = & \operatorname{ch}^\Sigma\left(H^\bullet(X, \xi|_X), h^{H(X,\xi|_X)}\right) \\ & - \int_{X\cup\Sigma X} \operatorname{Td}^\Sigma(TX, h^{TX}) \operatorname{ch}^\Sigma(\xi, h^\xi). \end{aligned} \tag{4.30}$$

Proof. By Theorems 4.1, 4.3, 4.5, we get (4.30). □

4.4. Anomaly formulas for the analytic torsion forms

Let now (ω', h'^ξ) be another couple of objects similar to (ω, h^ξ). We denote with a $'$ the objects associated to (ω', h'^ξ).

Theorem 4.8. *The following identity holds in $P^{B\cup\Sigma B}/P^{B\cup\Sigma B,0}$,*

$$\begin{aligned} & T(\omega', h'^\xi) - T(\omega, h^\xi) = \widetilde{\operatorname{ch}}^\Sigma\Big(H^\bullet(X, \xi|_X), h^{H(X,\xi|_X)}, h'^{H(X,\xi|_X)}\Big) \\ & - \int_{X\cup\Sigma X} \Big[\widetilde{\operatorname{Td}}^\Sigma(TX, h^{TX}, h'^{TX}) \operatorname{ch}^\Sigma(\xi, h^\xi) + \operatorname{Td}^\Sigma(TX, h'^{TX})\widetilde{\operatorname{ch}}^\Sigma(\xi, h^\xi, h'^\xi)\Big]. \end{aligned} \tag{4.31}$$

In particular, the class of $T(\omega, h^\xi) \in P^{B\cup\Sigma B}/P^{B\cup\Sigma B,0}$ depends only on (h^{TX}, h^ξ).

Proof. By (4.4), and by combining the proof of [33, Theorem 2.13], and Theorem 4.3, we have (4.31). □

5. The Quillen norm in the submersion case

Let $\pi : M \to B$ be a holomorphic orbifold submersion of M onto B with compact fibre X. Let ξ be a holomorphic orbifold vector bundle on M. In this section, we will calculate the Quillen norm of the canonical section of $\lambda_M(\xi) \otimes \lambda^{-1}(R^\bullet\pi_*\xi)$. This extends the result of [2, Theorem 3.1] to the orbifold case.

This section is organized as follows. In Section 5.1, we state a formula for the Quillen norm of the canonical section σ. In Section 5.2, we introduce a 1-form on $\mathbb{R}_+^* \times \mathbb{R}_+^*$ as in [2, §3a)]. In Section 5.3, we state eight intermediate results which we need for the proof of Theorem 5.1, whose proofs are delayed to Sections 5.5–5.8. In Section 5.4, we prove Theorem 5.1. In Section 5.5, we prove Theorems 5.7–5.11. In Section 5.6, we prove Theorem 5.12. In Section 5.7, we prove Theorem 5.13. In Section 5.8, we prove Theorem 5.14.

We use the notation of Sections 1, 4.

5.1. A formula for the Quillen norm of the canonical section σ

Let M, B be compact complex orbifolds. Let $\pi : M \to B$ be a holomorphic orbifold submersion of M onto B with compact fiber X. Let ξ be a holomorphic orbifold vector bundle on M.

We assume that the sheaves $R^k\pi_*\xi (0 \le k \le \dim X)$ are orbifold vector bundles on B. Set

$$\begin{aligned} &\lambda_M(\xi) = \otimes_j (\det H^j(M,\xi))^{(-1)^{j+1}}, \\ &\lambda(R^\bullet \pi_*\xi) = \otimes_{j,k} (\det H^j(B, R^k\pi_*\xi))^{(-1)^{j+k+1}}. \end{aligned} \tag{5.1}$$

By [28], the line $\lambda_M(\xi) \otimes \lambda^{-1}(R^\bullet\pi_*\xi)$ has a canonical nonzero section σ.

Let h^{TM}, h^{TB} be Kähler metrics on TM and TB. Let h^{TX} be the metric induced by h^{TM} on TX. Let h^ξ be a Hermitian metric on ξ.

On M, we have the exact sequence of holomorphic Hermitian proper orbifold vector bundles (cf. Definition 1.2)

$$0 \to TX \to TM \to \pi^*TB \to 0. \tag{5.2}$$

By a construction of [10, §1f)], there is a uniquely defined class of forms

$$\widetilde{\mathrm{Td}}^\Sigma(TM, TB, h^{TM}, h^{TB}) \in P^{M\cup\Sigma M}/P^{M\cup\Sigma M,0},$$

such that

$$\begin{aligned} &\frac{\overline{\partial}\partial}{2i\pi}\widetilde{\mathrm{Td}}^\Sigma(TM, TB, h^{TM}, h^{TB}) \\ &\quad = \mathrm{Td}^\Sigma(TM, h^{TM}) - \pi^*(\mathrm{Td}^\Sigma(TB, h^{TB}))\,\mathrm{Td}^\Sigma(TX, h^{TX}). \end{aligned} \tag{5.3}$$

Let ω^M be the Kähler form of h^{TM}. Let $\|\ \ \|_{\lambda_M(\xi)\otimes\lambda^{-1}(R^\bullet\pi_*\xi)}$ be the Quillen metric on the line $\lambda_M(\xi) \otimes \lambda^{-1}(R^\bullet\pi_*\xi)$ attached to the metrics h^{TM}, h^ξ, h^{TB}, $h^{H(X,\xi|_X)}$ on TM, ξ, TB, $R^\bullet\pi_*\xi$.

Recall that the integral $\int_{B\cup\Sigma B}$ is defined in (4.12).

Now we state the main result of this section, which extends [2, Theorem 3.1].

Theorem 5.1. *The following identity holds,*

$$\begin{aligned} \log\left(\|\sigma\|^2_{\lambda_M(\xi)\otimes\lambda^{-1}(R^\bullet\pi_*\xi)}\right) = &-\int_{B\cup\Sigma B} \mathrm{Td}^\Sigma(TB, h^{TB}) T(\omega^M, h^\xi) \\ &+ \int_{M\cup\Sigma M} \widetilde{\mathrm{Td}}^\Sigma(TM, TB, h^{TM}, h^{TB})\,\mathrm{ch}^\Sigma(\xi, h^\xi). \end{aligned} \tag{5.4}$$

Proof. The remainder of this section is devoted to the proof of Theorem 5.1. □

Remark 5.2. By Theorem 4.8, to prove Theorem 5.1 for any Kähler metrics h^{TM}, h^{TB}, we only need to establish (5.4) for one given metrics h^{TM}, h^{TB}. So by replacing h^{TM} by $h^{TM} + \pi^*h^{TB}$, we may and we will assume that $\widetilde{h}^{TM}$ is a Kähler metric on TM and

$$h^{TM} = \widetilde{h}^{TM} + \pi^*h^{TB}. \tag{5.5}$$

5.2. A fundamental closed 1-form

Let N_V, N_H be the number operators of $\Lambda(T^{*(0,1)}X)$, $\Lambda(T^{*(0,1)}B)$. As in [2, §4], the operators N_V and N_H act naturally on $\Lambda(T^{*(0,1)}M)$. Of course, $N = N_V + N_H$ defines the total grading of $\Lambda(T^{*(0,1)}M) \otimes \xi$ and $\Omega^\bullet(M, \xi)$.

Definition 5.3. For $T > 0$, let h_T^{TM} be the Kähler metric on TM

$$h_T^{TM} = \frac{1}{T^2}\widetilde{h}^{TM} + \pi^* h^{TB}. \tag{5.6}$$

Let $\langle \quad \rangle_T$ be the Hermitian product (1.18) on $\Omega^\bullet(M, \xi)$ attached to the metrics h_T^{TM}, h^ξ. Let D_T^M be the corresponding operator constructed in (1.19) acting on $\Omega^\bullet(M, \xi)$. Let $*_T$ be the Hodge operator with respect to the metric h_T^{TM}. Then $*_T$ acts on $\Lambda(T_{\mathbb{R}}^* M) \otimes \xi$.

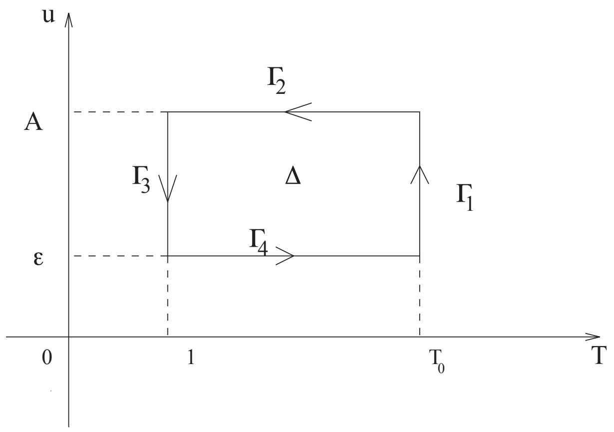
Theorem 5.4. *Let $\alpha_{u,T}$ be the 1-form on $\mathbb{R}_+^* \times \mathbb{R}_+^*$*

$$\alpha_{u,T} = \frac{2du}{u}\operatorname{Tr}_s\left[N \exp(-u^2 D_T^{M,2})\right] + dT \operatorname{Tr}_s\left[*_T^{-1}\frac{\partial *_T}{\partial T}\exp(-u^2 D_T^{M,2})\right]. \tag{5.7}$$

Then $\alpha_{u,T}$ is closed.

Proof. The proof of Theorem 5.4 is identical to the proof of [2, Theorem 4.3 and (4.30)]. □

Take $\epsilon, A, T, 0 < \epsilon \leq 1 \leq A < +\infty$, $1 \leq T_0 < +\infty$. Let $\Gamma = \Gamma_{\epsilon,A,T_0}$ be the oriented contour in $\mathbb{R}_+^* \times \mathbb{R}_+^*$



The contour Γ is made of the four oriented pieces $\Gamma_1, \ldots, \Gamma_4$ indicated above. For $1 \leq k \leq 4$, set

$$I_k^0 = \int_{\Gamma_k} \alpha. \tag{5.8}$$

Theorem 5.5. *The following identity holds,*

$$\sum_{k=1}^{4} I_k^0 = 0. \tag{5.9}$$

Proof. This follows from Theorem 5.4. □

5.3. Eight intermediate results

Let $\overline{\partial}^{B*}$ be the formal adjoint of the operator $\overline{\partial}^{B}$ acting on $\Omega^\bullet(B, R^\bullet\pi_*\xi)$, with respect to the metrics $h^{TB}, h^{H(X,\xi|_X)}$. Set

$$D^B = \overline{\partial}^{B} + \overline{\partial}^{B*}, \quad F = \operatorname{Ker}(D^B). \tag{5.10}$$

By the Hodge theory,

$$H^\bullet(B, R^\bullet\pi_*\xi) \simeq F. \tag{5.11}$$

Let Q be the orthogonal projection from $\Omega^\bullet(B, R^\bullet\pi_*\xi)$ on F with respect to the Hermitian product (1.18) attached to the metrics $h^{TB}, h^{H(X,\xi|_X)}$. Set $Q^\perp = 1 - Q$.

Let $a \in]0, 1]$ be such that the operator $D^{B,2}$ has no eigenvalues in $]0, 2a]$.

Definition 5.6. For $T > 0$, set

$$E_T = \operatorname{Ker}(D_T^{M,2}). \tag{5.12}$$

Let P_T be the orthogonal projection operator from $\Omega^\bullet(M, \xi)$ on E_T with respect to $\langle\ \rangle_T$.

Let $E_T^{[0,a]}$ (resp. $E_T^{]0,a]}$) be the direct sum of the eigenspaces of $D_T^{M,2}$ associated with eigenvalues $\lambda \in [0, a]$ (resp. $\lambda \in]0, a]$). Let $D_T^{M,2,[0,a]}$(resp. $D_T^{M,2,]0,a]}$) be the restriction of $D_T^{M,2}$ to $E_T^{[0,a]}$ (resp. $E_T^{]0,a]}$). Let $P_T^{[0,a]}$ (resp. $P_T^{]0,a]}$) be the orthogonal projection operator from $\Omega^\bullet(M, \xi)$ on $E_T^{[0,a]}$ (resp. $E_T^{]0,a]}$) with respect to $\langle\ \rangle_T$. Set $P^{]a,+\infty[} = 1 - P_T^{[0,a]}$. Set

$$\chi(\xi) = \sum_k (-1)^k \dim H^k(M, \xi), \quad \chi(R^j\pi_*\xi) = \sum_k (-1)^k \dim H^k(B, R^j\pi_*\xi).$$

We now state eight intermediate results contained in Theorems 5.7–5.14 which play an essential role in the proof of Theorem 5.1. The proof of Theorems 5.7–5.14 are deferred to Sections 5.5–5.8.

Theorem 5.7. *For any $u > 0$,*

$$\lim_{T\to+\infty} \operatorname{Tr}_s\left[N \exp(-u^2 D_T^{M,2})\right] = \operatorname{Tr}_s\left[N \exp(-u^2 D^{B,2})\right]. \tag{5.13}$$

For any $u > 0$, there exists $C > 0$ such that for $T \geq 1$,

$$\left|\operatorname{Tr}_s[N_V \exp(-u^2 D_T^{M,2})] - \sum_{j=0}^{\dim X} (-1)^j \chi(R^j\pi_*\xi)\right| \leq \frac{C}{T}. \tag{5.14}$$

For any $\varepsilon > 0$, there exists $C > 0$ such that for $u \geq \varepsilon$, $T \geq 1$,

$$\left| \mathrm{Tr}[\exp(-u^2 D_T^{M,2})] \right| \leq C. \tag{5.15}$$

Theorem 5.8. *For any $u > 0$,*

$$\lim_{T \to +\infty} \mathrm{Tr}_s \left[N \exp(-u^2 D_T^{M,2}) P^{]a,+\infty[} \right] = \mathrm{Tr}_s \left[N \exp(-u^2 D^{B,2}) Q^{\perp} \right]. \tag{5.16}$$

There exist $c > 0, C > 0$ such that for $u \geq 1, T \geq 1$,

$$\left| \mathrm{Tr}[N \exp(-u D_T^{M,2}) P^{]a,+\infty[}] \right| \leq c \exp(-Cu). \tag{5.17}$$

Theorem 5.9. *The following identity holds,*

$$\lim_{T \to +\infty} \mathrm{Tr} \left[D_T^{M,2,[0,a]} \right] = 0. \tag{5.18}$$

For $T \geq 1$ large enough, for $0 \leq k \leq \dim M$,

$$\dim E_T^{[0,a],k} = \sum_{j=0}^{k} \dim H^j(B, R^{k-j}\pi_* \xi). \tag{5.19}$$

Let (E_r, d_r) $(r \geq 2)$ be the spectral sequence of the Dolbeault complex $(\Omega^\bullet(M,\xi), \overline{\partial}^M)$ filtered as in [2, §1a)]. Then as in [2, §4], for $r \geq 2$, E_r is equipped with a metric h^{E_r} associated to h^{TM}, h^{TB}, h^ξ. For $r \geq 2$, let ${}_r|\quad|_{\lambda_M(\xi)}$ be the corresponding metric on $\lambda_M(\xi) \simeq (\det E_r)^{-1}$

For $r \geq 1$, let $N_{|E_r}, N_{H|E_r}, N_{V|E_r}$ be the restrictions of N, N_H, N_V to E_r.

Theorem 5.10. *The following identity holds,*

$$\begin{aligned} \lim_{T \to +\infty} \Big\{ \mathrm{Tr}_s[N \log(D_T^{M,2,]0,a]})] + 2 \sum_{r \geq 2} (r-1) \Big(\mathrm{Tr}_s[N_{|E_r}] - \mathrm{Tr}_s[N_{|E_{r+1}}] \Big) \log(T) \Big\} \\ = \log \left(\frac{{}_\infty|\quad|_{\lambda_M(\xi)}}{{}_2|\quad|_{\lambda_M(\xi)}} \right)^2. \end{aligned} \tag{5.20}$$

For $T \geq 1$, let $|\quad|_{\lambda_M(\xi),T}$ be the L_2 metric on the line $\lambda_M(\xi)$ associated to the metrics h_T^{TM}, h^ξ on TM, ξ.

Theorem 5.11. *The following identity holds,*

$$\begin{aligned} \lim_{T \to +\infty} \Big\{ \log \left(\frac{|\quad|_{\lambda_M(\xi),T}}{|\quad|_{\lambda_M(\xi)}} \right)^2 + 2 \Big(-\dim X \chi(\xi) + \mathrm{Tr}_s[N_{V|E_\infty}] \Big) \log(T) \Big\} \\ = \log \left(\frac{{}_\infty|\quad|_{\lambda_M(\xi)}}{|\quad|_{\lambda_M(\xi)}} \right)^2. \end{aligned} \tag{5.21}$$

For $u > 0$, let B_u be the Bismut superconnection on $\Omega^\bullet(X, \xi|_X)$ constructed in Section 1.3 which is attached to h^{TM}, h^ξ on TM, ξ. Let $\widetilde{N}_u$ be the operator defined in Section 1.3 associated with the metric $\widetilde{h}^{TM}$.

Theorem 5.12. *For any $T \geq 1$,*

$$\begin{aligned}\lim_{\varepsilon\to 0} \operatorname{Tr}_s \Big[*_{T/\varepsilon}^{-1} \frac{\partial}{\partial T}(*_{T/\varepsilon}) \exp(-\varepsilon^2 D_{T/\varepsilon}^{M,2}) \Big] \\ = \frac{2}{T} \int_{B\cup\Sigma B} \operatorname{Td}^\Sigma(TB, h^{TB}) \Phi \operatorname{Tr}_s^\Sigma \Big[\widetilde{N}_{T^2} \exp(-B_{T^2}^2) \Big] - \frac{2}{T} \dim X \chi(\xi).\end{aligned} \tag{5.22}$$

Let $\omega^M, \widetilde{\omega}^M, \omega^B$ be the Kähler forms associated with $h^{TM}, \widetilde{h}^{TM}, h^{TB}$. Let ∇_T^{TM} be the holomorphic Hermitian connection on (TM, h_T^{TM}), and let R_T^{TM} be its curvature.

Theorem 5.13. *There exists $C > 0$ such that for $\varepsilon \in]0,1], \varepsilon \leq T \leq 1$,*

$$\begin{aligned}\Big| \operatorname{Tr}_s \Big[*_{T/\varepsilon}^{-1} \frac{\partial}{\partial T}(*_{T/\varepsilon}) \exp\Big(- \varepsilon^2 D_{T/\varepsilon}^{M,2} \Big) \Big] \\ - \frac{2}{T^3} \int_{M\cup\Sigma M} \frac{\widetilde{\omega}^M}{2\pi} \operatorname{Td}^\Sigma(TM) \operatorname{ch}^\Sigma(\xi) \\ + \int_{M\cup\Sigma M} \frac{\partial}{\partial b} \operatorname{Td}^\Sigma \Big(\frac{-R_{T/\varepsilon}^{TM}}{2i\pi} - b(h_{T/\varepsilon}^{TM})^{-1} \frac{\partial}{\partial T}(h_{T/\varepsilon}^{TM}) \Big)_{b=0} \operatorname{ch}^\Sigma(\xi, h^\xi) \Big| \leq C.\end{aligned} \tag{5.23}$$

Theorem 5.14. *There exist $\delta \in]0,1], C > 0$ such that for $\varepsilon \in]0,1], T \geq 1$,*

$$\begin{aligned}\Big| \operatorname{Tr}_s \Big[*_{T/\varepsilon}^{-1} \frac{\partial}{\partial T}(*_{T/\varepsilon}) \exp(-\varepsilon^2 D_{T/\varepsilon}^{M,2}) \Big] \\ - \frac{2}{T} \Big(\sum_{j=0}^{\dim X} (-1)^j j \chi(R^j \pi_* \xi) - \dim X \chi(\xi) \Big) \Big| \leq \frac{C}{T^{1+\delta}}.\end{aligned} \tag{5.24}$$

Besides, at a formal level, Theorems 5.7–5.14 can be obtained formally from [2, Theorems 4.8–4.15]. This will permit us to transfer formally the discussion in [2, §4] to our situation.

5.4. A proof of Theorem 5.1

By Theorem 5.5, Theorems 5.7–5.14 and proceeding as in [2, §4c), d)], we get (5.4).

5.5. A proof of Theorems 5.7–5.11

The proof of Theorems 5.7–5.11 is essentially the same as the proof of [2, Theorems 4.8–4.12] given in [2, §5, §6], where the corresponding results were established when M, B are manifolds. Now we use the notation of [2, §5].

By Proposition 1.4, for each $b \in B$, there exists a small neighbourhood $(G_b, \widetilde{V}_b) \to V_b$, an orbifold $\widetilde{M}_b$, such that π is induced by a G_b-equivariant orbifold submersion $\widetilde{\pi}_b : \widetilde{M}_b \to \widetilde{V}_b$ with compact fiber X.

Then $\operatorname{Ker}(D_T^X)$ is a G_b-equivariant vector bundle on $\widetilde{V}_b$. This defines an orbifold Hermitian vector bundle $\operatorname{Ker}(D_T^X)$ on B.

For $T \in [1, +\infty]$, let $E_{1,T}$ be the vector space of the smooth sections on B of $\operatorname{Ker}(D_T^X)$. As in [2, (5.26)], we have

$$E_{1,T} \simeq E_1. \tag{5.25}$$

The proof of Theorems 5.7–5.11 then proceeds as in [2, §5, §6] by using (3.15).

5.6. A proof of Theorem 5.12

Now we use the notation of [33, §7].

By Proposition 1.4, for each $b \in B$, there exists a small neighbourhood $(G_b, \widetilde{V}_b) \to V_b$ ($\widetilde{V}_b$ is a neighbouhood of $0 \in \mathbb{C}^m$ and G_b acts linearly on $\mathbb{C}^m$), an orbifold $\widetilde{M}_b$, such that π is induced by a G_b-equivariant orbifold submersion $\widetilde{\pi}_b : \widetilde{M}_b \to \widetilde{V}_b$ with compact fibre X.

Let $(G_{b_i}, \widetilde{V}_{b_i})_{i\in I}$ be a cover of B such that $(G_{b_i}, \frac{1}{2}\widetilde{V}_{b_i})_{i\in I}$ also is a cover of B. Let $\beta = \inf_{i\in I}\{\text{injectivity radius of } b_i \text{ on } \widetilde{V}_{b_i}\}$. Let $\alpha \in]0, \beta/8]$.

If $b \in B$, let $B^B(b, r)$ be the open ball of center b and radius r in B.

Proposition 5.15. *For $\delta > 0$, there exist $c > 0$, $C > 0$ such that for $0 < \varepsilon \le \delta$, $T \ge 1$,*

$$\Big|\operatorname{Tr}_s \Big[*_T^{-1} \frac{\partial}{\partial T}(*_T) G_{\frac{\varepsilon}{T}}\big(\frac{\varepsilon}{T} D_T^M\big)\Big]\Big| \le c \exp\Big(-\frac{CT^2}{\varepsilon^2}\Big). \tag{5.26}$$

Proof. The proof of (5.26) is essentially the same as the proof of [2, Proposition 8.3]. □

For $T \ge 1$ fixed, we use (5.26) with $\varepsilon = T$ and T replace by T/ε, we find

$$\Big|\operatorname{Tr}_s \Big[*_{T/\varepsilon}^{-1} \frac{\partial}{\partial T}(*_{T/\varepsilon}) G_{\varepsilon}(\varepsilon D_{T/\varepsilon}^M)\Big]\Big| \le c \exp\Big(-\frac{C}{\varepsilon^2}\Big). \tag{5.27}$$

Set

$$A'_{\varepsilon,T} = \Big(\frac{T}{\varepsilon}\Big)^{N_V} \varepsilon D_{T/\varepsilon}^M \Big(\frac{T}{\varepsilon}\Big)^{-N_V}. \tag{5.28}$$

Let $F_\varepsilon(\varepsilon D_{T/\varepsilon}^M)(x, x')$, $F_\varepsilon(A'_{\varepsilon,T})(x, x')(x, x' \in M)$ be the smooth kernel associated with $F_\varepsilon(\varepsilon D_{T/\varepsilon}^M)$, $F_\varepsilon(A'_{\varepsilon,T})$ with respect to the volume form $\frac{dv_M(x')}{(2\pi)^{\dim M}}$. Using (2.11), (4.9) and finite propagation speed [20, §7.8], [35, Appendix D. 2], it is clear that for $\varepsilon \in]0, 1]$, $T \ge 1$, $x, x' \in M$, if $d^B(\pi(x), \pi(x')) \ge \alpha$, then

$$F_\varepsilon(\varepsilon D_{T/\varepsilon}^M)(x, x') = 0$$

and moreover, given $x \in M$, $F_\varepsilon(\varepsilon D_{T/\varepsilon}^M)(x, \cdot)$ only depends on the restriction of $D_{T/\varepsilon}^M$ to $\pi^{-1}(B^B(\pi(x), \alpha))$.

Let ρ_i be a partition of unity subordinate to the cover $(G_{b_i}, \frac{1}{2}\widetilde{V}_{b_i})_{i\in I}$ of B.
Then by (5.28), we get as in [2, (7.8)]

$$\operatorname{Tr}_s \Big[*_{T/\varepsilon}^{-1} \frac{\partial}{\partial T}(*_{T/\varepsilon}) F_\varepsilon(\varepsilon D_{T/\varepsilon}^M)\Big] = \operatorname{Tr}_s \Big[*_{T/\varepsilon}^{-1} \frac{\partial}{\partial T}(*_{T/\varepsilon}) F_\varepsilon(A'_{\varepsilon,T})\Big]. \tag{5.29}$$

We replace $\widetilde{M}_{b_i}$ by $(\widetilde{TB})_{b_i} \times X_{b_i} = \mathbb{C}^m \times X_{b_i}$ and trivialize the vector bundles as indicated in [33, §7b)].

As in [2, §9b)], for $\alpha > 0$ small enough, there is also a smooth $\mathbb{Z}$-graded vector bundle $K \subset \Omega_{b_i}$ over $(\widetilde{TB})_{b_i} \simeq \mathbb{R}^{2m}$ which coincides with $\operatorname{Ker}(D^X)$ on $B(0, 4\alpha)$, with $\operatorname{Ker}(D^X_{b_i})$ over $(\widetilde{TB})_{b_i}\backslash B(0,6\alpha)$ and such that if $K^\perp$ is the orthogonal bundle to K in Ω_{b_i},

$$K^\perp \cap \operatorname{Ker}(D^X_{b_i}) = \{0\}. \tag{5.30}$$

Let P_b be the orthogonal projection operator from Ω_{b_i} on K_b. Set $P_b^\perp = 1 - P_b$.

Let Δ^{TB} be the standard Laplacian on the vector space $(\widetilde{TB})_{b_i}$ with respect to the metric $h^{TB}_{b_i}$. Let $dv_{T_{b_i}B}$ be the Riemannian volume form on $((\widetilde{TB})_{b_i}, h^{TB}_{b_i})$.

Let $\varphi : \mathbb{R} \to [0,1]$ be a $\mathscr{C}^\infty$ function which is equal 1 if $|t| \le 2\alpha$, equal 0 if $|t| \ge 4\alpha$. Let $L^1_{\varepsilon,T}$ be the operator on $\mathbb{C}^m \times X_{b_0}$

$$L^1_{\varepsilon,T} = \varphi^2(|Y|)A'^2_{\varepsilon,T} + (1-\varphi^2(|Y|))\Big(\frac{-\varepsilon^2\Delta^{TB}}{2} + T^2 P^\perp_Y D^{X,2}_{b_i} P^\perp_Y\Big). \tag{5.31}$$

Let $\widetilde{F}_\varepsilon(L^1_{\varepsilon,T})\Big((Y,x),(Y',x')\Big)$ $((Y,x),(Y',x') \in (\widetilde{TB})_{b_i} \times X_{b_0})$ be the smooth kernels associated with $\widetilde{F}_\varepsilon(L^1_{\varepsilon,T})$ with respect to $\frac{dv_{T_{b_i}B}(Y')dv_{X_{b_i}}(x')}{(2\pi)^{\dim M}}$.

For $(Y,x) \in (\widetilde{TB})_{b_i} \times X_{b_i}$, $|Y| < \beta/4$, set

$$dv_M(Y,x) = k(Y,x) dv_{T_{b_i}B} dv_{X_{b_i}}. \tag{5.32}$$

Using finite propagation speed and (2.11), we see that if $(Y,x) \in (\widetilde{TB})_{b_i} \times X_{b_i}$, $|Y| < \alpha$, then

$$F_\varepsilon(A'_{\varepsilon,T})\Big((Y,x),(Y,x)\Big) = \sum_{h \in G_{b_i}} k(Y,x) h \widetilde{F}_\varepsilon(L^1_{\varepsilon,T})\Big(h^{-1}(Y,x),(Y,x)\Big). \tag{5.33}$$

By (5.33), and proceeding as in [33, §7], we have Theorem 5.12.

5.7. A proof of Theorem 5.13

As in [2, §8] or [33, §8], the following theorem implies Theorem 5.13.

Theorem 5.16. *There exists $C > 0$ such that for $0 < u \le 1, T \ge 1$,*

$$\begin{aligned}\Big| \operatorname{Tr}_s \Big[*_T^{-1} \frac{\partial *_T}{\partial T} \exp(-\frac{u^2}{T^2} D^{M,2}_T)\Big] - \frac{2}{u^2}\int_{M \cup \Sigma M} \frac{\widetilde{\omega}^M}{2\pi T} \operatorname{Td}^\Sigma(TM) \operatorname{ch}^\Sigma(\xi) \qquad (5.34)\\ + \int_{M\cup\Sigma M} \frac{\partial}{\partial b} \operatorname{Td}^\Sigma \Big(\frac{-R^{TM}_T}{2i\pi} - b (h^{TM}_T)^{-1} \frac{\partial}{\partial T}(h^{TM}_T)\Big)_{b=0} \operatorname{ch}^\Sigma(\xi, h^\xi) \Big| \le \frac{Cu^2}{T}.\end{aligned}$$

Proof. By (5.28)

$$A'_{1/T,1} = T^{N_V} \frac{1}{T} D^M_T T^{-N_V}. \tag{5.35}$$

Therefore

$$\operatorname{Tr}_s \Big[*_T^{-1} \frac{\partial *_T}{\partial T} \exp(-\frac{u^2}{T^2} D^{M,2}_T)\Big] = \operatorname{Tr}_s \Big[*_T^{-1} \frac{\partial *_T}{\partial T} \exp(-u^2 A'^2_{1/T,1})\Big]. \tag{5.36}$$

By (3.15), we can replace M by $(\mathbb{C}^m \times X_{b_i})/K_{b_i}$, and trivialize the vector bundles as indicated in [33, §7b)]. Then we will prove (5.34) in this situation.

Let $P_{u,T}(x,x')$ be the smooth kernel associated with the operator $\exp(-u^2A'^2_{1/T,1})$ with respect to $\frac{dv_M(x')}{(2\pi)^{\dim M}}$. Let

$$P^1_{\varepsilon,T,u}((Y,x),(Y',x'))\ ((Y,x),(Y',x')\in(\widetilde{TB})_{b_i}\times X_{b_i})$$

be the smooth kernel associated with the operator $\exp(-u^2L^1_{\varepsilon,T})$ with respect to $\frac{dv_{T_{b_i}B}(Y')\times dv_{X_{b_i}}(x')}{(2\pi)^{\dim M}}$. By Proposition 5.15, as $u\to 0$, uniformly on $T\geq 1$, the asymptotics of the following three terms is the same

$$\begin{aligned}
&\int_M \rho_i \operatorname{Tr}_s\Big[*_T^{-1}\frac{\partial *_T}{\partial T}F_{u/T}(uA'_{1/T,1})(x,x')\Big]dv_M/(2\pi)^{\dim M},\\
&\quad\int_M \rho_i \operatorname{Tr}_s\Big[*_T^{-1}\frac{\partial *_T}{\partial T}P_{u,T}(x,x')\Big]dv_M/(2\pi)^{\dim M},\\
&\int_{(\widetilde{TB})_{b_i}\times X_{b_i}}\rho_i\sum_{h\in G_{b_i}}\frac{1}{|G_{b_i}|}\operatorname{Tr}_s\Big[h *_T^{-1}\frac{\partial *_T}{\partial T}P^1_{1/T,1,u}(h^{-1}(Y,x),(Y,x))\Big]\\
&\qquad\qquad k(Y,x)dv_{T_{b_i}B}(Y)\times dv_{X_{b_i}}(x)/(2\pi)^{\dim M}.
\end{aligned}\tag{5.37}$$

By [33, §8], (5.36), (5.37), we get Theorem 5.16. □

5.8. A proof of Theorem 5.14

Proposition 5.17. *There exists $C>0$, such that for $0<\varepsilon\leq 1$, $T\geq 1$*

$$\begin{aligned}
\Big|&\operatorname{Tr}_s\Big[*_{T/\varepsilon}^{-1}\frac{\partial}{\partial T}(*_{T/\varepsilon})G_\varepsilon(\varepsilon D^M_{T/\varepsilon})\Big]\\
&-\frac{2}{T}\Big(\sum_{j=0}^{\dim X}(-1)^j j\chi(R^j\pi_*\xi)-\dim X\chi(\xi)\Big)G_\varepsilon(0)\Big|\leq\frac{C}{T^2}.
\end{aligned}\tag{5.38}$$

Proof. By an analogue of the McKean–Singer formula [1, Theorem 3.50], we find that

$$\operatorname{Tr}_s[N_V G_\varepsilon(\varepsilon D^B)]=\sum_{j=0}^{\dim X}(-1)^j j\chi(R^j\pi_*\xi)G_\varepsilon(0).\tag{5.39}$$

Using (5.39) and proceeding as in [2, Proposition 9.1], we have (5.38). □

By (4.10) and (5.38), to establish Theorem 5.14, we only need to establish the following result,

Theorem 5.18. *If $\alpha>0$ is small enough, there exist $\delta>0, C>0$, such that for $0<\varepsilon\leq 1$, $T\geq 1$*

$$\begin{aligned}
\Big|&\operatorname{Tr}_s\Big[*_{T/\varepsilon}^{-1}\frac{\partial}{\partial T}(*_{T/\varepsilon})F_\varepsilon(\varepsilon D^M_{T/\varepsilon})\Big]\\
&-\frac{2}{T}\Big(\sum_{j=0}^{\dim X}(-1)^j j\chi(R^j\pi_*\xi)-\dim X\chi(\xi)\Big)F_\varepsilon(0)\Big|\leq\frac{C}{T^{1+\delta}}.
\end{aligned}\tag{5.40}$$

Proof. Using (5.28), we deduce that

$$\mathrm{Tr}_s\left[*_{T/\varepsilon}^{-1}\frac{\partial}{\partial T}(*_{T/\varepsilon})F_\varepsilon(\varepsilon D^M_{T/\varepsilon})\right]=\mathrm{Tr}_s\left[*_{T/\varepsilon}^{-1}\frac{\partial}{\partial T}(*_{T/\varepsilon})\widetilde{F}_\varepsilon(A'^2_{\varepsilon,T})\right]. \tag{5.41}$$

Let $\widetilde{F}_\varepsilon(A'^2_{\varepsilon,T})(x,x')(x,x'\in M)$ be the smooth kernel associated with $\widetilde{F}_\varepsilon(A'^2_{\varepsilon,T})$ with respect to $dv_M(x')/(2\pi)^{\dim M}$. Using finite propagation speed, it is clear that if $x\in M$, $\widetilde{F}_\varepsilon(A'^2_{\varepsilon,T})(x,\cdot)$ only depends on the restriction of $A'_{\varepsilon,T}$ to $\pi^{-1}(B^B(\pi(x),\alpha))$.

We use the same trivialization and notation as in Section 5.6. If $(Y,x)\in(\widetilde{TB})_{b_i}\times X_{b_i}$, $|Y|<\alpha$, then

$$\begin{aligned}&\rho_i(Y,x)\widetilde{F}_\varepsilon(A'^2_{\varepsilon,T})((Y,x),(Y,x))\\&\quad=\rho_i\sum_{h\in G_{b_i}}k(Y,x)h\widetilde{F}_\varepsilon(L^1_{\varepsilon,T})(h^{-1}(Y,x),(Y,x)).\end{aligned}\tag{5.42}$$

By [33, §9], (5.41), (5.42), we get Theorem 5.18. □

The proof of Theorem 5.14 is completed. □

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