GAGA TYPE RESULTS FOR SINGULARITY CATEGORIES

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Abstract. Several GAGA type results for singularity categories are presented. Firstly, as an easy consequence of Serre's GAGA theorem, for a complex projective variety, its singularity category is naturally equivalent to that of its analytificaton. Secondly, the torsion singularity category of a formal scheme is introduced, and under Orlov's condition (ELF), for the formal completion of a Noetherian scheme along a closed subset, its torsion singularity category is shown to be equivalent to the singularity of the original scheme with support in the closed subset. Lastly, using Artin Approximation Theorem and the previous result, an alternative result of a result of Orlov is provided, which says that for a finitely generated local algebra with isolated singularity, its singularity category is equivalent, up to direct summands, to the singularity category of its Henselization, while the latter is equivalent to the singularity category of its completion.

1. Introduction

Let X be a Noetherian scheme over a field k. Let $\mathcal{D}(\mathrm{Qcoh}X)$ be the unbounded derived category of quasi-coherent sheaves on X and $\mathcal{D}^b(\mathrm{coh}X)$ the bounded derived category of coherent sheaves on X. Since X is Noetherian, $\mathcal{D}^b(\mathrm{coh}X)$ can be viewed as the full subcategory $\mathcal{D}^{\emptyset,b}_{\mathrm{coh}}(\mathrm{Qcoh}X)$ consisting of all cohomologically bounded complexes with coherent cohomologies [3, Ex. II, 2.2.2].

We say that a complex on X is *perfect* if it is locally quasi-isomorphic to a bounded complex of locally free sheaves of finite rank. We denote by $\mathrm{per}(X) \subseteq \mathcal{D}(\mathrm{Qcoh}X)$ the full triangulated subcategory of perfect complexes. By a result of Neeman [9], the unbounded derived category $\mathcal{D}(\mathrm{Qcoh}X)$ admits all coproducts and the subcategory of perfect complexes $\mathrm{per}(X)$ coincides with subcategory of compact objects in $\mathcal{D}(\mathrm{Qcoh}X)$. The category $\mathrm{per}(X)$ can be considered as a full triangulated subcategory of $\mathcal{D}^b(\mathrm{coh}X)$.

Definition 1.1. [11] The triangulated category of singularities of X, denoted by $\mathcal{D}_{sg}(X)$, is defined as the quotient of the triangulated category $\mathcal{D}^b(\mathrm{coh}X)$ by the full triangulated subcategory $\mathrm{per}(X)$.

Remark 1.2. The algebraic version of singularity category was first introduced by Buchweitz in [4].

The main goal of this paper is to give several GAGA type results for singularity categories. The famous GAGA theorem [12] gives a comparison between the category of projective $\mathbb{C}$-schemes of finite type and the category of complex analytic geometry. For example, one can compute sheaf cohomology either in the Zariski topology or in the analytic topology, obtaining the same result. As an easy consequence of GAGA, we show that the singularity category is invariant under the analytification functor. More precisely, let X be a scheme of finite type over $\mathbb{C}$ and X^{an} its analytification. Then their singularity categories are equivalent, see Corollary 2.4.

For any closed subscheme $Z \subseteq X$ one can associate a formal scheme $\mathfrak{X}_Z$, i.e. the formal completion of X along Z. In [11], Orlov proved that any two schemes X and X' satisfying (ELF), if the formal completions X and X' along singularities are isomorphic, then the idempotent completions of the triangulated categories of singularities $\mathcal{D}_{sg}(X)$ and $\mathcal{D}_{sg}(X')$ are equivalent. We say that a scheme X satisfies condition (ELF) if X is separated Noetherian of finite Krull dimension and has enough locally free sheaves, i.e. for any coherent sheaf $\mathcal{F}$ there is an epimorphism $\mathcal{E} \twoheadrightarrow \mathcal{F}$ with a locally free sheaf $\mathcal{E}$.

Let X be a scheme which satisfies (ELF) condition and Z a closed subset. Let $\kappa : \mathfrak{X}_Z \to X$ be the completion of X along Z. In Section 3, we give the definition of torsion singularity category of $\mathfrak{X}_Z$, see Definition 3.7. It turns out that the torsion singularity category is equivalent to the singularity of the original scheme with support in Z, see Corollary 3.8.

In the last section, we use the notation of torsion singularity category to give a proof of [11, Remark 3.6]. Let $(A, \mathfrak{m})$ a finitely generated local algebra over a filed k. Let $(A_h, \mathfrak{m}_h)$ be the henselization of

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$(A, \mathfrak{m})$ and $\hat{A}$ the $\mathfrak{m}$-adic completion of A. The natural inclusion $A \hookrightarrow A_h \hookrightarrow \hat{A}$ induces the following equivalences of triangulated categories (see Theorem 4.5)

$$\mathcal{D}_{sg}(A) \xrightarrow{\sim d.s.} \mathcal{D}_{sg}(A_h) \xrightarrow{\simeq} \mathcal{D}_{sg}(\hat{A}),$$

where the notation $\xrightarrow{\sim d.s}$ means an equivalence up to direct summands.

2. Serre's GAGA for singularity categories

In this section, we prove that the singularity category is invariant under the analytification functor. More precisely, let X be be a scheme of finite type over $\mathbb{C}$ and X^{an} its analytification. Then their singularity categories are equivalent. For the detail of analytification of a complex algebraic variety, we refer the reader to [13] and [10, Chapter 5].

Let $(X, \mathcal{O}_X)$ be a scheme of finite type over $\mathbb{C}$. Let X^{an} be the closed points of X. There is a sheaf of ring $\mathcal{O}^{an}$ on X^{an}, and a map of ringed spaces over $\mathbb{C}$

$$h : (X^{an}, \mathcal{O}_X^{an}) \to (X, \mathcal{O}_X).$$

The ringed space $(X^{an}, \mathcal{O}_X^{an})$ is an analytic space and is called the analytification of the scheme $(X, \mathcal{O}_X)$. We get a functor $(\ )^{an}$ from the category of schemes of finite type over $\mathbb{C}$ to analytic spaces.

Definition 2.1. If $\mathcal{M}$ is any sheaf of $\mathcal{O}_X$-module on X, the corresponding analytic sheaf $\mathcal{M}^h$ on X^{an} is defined as

$$\mathcal{M}^h := \mathcal{M} \otimes_{h^{-1}\mathcal{O}_X} h^{-1}\mathcal{M}.$$

Theorem 2.2. [13, Theorem 13.5.8] *Let X be a projective algebraic variety over $\mathbb{C}$. Then the functor $M \mapsto M^h$ is a cohomology preserving equivalence of categories from the category of coherent algebraic sheaves on X to the category of coherent analytic sheaves on X^h.*

Proposition 2.3. [13, Corollary 13.6.6] *Let X be a projective variety over $\mathbb{C}$. Then the above equivalence induces an equivalence from the category of algebraic vector bundles on X to the category of holomorphic vector bundles on X^h.*

Corollary 2.4. *Let X be a projective variety over $\mathbb{C}$. Then we have equivalences $\mathcal{D}^b(\mathrm{coh}X) \xrightarrow{\simeq} \mathcal{D}^b(\mathrm{coh}X^{an})$ and $\mathrm{per}X \xrightarrow{\simeq} \mathrm{per}X^{an}$. Hence we have the equivalence between their singularity categories, i.e. $\mathcal{D}_{sg}(X) \simeq \mathcal{D}_{sg}(X^{an})$.*

Proof. By the above theorem and proposition, we have the following commutative diagram of categories

$$\begin{array}{ccc} \mathrm{coh}X & \xrightarrow{\simeq} & \mathrm{coh}X^{an} \\ \uparrow & & \uparrow \\ \mathrm{Vec}X & \xrightarrow{\simeq} & \mathrm{Vec}X^{an}, \end{array}$$

where the horizontal arrows are equivalences. It follows that $\mathcal{D}^b(X) \xrightarrow{\simeq} \mathcal{D}^b(X^{an})$ and $\mathrm{per}(X) \xrightarrow{\simeq} \mathrm{per}(X^{an})$. Thus, we have an equivalence $\mathcal{D}_{sg}(X) \xrightarrow{\simeq} \mathcal{D}_{sg}(X^{an})$.

$\surd$

3. Torsion singularity categories for formal schemes

Let X be a scheme which satisfies consdition (ELF). We denote by $\mathrm{Qcoh}X$ the abelian category of quasi-coherent sheaves on X and $\mathrm{coh}X$ the abelian subcategory of coherent sheaves. Let $i : Z \to X$ be a closed subspace of X.

Consider the abelian subcategory $\mathrm{Qcoh}_Z X$ of quasi-coherent sheaves on X with support on Z, i.e. all quasi-coherent sheaves $\mathcal{F}$ such that $j^*\mathcal{F} = 0$, where $j : U \to X$ is the open embedding of the complement $U = X \setminus Z$. Let $i : \mathrm{Qcoh}_Z X \to \mathrm{Qcoh}X$ be the inclusion functor. It admits a right adjoint Γ_Z which is described as follows: For any quasi-coherent sheaf $\mathcal{F}$ on X, let

$$\Gamma_Z(\mathcal{F}) = \varinjlim_n \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X/\mathcal{J}^n, \mathcal{F}),$$

where $\mathcal{J}$ is a some ideal sheaf such that $Z = \mathrm{Supp}(\mathcal{O}_X/\mathcal{J})$. We say that $\mathcal{F} \in \mathrm{Qcoh}X$ is a *Z-torsion sheaf* if $\Gamma_Z(\mathcal{F}) = \mathcal{F}$.

Lemma 3.1. [11, Lemma 2.1] *Let X be a Noetherian scheme Z a closed subspace. Then the natural functor $\mathcal{D}^b(\mathrm{coh}_Z X) \to \mathcal{D}^b(\mathrm{coh}X)$ is fully faithful and gives an equivalence with the full subcategory $\mathcal{D}^b_Z(\mathrm{coh}X) \subseteq \mathcal{D}^b(\mathrm{coh}X)$ consisting of all complexes whose cohomology is supported on Z.*

Denote by $\mathrm{per}_Z(X)$ the intersection $\mathrm{per}(X) \cap \mathcal{D}^b_Z(\mathrm{coh}X)$.

Lemma 3.2. [11, Lemma 2.4] *Assume that X satisfies (ELF). Then an object $M \in \mathcal{D}^b_Z(\mathrm{coh}X)$ belongs to $\mathrm{per}_Z(X)$ iff for any object $N \in \mathcal{D}^b_Z(\mathrm{coh}X)$ all $\mathrm{Hom}(M, N[i])$ are trivial except for finite number of $i \in \mathbb{Z}$.*

Definition 3.3. [11, pp.210] The *singularity category support on Z* is defined as the following Verdier quotient

$$\mathcal{D}_{sg,Z}(X) := \mathcal{D}^b_Z(X)/\mathrm{per}_Z(X).$$

3.1. Formal completions along a closed subscheme.

For any closed subscheme $Z \subseteq X$, we define the formal completion $(\mathfrak{X}_Z, \widehat{\mathcal{O}_X})$ of X along Z as a ringed space $(Z, \varprojlim \mathcal{O}_X/\mathcal{J}^n)$, where $\mathcal{J}$ is the ideal sheaf corresponding to Z. The formal completion depends only on the closed subset $\mathrm{Supp}Z$ and does not depend on a scheme structure on Z. We denote by $\mathfrak{X}$ the formal completion of X along its singularities $\mathrm{Sing}(X)$.

The canonical morphism $\mathcal{O}_X \to \varprojlim \mathcal{O}_X/\mathcal{J}$ induces a canonical morphism of ringed spaces

$$\kappa : (\mathfrak{X}_Z, \widehat{\mathcal{O}_X}) \to (X, \mathcal{O}_X).$$

Let $\mathfrak{I}$ be a corresponding ideal of definition of the formal Noetherian scheme $\mathfrak{X}_Z$. For any quasi-coherent sheaf $\mathfrak{F}$ on $\mathfrak{X}_Z$, we set

$$\varGamma_{\mathfrak{X}}(\mathfrak{F}) = \varinjlim_n \mathcal{H}om_{\mathfrak{X}_Z}(\mathcal{O}_{\mathfrak{X}_Z}/\mathfrak{I}^n, \mathfrak{F}).$$

We say that $\mathfrak{F} \in \mathrm{Qcoh}\mathfrak{X}_Z$ is a *$\mathfrak{X}_Z$-torsion sheaf* if $\varGamma_{\mathfrak{X}}(\mathfrak{F}) = \mathfrak{F}$. Denote by $\mathrm{coh}_t\mathfrak{X}_Z$ (resp. $\mathrm{Qcoh}_t\mathfrak{X}_Z$) the full subcategory of $\mathrm{Qcoh}\mathfrak{X}_Z$ whose objects are the (quasi)-coherent torsion sheaves. Let $\mathcal{D}^b_t(\mathrm{coh}\mathfrak{X}_Z)$ be the full subcategory of $\mathcal{D}^b(\mathrm{coh}\mathfrak{X}_Z)$ consisting of all complexes whose cohomologies are $\mathfrak{X}_Z$-torsion sheaves.

Proposition 3.4. [14, Proposition 5.2.4] *Let Z be a closed subset of a locally Noetherian scheme X, and let $\kappa : \mathfrak{X} \to X$ be the completion of X along Z. Then the exact functors κ^* and κ_* restrict to inverse isomorphisms between the categories $\mathcal{D}_Z(X)$ and $\mathcal{D}_t(\mathfrak{X}_Z)$, and between the categories $\mathcal{D}_{qc,Z}(X)$ and $\mathcal{D}_{qc,t}(\mathfrak{X})$; and if $M \in \mathcal{D}_{qc,t}(\mathfrak{X})$ has coherent homology, then so does $\kappa_* M$.*

Let $\mathfrak{X}$ be a formal scheme. We say that an object $\mathfrak{J}$ of $\mathcal{D}(\mathrm{Qcoh}\mathfrak{X})$ is called *perfect* if for every $x \in \mathfrak{X}$ there is an open neighborhood $\mathfrak{U}$ of x and a bounded complex of locally-free finite type $\mathcal{O}_{\mathfrak{U}}$-Modules $\mathcal{F}$ together with an isomorphism $\mathcal{F} \xrightarrow{\sim} \mathcal{E}^{\cdot}|_{\mathfrak{U}}$ in $\mathcal{D}(\mathrm{Qcoh}\mathfrak{U})$(cf. [8, Définition 4.7]). The full subcategory of perfect complex on $\mathfrak{X}$ is denoted by $\mathrm{per}(\mathfrak{X})$.

Denote by $\mathrm{per}_t(\mathfrak{X}_Z)$ the intersection $\mathrm{per}(\mathfrak{X}) \cap \mathcal{D}^b_t(\mathrm{coh}\mathfrak{X}_Z)$. By combining Proposition 3.4 and Lemma 3.2, we have the following Corollary.

Corollary 3.5. An object $\mathfrak{J} \in \mathcal{D}^b_t(\mathfrak{X}_Z)$ is perfect if and only if for any object $\mathfrak{K} \in \mathcal{D}^b_t(\mathfrak{X}_Z)$ all $\mathrm{Hom}(\mathfrak{J}, \mathfrak{K}[i]) = 0$ are trivial except for finite number of $i \in \mathbb{Z}$.

Proposition 3.6. *The image of $\mathrm{per}_Z X$ under the equivalence $\kappa_* : \mathcal{D}_Z(X) \to \mathcal{D}_t(\mathfrak{X}_Z)$ is $\mathrm{per}_t\mathfrak{X}_Z$.*

Proof. It follows from Lemma 3.2 and Corollary 3.5. $\surd$

3.2. Torsion singularity categories.

Definition 3.7. Let X be a scheme and Z a closed subset of X. Let $\mathfrak{X}_Z$ be the formal completion along the closed subset $\mathrm{Supp}Z$. We define *torsion singularity category* $\mathcal{D}_{sg,t}(\mathfrak{X}_Z)$ as the Verdier quotient

$$\mathcal{D}^b_t(\mathrm{coh}\mathfrak{X}_Z)/\mathrm{per}_t(\mathfrak{X}_Z).$$

Combining Proposition 3.4 and 3.6, we obtain the following Corollary.

Corollary 3.8. *Let X be a scheme which satisfies ELF condition and Z a closed subset and let $\kappa : \mathfrak{X} \to X$ be the completion of X along Z. We have an equivalence of triangulated categories*

$$\kappa^* : \mathcal{D}_{sg,Z}(X) \xrightarrow{\simeq} \mathcal{D}_{sg,t}(\mathfrak{X}_Z).$$

Theorem 3.9. [5] *Let X be a scheme which satisfies ELF condition and Z a closed subset. We denote by $U = X - Z$ the corresponding open subscheme. Then we have the following short exact sequence (up to direct summands)*

$$0 \to \mathcal{D}_{sg,Z}(X) \to \mathcal{D}_{sg}(X) \to \mathcal{D}_{sg}(U) \to 0.$$

By using the language of torsion singularity categories, we obtain the following Corollary.

Corollary 3.10. *Let X be a scheme which satisfies ELF condition and Z a closed subset and let $\kappa : \mathfrak{X} \to X$ be the completion of X along Z. We denote by $U = X - Z$ the corresponding open subscheme. Then we have the following short exact sequence (up to direct summands)*

$$0 \to \mathcal{D}_{sg,t}(\mathfrak{X}_Z) \to \mathcal{D}_{sg}(X) \to \mathcal{D}_{sg}(U) \to 0.$$

Recall that a triangulated category $\mathcal{T}$ is said to be *idempotent complete* if any idempotent $e : M \to M$, $e^2 = e$, arises from a splitting of M,

$$M = \ker(e) \oplus \mathrm{Im}(e).$$

Let $\mathcal{T}$ be a triangulated category. Balmer–Schlichting [2] showed that the idempotent completion $\overline{\mathcal{T}}$ of $\mathcal{T}$ has a natural structure of a triangulated category. Moreover, $\overline{\mathcal{T}}$ is idempotent complete.

Proposition 3.11. [11, Proposition 2.7] *Let Z be the set of singularities of X, i.e. $Z = \mathrm{Sing}(X)$. Any object of $\mathcal{D}_{sg}(X)$ is a direct summand of an object in its full subcategory $\mathcal{D}_{sg,Z}(X)$. In particular, the idempotent completions of these categories are equivalent, i.e. $\overline{\mathcal{D}_{sg}(X)} \simeq \overline{\mathcal{D}_{sg,Z}(X)}$.*

Theorem 3.12. [11, Theorem 2.10] *Let X and X' be two schemes satisfying (ELF). Assume that the formal completions $\mathfrak{X}$ and $\mathfrak{X}'$ along singularities are isomorphic. Then the idempotent completions of the triangulated categories of singularities $\overline{\mathcal{D}_{sg}(X)}$ and $\overline{\mathcal{D}_{sg}(X')}$ are equivalent.*

4. Singularity categories for isolated singularities

Let A a finitely generated local algebra over a filed k. The unique maximal ideal of A is denoted by $\mathfrak{m}$. We suppose that $Z = \{\mathfrak{m}\}$ is an isolated singularity of $X = \mathrm{Spec}(A)$.

Consider the henselization $(A_h, \mathfrak{m}_h)$ of $(A, \mathfrak{m})$. By definition [6, Section 18], $A_h = \varinjlim B$, where the limit is taking by the category of all Nisnevich neighborhoods $\mathrm{Spec}B \to \mathrm{Spec}A$ of $\mathrm{Spec}A/\mathfrak{m}$ in $\mathrm{Spec}A$ and $\mathfrak{m}_h = \mathfrak{m}A_h$. Let $\hat{A}$ be the $\mathfrak{m}$-adic completion of A. The natural inclusion $A \hookrightarrow A_h \hookrightarrow \hat{A}$ [7, pp.51] induces the following commutative diagram of singularity categories

$$\begin{array}{ccccc} \mathcal{D}_{sg}(A) & \longrightarrow & \mathcal{D}_{sg}(A_h) & \longrightarrow & \mathcal{D}_{sg}(\hat{A}) \\ \uparrow & & \uparrow & & \uparrow \\ \mathcal{D}_{sg,Z}(A) & \longrightarrow & \mathcal{D}_{sg,Z}(A_h) & \longrightarrow & \mathcal{D}_{sg,Z}(\hat{A}). \end{array}$$

Proposition 4.1. *We have the following equivalences of triangulated categories*

$$\overline{\mathcal{D}_{sg}(A)} \simeq \overline{\mathcal{D}_{sg,Z}(A)},\ \overline{\mathcal{D}_{sg}(A_h)} \simeq \overline{\mathcal{D}_{sg,Z}(A_h)},\ \text{and}\ \overline{\mathcal{D}_{sg}(\hat{A})} \simeq \overline{\mathcal{D}_{sg,Z}(\hat{A})}.$$

By [7, pp.51], the formal completions of $\mathrm{Spec}A$, $\mathrm{Spec}A_h$, $\mathrm{Spec}\hat{A}$ along $Z = \{\mathfrak{m}\}$ are isomorphic. We denote it by $\mathfrak{X}_Z$. Thus, by Corollary 3.8, we have the following result.

Proposition 4.2. *We have the following equivalences of triangulated categories*

$$\mathcal{D}_{sg,Z}(A) \simeq \mathcal{D}_{sg,Z}(A_h) \simeq \mathcal{D}_{sg,Z}(\hat{A}) \simeq \mathcal{D}_{sg,t}(\mathfrak{X}_Z).$$

By Proposition 4.1, we have the following commutative diagram of singularity categories

$$\begin{array}{ccccccc} \mathcal{D}_{sg}(A) & \longrightarrow & \mathcal{D}_{sg}(A_h) & \longrightarrow & \mathcal{D}_{sg}(\hat{A}) & & \\ {\scriptstyle \sim d.s}\uparrow & & {\scriptstyle \sim d.s}\uparrow & & {\scriptstyle \sim d.s}\uparrow & & \\ \mathcal{D}_{sg,Z}(A) & \longrightarrow & \mathcal{D}_{sg,Z}(A_h) & \longrightarrow & \mathcal{D}_{sg,Z}(\hat{A}) & \xrightarrow{\simeq} & \mathcal{D}_{sg,t}(\mathfrak{X}_Z), \end{array}$$

where the notation $\xrightarrow{\sim d.s}$ means an equivalence up to direct summands.

Corollary 4.3. *The following triangulated categories are equivalent*

$$\overline{\mathcal{D}_{sg}(A)} \simeq \overline{\mathcal{D}_{sg}(A_h)} \simeq \overline{\mathcal{D}_{sg}(\hat{A})}.$$

Corollary 4.4. *The canonical functor*

$$\sigma : \mathcal{D}_{sg}(A_h) \to \mathcal{D}_{sg}(\hat{A})$$

is dense.

Proof. It follows from [1, Theorem 3.10]. $\surd$

By Corollary 4.3, functor σ is also fully faithful. Combining Corollary 4.3 and 4.4, we obtain the following Theorem.

Theorem 4.5. *We have the following equivalences of triangulated categories*

$$\mathcal{D}_{sg}(A) \xrightarrow{\sim d.s.} \mathcal{D}_{sg}(A_h) \xrightarrow{\sim} \mathcal{D}_{sg}(\hat{A}).$$

Remark 4.6. This result has been mentioned in [11, Remark 3.6]. We use the notation of torsion singularity categories to reprove this result.

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