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The Grothendieck-Riemann-Roch theorem

Mathjeunes

This talk was given for the seminar Mathjeunes in Orsay. Questions, comments, remarks, etc can all be mailed to me at ariyanjavan at gmail.com

Introduction

The classical Riemann-Roch problem can be stated as follows in modern language. For a compact Riemann surface X of genus g and a divisor D on X, how can we calculate $\dim H^0(X, \mathcal{O}_X(D))$? There is no general answer to this question. Instead, writing $\mathcal{L} = \mathcal{O}_X(D)$, we can show that

$$\dim H^0(X,\mathcal{L}) - \dim H^1(X,\mathcal{L}) = \deg D + 1 - g = \int_X \left(\frac{c_1(K^*)}{2} + c_1(\mathcal{L}) \right),$$

where K is the cotangent bundle of X and $\deg D$ is the degree of D. This is the Riemann-Roch theorem for compact Riemann surfaces. The left-hand side of this equation is the Euler characteristic $\chi(X, \mathcal{L})$. This is a cohomological invariant of $\mathcal{L}$. We see that the Riemann-Roch theorem can be interpreted as

$$\text{Euler characteristic of } \mathcal{L} = \int_X (\text{some characteristic class } X \text{ and } \mathcal{L}).$$

Now, Hirzebruch generalized this to to compact complex analytic manifolds X of any dimension ([Hirz]): for any holomorphic vector bundle $\mathcal{E}$ on a compact complex analytic manifold X, we have that

$$\chi(X,\mathcal{E}) = \int_X \mathrm{ch}(\mathcal{E})\,\mathrm{td}(X),$$

where $\mathrm{ch}(\mathcal{E})$ is the Chern character of $\mathcal{E}$ and $\mathrm{td}(X)$ is the Todd class of the tangent bundle $\mathcal{T}_X$ of X. By now, people had noticed the importance of the Euler characteristic

$$\chi(X,E) = \sum (-1)^i \dim H^i(X,E).$$

The next step is to generalize this theorem to Grothendieck's scheme theory. For starters, this means replacing the base field $\mathbf{C}$ by a field of any characteristic. Grothendieck proved a "relativized version" of the Riemann-Roch theorem which is much more powerful than Hirzebruch's theorem.

We can explain Grothendieck's approach by looking a bit closer at Hirzebruch's theorem. Let X be a compact complex variety and let f be a morphism from X to a point. We can rewrite Hirzebruch's theorem as

$$\sum (-1)^i \dim R^i f_* \mathcal{E} = f_* \left(\mathrm{ch}(\mathcal{E}) \, \mathrm{td}(X) \right). \tag{1}$$

Grothendieck showed that a generalization of equality (1) holds for any proper morphism of smooth projective varieties. Let us list some of his main ideas.

1. The additivity of the Euler characteristic plays an important role. This lead Grothendieck to the Grothendieck groups $K_0(X)$ and $K^0(X)$ associated to a noetherian scheme.
2. Grothendieck considered coherent sheaves, and not just vector bundles (which correspond to locally free coherent sheaves).
3. The cohomology ring is replaced by the Chow ring. This is an "intersection ring" and contains more information then the cohomology ring.
4. Grothendieck "relativized" the theorem.
5. The Riemann-Roch equality is an equality of classes in the Chow ring and no longer an equality of numbers.

Example 0.1. If $f : X \longrightarrow Y$ is a finite, separable morphism of smooth projective curves, it holds that

$$2g(X) - 2 = \deg f \cdot (2g(Y) - 2) + \deg R,$$

where R is the ramification divisor. This is the Riemann-Hurwitz theorem. But one shouldn't forget that we have much more precise information. Namely, the class of the canonical divisor K_X equals the class of the divisor $f^* K_Y + R$ in the divisor class group $\mathrm{Pic}(X) = \mathbf{Z} \oplus \mathrm{Pic}^0(X)$. What does this mean on $\mathrm{Pic}^0(X)$?

The Grothendieck-Riemann-Roch theorem states that if $f : X \longrightarrow Y$ is a proper morphism of smooth quasi-projective varieties, the following diagram

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\mathrm{ch} \cdot \mathrm{td}(X)} & A^{\cdot}(X) \otimes_{\mathbf{Z}} \mathbf{Q} \\ \downarrow f_! & & \downarrow f_* \\ K_0(Y) & \xrightarrow{\mathrm{ch} \cdot \mathrm{td}(Y)} & A^{\cdot}(Y) \otimes_{\mathbf{Z}} \mathbf{Q} \end{array}$$

is commutative. This statement implies that for every locally free coherent sheaf $\mathcal{E}$ on X, we have that

$$\mathrm{ch}(\sum (-1)^i R^i f_* \mathcal{E}) \, \mathrm{td}(Y) = f_*(\mathrm{ch}(\mathcal{E}) \, \mathrm{td}(X))$$

in $\mathbf{A}^{\cdot}(Y)_{\mathbf{Q}}$.

The first talk is divided into five parts:

1. Grothendieck groups for abelian categories and their additive subcategories;
2. Grothendieck groups for regular schemes;

3. Intersection theory on smooth quasi-projective varieties;

4. Characteristic classes;

5. Grothendieck-Riemann-Roch and some examples.

Remark 0.2. In this talk we identify vector bundles and locally free coherent sheaves.

1 Definition of Grothendieck group

Let $\mathcal{C}$ be an additive category embedded in an abelian category $\mathcal{A}$. Let $\mathrm{Ob}(\mathcal{C})$ denote the class of objects and let $\mathrm{Ob}(\mathcal{C})/\cong$ be the set of isomorphism classes[1]. Let $F(\mathcal{C})$ be the free abelian group on $\mathrm{Ob}(\mathcal{C})/\cong$, i.e. an element $T \in F(\mathcal{C})$ is a finite formal sum

$$\sum n_X[X],$$

where $[X]$ denotes the isomorphism class of $X \in \mathrm{Ob}(\mathcal{C})$ and n_X is an integer which is almost always zero.

Definition 1.1. To any sequence

$$(E) \qquad 0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

in $\mathcal{C}$, which is exact in $\mathcal{A}$, we associate the element $Q(E) = [A] - [A'] - [A'']$ in $F(\mathcal{C})$. Let $H(\mathcal{C})$ be the subgroup generated by the elements $Q(E)$ where E runs through all short exact sequences.

Definition 1.2. We define the *Grothendieck group*, denoted by $K(\mathcal{C})$, as the quotient group

$$K(\mathcal{C}) = F(\mathcal{C})/H(\mathcal{C}).$$

Remark 1.3. Since $\mathcal{C}$ is additive, it has finite direct sums. The fact that the sequence

$$0 \longrightarrow A \longrightarrow A \oplus B \longrightarrow B \longrightarrow 0$$

is exact (in $\mathcal{A}$) shows that the addition is given by $[A \oplus B] = [A] + [B]$.

Examples 1.4. Let A be a commutative ring.

1. Let $\mathcal{C}$ be the category of A-modules. (This is not a small category. To avoid this the reader may consider only countably generated A-modules.) Let us show that $K(\mathcal{C}) = (0)$. To this extent, let M be an A-module and note that $M \oplus \bigoplus_{n\in\mathbf{N}} M \cong \bigoplus_{n\in\mathbf{N}} M$. We see that $\mathrm{cl}(M) + \mathrm{cl}(\bigoplus_{n\in\mathbf{N}} M) = [\bigoplus_{n\in\mathbf{N}} M]$, which shows that $\mathrm{cl}(M) = 0$ in $K(\mathcal{C})$.

2. Let A be a principal ideal domain and $\mathcal{C}$ be the abelian category of finitely generated A-modules. By the structure theorem of A-modules, any R-module is isomorphic to the direct sum of a free module and a torsion part which is the direct sum of cyclic modules. The rank of an A-module is defined as the rank of its free part. That gives us a surjective map $\mathrm{rk} : \mathrm{Ob}(\mathcal{C})/\cong \longrightarrow \mathbf{Z}$ which induces a surjective homomorphism from

[1] The reader should actually consider small categories.

$F(\mathcal{C})$ to $\mathbf{Z}$. Since the rank is trivial on the elements $Q(E)$, it induces a group morphism $\widetilde{\text{rk}}$ from $K(\mathcal{C})$ to $\mathbf{Z}$. Note that for any nonzero ideal $I = (x)$, we have a short exact sequence

$$0 \longrightarrow A \xrightarrow{\cdot x} A \longrightarrow A/I \longrightarrow 0\ ,$$

which shows that $\text{cl}(A/I) = 0$ in $K(\mathcal{C})$. Therefore the kernel of $\widetilde{\text{rk}}$ is trivial. Furthermore, since any short exact sequence of free A-modules is split and $\text{rk}(\text{cl}(R)) = 1$, the rank induces an isomorphism from $K(\mathcal{C})$ to $\mathbf{Z}$.

3. Let A be a principal ideal domain and let $\mathcal{C}_m$ be the category of finitely generated free A-modules of rank 0 or rank greater than or equal to some fixed positive integer m. Since it has finite direct sums and the zero object, it is an additive subcategory of the abelian category $\mathcal{C}$ of finitely generated free A-modules. The reasoning above shows that the rank map induces an isomorphism $K(\mathcal{C}_m) \cong \mathbf{Z}$ with generator $\text{cl}(A^{m+1}) - \text{cl}(A^m)$. In particular, the natural inclusion $\mathcal{C}_m \subset \mathcal{C}$ induces an isomorphism on the level of Grothendieck groups.

2 $\mathcal{O}_X$-modules

References for this section are Hartshornes book [Har] and Weibels book [Weibel].

For simplicity, we will suppose that the base field k is algebraically closed.

Let X be a variety and let $\mathcal{O}_X$ be its structure sheaf. Recall that the affine open subsets of X form a basis for the topology on X and that $\mathcal{O}_X$ is determined by the rule

$$\mathcal{O}_X(U) = \Gamma(U, \mathcal{O}_X) = \Gamma(U, \mathcal{O}_U) = k[T_1, \ldots, T_n]/I,$$

if $U \subset X$ is isomorphic to the affine variety determined by the prime ideal $I \subset k[T_1, \ldots, T_n]$.

Definition 2.1. A *coherent sheaf* is a sheaf of abelian groups $\mathcal{F}$ on X endowed with a multiplication $\mathcal{O}_X \times \mathcal{F} \longrightarrow \mathcal{F}$ such that the following properties hold.

$\mathcal{O}_X$-module structure: For each open $U \subset X$, the abelian group of sections $\mathcal{F}(U)$ becomes a module over $\mathcal{O}_X(U)$.

Quasi-coherence: For every open affine subsets $U \subset V \subset X$, $\mathcal{F}(U) = \mathcal{F}(V) \otimes_{\mathcal{O}_X(V)} \mathcal{O}_X(U)$.

Coherence: For each open affine $U \subset X$ the module $\mathcal{F}(U)$ is finitely generated over $\mathcal{O}_X(U)$.

Let $\text{Coh}(X)$ be the category of coherent sheaves on X. (A morphism of coherent sheaves is a morphism of sheaves which respects the module structure.) It is a full abelian subcategory of the abelian category of $\mathcal{O}_X$-modules. If $X = \text{Spec}\, A$ is affine, the global sections functor $\Gamma(X, -)$ gives an equivalence of categories from $\text{Coh}(X)$ to the category of finitely generated A-modules. Its quasi-inverse assigns to each finitely generated A-module M the coherent sheaf $\widetilde{M}$.

Definition 2.2. The *Grothendieck group of coherent sheaves*, denoted by $K_0(X)$, is defined as

$$K_0(X) = K(\text{Coh}(X)).$$

For a ring A, we write $K_0(A) = K_0(\text{Spec}\, A)$.

Example 2.3. Let K be the function field of $\mathbf{P}^1_k$ and let η be its generic point. The map $K_0(\mathbf{P}^1_k) \longrightarrow \mathbf{Z} \oplus \mathbf{Z}$ given by $\mathcal{F} \mapsto (\dim_K \mathcal{F}_\eta, \chi(\mathbf{P}^1, \mathcal{F}))$ is an isomorphism.

Example 2.4. Let us show that $K_0(A) \cong \mathbf{Z} \cdot k[x] \oplus \mathbf{Z} \cdot k[y]$, where $A = k[x,y]/(xy)$. We claim that $K_0(A)$ is generated by the classes of the A-modules

$$k[x] = A/(y), \quad k[y] = A/(x), \quad k[x]/(f) = A/(y,f), \quad k[y]/(g) = A/(x,g),$$

where $f \in k[x]$ is an irreducible polynomial and $g \in k[y]$ is an irreducible polynomial. Let us verify this. Take a finitely generated nonzero A-module M. We have precisely two generic points: η_x and η_y. The residue field of η_x is $k(y)$ and the residue field of η_y is $k(x)$. Let $r = \operatorname{rk} M_{(x)}$ be the rank of M at η_x and let $s = \operatorname{rk} M_{(y)}$. Clearly, we have an injective homomorphism

$$k[x]^r \oplus k[y]^s \longrightarrow M$$

whose cokernel N is torsion. Since N is torsion, it has finite support. Therefore, its support must consist of only maximal ideals. (It can't contain a generic point. Else it would be infinite.) Thus, it has a composition series by the above Theorem. Thus N is a finite sum of the form

$$\sum_{f \text{ irreducible}} n_f \cdot \operatorname{cl}(k[x]/(f)) + \sum_{g \text{ irreducible}} m_g \cdot \operatorname{cl}(k[y]/(g))$$

in the Grothendieck group. (In $K_0(A)$ write N as the sum of the simple quotients that appear in its composition series.) This proves the claim. Now, for any nonzero $f \in k[x]$, the short exact sequence of A-modules

$$0 \longrightarrow k[x] \longrightarrow k[x] \longrightarrow k[x]/(f) \longrightarrow 0$$

shows that the class of $k[x]/(f)$ is zero in $K_0(A)$. Similarly, for any nonzero $g \in k[y]$, the class of $k[y]/(g)$ is zero in $K_0(A)$. Hence $K_0(A)$ is generated by (the classes of) $k[x]$ and $k[y]$. These are linearly independent over $\mathbf{Z}$. In fact, suppose that $a \cdot k[x] + b \cdot k[y] = 0$, where $a, b \in \mathbf{Z}$. Take the rank at (y) to see that $a = 0$. Similarly, take the rank at (x) to see that $b = 0$. Thus, we conclude that $K_0(A) \cong \mathbf{Z} \cdot k[x] \oplus \mathbf{Z} \cdot k[y]$.

Definition 2.5. A *vector bundle (of rank r)* is a coherent sheaf $\mathcal{F}$ where every point $x \in X$ has an affine neighborhood $U \subset X$ such that $\mathcal{F}(U)$ is a free $\mathcal{O}_X(U)$-module of rank r. A *line bundle* is a vector bundle of rank 1. Let $\operatorname{Vect}(X)$ be the category of vector bundles on X.

The category $\operatorname{Vect}(X)$ is additive and embedded in $\operatorname{Coh}(X)$. It is not an abelian category in general. We define its Grothendieck group via this embedding.

Definition 2.6. We define the *Grothendieck group of vector bundles* on X, denoted by $K^0(X)$, as

$$K^0(X) = K(\operatorname{Vect}(X)).$$

For a ring A, we write $K^0(A) = K^0(\operatorname{Spec} A)$.

Proposition 2.7. The tensor product (over $\mathcal{O}_X$) defines a commutative ringstructure on $F(\operatorname{Vect}(X))$.

Proof. The tensor product of vector bundles is a vector bundle. The tensor product is associative and commutative as follows from its universal property and $\mathcal{O}_X$ is clearly the identity element. The tensor product is also distributive with respect to the direct sum. $\square$

Proposition 2.8. The tensor product defines a commutative ring structure on $K^0(X)$.

Proof. We need to show that the subgroup $H(\mathrm{Vect}(X))$ is an ideal of $F(\mathrm{Vect}(X))$. But this follows from the fact that any vector bundle is flat. $\square$

Example 2.9. Let k be a field. Let $X = \mathbf{P}_k^n$ and let $\mathcal{F}$ be a coherent sheaf on X. Write $n = \dim X$ and $\mathcal{O} = \mathcal{O}_X$. There exists an integer $m \in \mathbf{Z}$ and a positive integer $r > 0$, such that $\mathcal{F}$ is a quotient sheaf of $\bigoplus^r \mathcal{O}(m) = \mathcal{O}(m)^{\oplus r}$. Therefore, we have a resolution of vector bundles

$$0 \longrightarrow \mathcal{E}_n \longrightarrow \mathcal{E}_{n-1} \longrightarrow \ldots \longrightarrow \mathcal{E}_1 \longrightarrow \mathcal{E}_0 \longrightarrow \mathcal{F} \longrightarrow 0,$$

where $\mathcal{E}_i$ is a direct sum of line bundles of the form $\mathcal{O}(m)$. Thus, the Grothendieck group of vector bundles $K^0(X)$ is generated by the classes of the line bundles $\mathcal{O}(m)$, where $m \in \mathbf{Z}$. We have an exact sequence

$$0 \longrightarrow \mathcal{O} \longrightarrow \oplus^{n+1}\mathcal{O}(1) \longrightarrow \oplus^{\binom{n+1}{2}}\mathcal{O}(2) \longrightarrow \cdots \longrightarrow \oplus^{n+1}\mathcal{O}(n) \longrightarrow \mathcal{O}(n+1) \longrightarrow 0\,.$$

In fact, this is a dualized Koszul complex. From this exact sequence, we get two important facts. Firstly, writing ξ for the class of $\mathcal{O}(1)$ in $K^0(X)$, we see that $(1-\xi)^{n+1} = 0$. Here we invoked the ringstructure on $K^0(X)$. Furthermore, we see that $K^0(X)$ is generated by $(1, \xi, \ldots, \xi^n)$. We will show that the surjective homomorphism $\mathbf{Z}^{n+1} \longrightarrow K^0(X)$ given by $(a_0, \ldots, a_n) \mapsto \sum_{i=0}^n a_i\xi^i$ is an isomorphism of abelian groups. To prove this, it suffices to show that the above surjective homomorphism $\mathbf{Z}^{n+1} \longrightarrow K^0(X)$ is injective. In fact, suppose that $\alpha = \sum_{i=0}^n a_i\xi^i = 0$, where $(a_0, \ldots, a_n) \in \mathbf{Z}^{n+1}\backslash\{0\}$. Choose i maximal with $a_i \neq 0$. Then

$$a_i = \chi(X, \alpha \cdot \xi^{-i}) = 0.$$

(This follows from the computations involving Cech cohomology.) Contradiction. We conclude that

$$(1, \xi, \ldots, \xi^n)$$

is a $\mathbf{Z}$-basis for the abelian group $K^0(X)$. Also, the map $\mathbf{Z}[x]/(1-x)^n \longrightarrow K^0(X)$ given by $x \mod (1-x)^n \mapsto \xi$ is an isomorphism of rings with inverse given by

$$\mathrm{cl}(\mathcal{E}) \mapsto \sum_{i=0}^n \chi(X, \mathcal{E}(i))x^i \mod (1-x)^n.$$

The (exact) embedding $\mathrm{Vect}(X) \longrightarrow \mathrm{Coh}(X)$ of categories induces a natural homomorphism $K^0(X) \longrightarrow K_0(X)$.

Theorem 2.10. If X is nonsingular, quasi-projective and irreducible, the canonical homomorphism $K^0(X) \longrightarrow K_0(X)$ is an isomorphism of groups. $\square$

Let us illustrate the importance of nonsingularity.

Example 2.11. Let k be a field, $A = k[x,y]/(xy)$ and $I = (x,y) \subset A$. Note that A is non-regular. Consider the infinite resolution of free A-modules for $k = A/I$ given by

$$\cdots \xrightarrow{g} A^2 \xrightarrow{h} A^2 \xrightarrow{g} A^2 \xrightarrow{h} A^2 \xrightarrow{g} A^2 \xrightarrow{f} A \longrightarrow k \longrightarrow 0\,.$$

Here

$$f : (s,t) \mapsto sx + ty,\; g : (s,t) \mapsto (sy, tx) \text{ and } h : (s,t) \mapsto (sx, ty).$$

It is easy to see that

$$\mathrm{Tor}_i^A(k,k) = \begin{cases} k & \text{if } \; i = 0 \\ k^2 & \text{if } \; i > 0 \end{cases}.$$

To this extent it suffices to note that after tensoring the above resolution with $k \otimes_A -$ all the maps are zero. This shows that there can not be a finite (projective) resolution of A-modules for k. (Since then the $\mathrm{Tor}_i^A(k,-)$ functors would be identically zero for $i \gg 0$.) We have shown that $K_0(A) \cong \mathbf{Z}^2$. It is easy to see that $K^0(A) \cong \mathbf{Z}$. The Cartan homomorphism $\mathbf{Z} \longrightarrow \mathbf{Z} \oplus \mathbf{Z}$ is given by the diagonal embedding.

Let $f : X \longrightarrow Y$ be a morphism of varieties and let $\mathcal{F}$ be a coherent sheaf on X.

Suppose that f is a proper morphism. Then the direct image of $\mathcal{F}$ is again coherent. Note that f_* is left exact. Since f is proper, we have that $R^i f_* \mathcal{F}$ is also coherent on Y. The morphism $f_! : K_0(X) \longrightarrow K_0(Y)$ given by

$$\mathrm{cl}(\mathcal{F}) \mapsto \sum (-1)^i \mathrm{cl}(R^i f_* \mathcal{F})$$

is well-defined and makes K_0 into a covariant functor from the category of varieties to the category of abelian groups.

We see that K_0 is a covariant functor from the category of varieties to the category of abelian groups.

Using the isomorphism in Theorem 2.10 we get a ringstructure on $K_0(X)$. We can show that, for any two coherent sheaves $\mathcal{F}$ and $\mathcal{G}$, we have that

$$\mathrm{cl}(\mathcal{F}) \cdot \mathrm{cl}(\mathcal{G}) = \sum_{i=0}^{\dim X} (-1)^i \; \mathrm{cl}(\mathrm{Tor}_i^{\mathcal{O}_X}(\mathcal{F},\mathcal{G}))$$

in $K_0(X)$.

Remark 2.12. There is some intersection theory hidden here. See Remark 3.6.

3 Intersection theory

References for this section are Serre's book [Ser] and Fulton's book [**?**].

One way to do intersection theory on a smooth projective variety is to define intersection multiplicities and use the moving lemma.

Let us begin with an example.

Example 3.1. Let k be a field. Suppose that $X = \mathbf{A}_k^2 = \operatorname{Spec} k[x,y]$, $V = \operatorname{Spec} k[x,y]/(y^2 - x)$ and $W = \operatorname{Spec} k[x,y]/(x-a)$, where $a \in k$. Intuitively, the intersection of V and W should consist of two points. By definition, the scheme-theoretic intersection is given by

$$V \cap W = \operatorname{Spec} k[x,y]/(y^2 - x, x - a).$$

Clearly, we have that $k[x,y]/(y^2 - x, x - a) \cong k[y]/(y^2 - a)$. If $a \neq 0$ and k is algebraically closed, the Chinese remainder theorem shows that

$$V \cap W = \operatorname{Spec} k[y]/(y - \sqrt{a}) \amalg \operatorname{Spec} k[y]/(y + \sqrt{a}).$$

This clearly coincides with our intuition. Even for $a = 0$, we get that $V \cap W = \operatorname{Spec} k[y]/(y^2)$ which we can interpret as a double point. Of course, when k is not algebraically closed, we also want the multiplicity to be 2.

Let X be a quasi-projective smooth variety. (The base field is algebraically closed and by a variety we mean a separated integral scheme of finite type over the base field. A subvariety of X is a closed subscheme of X which is a variety.)

Serre gave the following definition for the intersection multiplicity and proved that it is the correct one.

Definition 3.2. Let V and W be subvarieties of X. We say that V and W *intersect properly* if, for any irreducible component Z of $V \cap W$, we have that

$$\operatorname{codim} Z = \operatorname{codim} V + \operatorname{codim} W.$$

It is clear that this is a local property. Now, suppose that V and W intersect properly and let Z be an irreducible component of $V \cap W$. We define the *(local) intersection multiplicity* of V and W along Z, denoted by $i_Z(V, W)$, as

$$i_Z(V, W) = \sum (-1)^i \operatorname{length}_A(\operatorname{Tor}_i^A(A/I, A/J)).$$

Here A is the local ring $\mathcal{O}_{X,z}$ at the generic point z of Z, I is the ideal defining V and J is the ideal defining W.

Example 3.3. Let V be the subscheme defined by the ideal $I := (x,y) \cap (z,w) = (xz, xw, yz, yw)$ in $\mathbf{A}^4 = \operatorname{Spec} k[x,y,z,w]$. Geometrically, V is the union of two planes meeting at the origin in $\mathbf{A}^4$. Let W be the plane $J := (x - z, y - w)$. Note that W meets every component of V in precisely one point P. By linearity, we have that $i_P(V, W) = 2$. Note that

$$A/(I + J) = \operatorname{Spec} k[x,y,z,w]/(xz, xw, yz, yw, x - z, y - w) = k[x,y]/(x^2, xy, y^2)$$

has length 3. We see that the higher Tor's are necessary.

Let $Z(X)$ denote the abelian group of cycles on X. For any integral closed subscheme $V \subset X$, we let $[V]$ denote its class in $Z(X)$. The following theorem reveals the geometric nature of $K_0(X)$.

Theorem 3.4. The homomorphism $Z(X) \longrightarrow K_0(X)$ defined by $[V] \mapsto \operatorname{cl}(\mathcal{O}_V)$ is surjective.

Definition 3.5. Let V and W be subvarieties of X which intersect properly. We define the *product cycle* of V and W, denoted by $[V]\cdot[W] \in Z(X)$, as

$$[V]\cdot[W] = \sum_Z i_Z(V,W)[Z].$$

Here the sum is over all irreducible components of $V \cap W$.

Example 3.6. Suppose that V and W are in general position. That is, for every $x \in V \cap W$ there is an affine open subset $U \subset X$ such that $\mathcal{J}_1(U)$ is generated by a regular sequence $(x_1,\dots,x_r)$ and $\mathcal{J}_2(U)$ is generated by a regular sequence $(g_1,\dots,g_s)$ with $(f_1,\dots,f_r,g_1,\dots,g_s)$ a regular sequence in $\mathcal{O}_X(U)$. Note that V and W intersect properly. In fact, by the Hauptidealsatz, if $A = \mathcal{O}_{X,c}$ is the local ring at a generic point c of $V \cap W$, we have that $\operatorname{codim} A/(I+J) = r+s$ with $I = (f_1,\dots,f_r)$ the ideal defining V at x and $J = (g_1,\dots,g_s)$ the ideal defining W at x. One can show (assuming V and W are in general position) that

$$\mathrm{cl}(\mathcal{O}_V)\cdot\mathrm{cl}(\mathcal{O}_W) = \sum(-1)^i\mathrm{cl}\Big(\operatorname{Tor}_i^{\mathcal{O}_X}(\mathcal{O}_V,\mathcal{O}_W)\Big) = \mathrm{cl}(\mathcal{O}_V \otimes_{\mathcal{O}_X} \mathcal{O}_W) = \mathrm{cl}(\mathcal{O}_{V\cap W})$$

in $K_0(X)$. Also, we have that

$$[V]\cdot[W] = \sum_Z i_Z(V,W)[Z] = \sum_Z \operatorname{length}\mathcal{O}_{V\cap W,z}[Z] = [\mathcal{O}_{V\cap W}] = [V\cap W].$$

Therefore, in this case, the product in K_0-theory coincides with the product cycle under the homomorphism $Z(X) \longrightarrow K_0(X)$ and they both coincide with taking intersections.

For any $r \geq 0$, we let $Z^r(X)$ denote the cycles of codimension r on X. So $Z^0(X) = \mathbf{Z}\cdot[X]$ and $Z^1(X) = \mathrm{Div}(X)$. The Chow group of codimension r, denoted by $A^r(X)$, is the group $Z^r(X)$ modulo the group of cycles "rationally equivalent" to zero. For example, $A^1(X) = \mathrm{Cl}(X) = \mathrm{Pic}(X)$. The Chow group is $A^{\cdot}(X) = \oplus A^r(X)$.

Let $f : X \longrightarrow Y$ be a proper morphism of varieties. Let V be a subvariety of X with image $W = f(V)$. If $\dim W < \dim V$, we set $f_*(V) = 0$. If $\dim V = \dim W$, the function field $K(V)$ is a finite extension field of $K(W)$, and we set

$$f_*(V) = [K(V) : K(W)]W.$$

Extending by linearity defines a homomorphism f_* of $Z(X)$ to $Z(Y)$. These homomorphisms are functorial, as follows from the multiplicativity of degrees of field extension. One can show that f_* induces a morphism of groups $f_* : A^{\cdot}(X) \longrightarrow A^{\cdot}(Y)$.

From the "moving lemma" and general properties of the product cycle, one can deduce that the product cycle induces a commutative associative unitary graded ring structure on $A^{\cdot}(X)$. Equipped with this graded ringstructure we call $A^{\cdot}(X)$ the Chow ring of X.

So now we "understand" what the following objects and arrows

$$\begin{array}{ccc} K_0(X) & \qquad & A^{\cdot}(X)\otimes_{\mathbf{Z}}\mathbf{Q} \\ \big\downarrow f_! & & \big\downarrow f_* \\ K_0(Y) & & A^{\cdot}(Y)\otimes_{\mathbf{Z}}\mathbf{Q} \end{array}$$

mean if $f : X \longrightarrow Y$ is a proper morphism of smooth quasi-projective varieties. It remains to connect the world of Grothendieck groups and intersection theory. This is done by characteristic classes.

4 Characteristic classes

For $f : X \longrightarrow Y$ a morphism of smooth quasi-projective varieties, we can define pull-back morphisms $f^! : K_0(Y) \longrightarrow K_0(X)$ and $f^* : A^{\cdot}(Y) \longrightarrow A^{\cdot}(X)$.

In K_0-theory, we use that the pull-back of a vector bundle on X via f is again a vector bundle. Since vector bundles are flat, the functor f^* from $\mathrm{Vect}(Y)$ to $\mathrm{Vect}(X)$ is exact. We have a homomorphism $f^* : K^0(Y) \longrightarrow K^0(X)$. It is a morphism of rings. Identifying $K^0(X)$ with K_0, we get a pull-back for K_0 (when X is smooth and quasi-projective).

When f is flat, the pull-back $f^* : A^{\cdot}(Y) \longrightarrow A^{\cdot}(X)$ is given by $f^*[V] = [f^{-1}V]$. (General case is a bit harder to do.)

Let X be a smooth quasi-projective variety over a field k. Let $\mathrm{Pic}(X)$ be the group of invertible sheaves on X and let $\mathrm{Cl}(X) = A^1(X)$ be the divisor class group. Every divisor D on X determines up to isomorphism an invertible sheaf $\mathcal{O}_X(D)$ (denoted by $\mathcal{L}(D)$ in Hartshorne) and every invertible sheaf is of this type. This induces an isomorphism $\mathrm{Cl}(X) \longrightarrow \mathrm{Pic}(X)$. See [Har, Chapter II, Proposition 6.16].

Definition 4.1. For any $\mathcal{L} \in \mathrm{Pic}(X)$, we define the *first Chern class* of $\mathcal{L}$ in $\mathrm{Cl}(X)$ by $c_1(\mathcal{L}) = [D]$, where $[D] \in \mathrm{Cl}(X)$ is such that $\mathcal{O}_X(D) = \mathcal{L}$ in $\mathrm{Pic}(X)$. By definition, the homomorphism $c_1 : \mathrm{Pic}(X) \longrightarrow \mathrm{Cl}(X)$ is the inverse of the homomorphism $\mathrm{Cl}(X) \longrightarrow \mathrm{Pic}(X)$ described above.

Let $\mathcal{E}$ be a vector bundle of rank r and let $\pi : \mathbf{P}(\mathcal{E}) \longrightarrow X$ be the associated projective bundle. Let $\mathcal{O}(1) = \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ be the canonical invertible sheaf on $\mathbf{P}(\mathcal{E})$. Note that the pull-back $\pi^* : A(X) \longrightarrow A(\mathbf{P}(\mathcal{E}))$ makes $A(\mathbf{P}(\mathcal{E}))$ into an $A(X)$-module.

Example 4.2. Suppose that $\mathcal{E} = \mathcal{O}_X^r$. Then $\mathbf{P}(\mathcal{E}) = \mathbf{P}_X^{r-1} = \mathbf{P}_k^{r-1} \times_k X$. We have that $K_0(\mathbf{P}(\mathcal{E})) = K_0(X) \otimes_{\mathbf{Z}} K_0(\mathbf{P}_k^{r-1})$ is a free $K_0(X)$-module. If ξ is the class of $\mathcal{O}(1)$ in $K_0(\mathbf{P}_k^{r-1})$, we have seen that $K_0(\mathbf{P}_k^{r-1})$ is a free abelian group with basis $(1, \xi, \ldots, \xi^{r-1})$. In particular, $K_0(\mathbf{P}(\mathcal{E}))$ is a free $K_0(X)$-module with the same basis.

The Chow ring is not that different.

Theorem 4.3. The Chow ring $A^{\cdot}(\mathbf{P}(\mathcal{E}))$ is a free $A^{\cdot}(X)$-module with basis $(1, \xi, \ldots, \xi^{r-1})$, where $\xi = c_1(\mathcal{O}(1)) \in A^1(\mathbf{P}(\mathcal{E}))$.

Definition 4.4. There exist unique elements $a_i \in A^i(X)$ $(1 \leq i \leq r)$ such that

$$\xi^r - \pi^*(a_1) \cdot \xi^{r-1} + \pi^*(a_2) \cdot \xi^{r-2} - \ldots + (-1)^r \pi^*(a_r) = 0.$$

We define the *i-th Chern class* of $\mathcal{E}$, denoted by $c_i(\mathcal{E}) \in A(X)$, as $c_i(\mathcal{E}) = a_i$ for $1 \leq i \leq r$. We put $c_0(\mathcal{E}) = 1$. By convention, $c_i(\mathcal{E}) = 0$ for $i > r$.

Example 4.5. Let $\mathcal{E} = \mathcal{O}_X(D)$ be an invertible sheaf where D is in $\mathrm{Cl}(X)$. Then $\mathbf{P}(\mathcal{E}) = X$, $\mathcal{O}(1) = \mathcal{O}_X(D)$ and π is the identity map. Therefore, we have that $\xi - \pi^*(a_1) = 0$ showing that $c_1(\mathcal{E}) = [D]$ as one would expect.

Definition 4.6. We define the *Chern polynomial* of $\mathcal{E}$, denoted by $c_t(\mathcal{E})$, as the element

$$c_t(\mathcal{E}) = 1 + c_1(\mathcal{E})t + \ldots + c_r(\mathcal{E})t^r$$

in the ring $A(X)[t]$.

Theorem 4.7. There is a unique theory of Chern classes for X, which assigns to each vector bundle $\mathcal{E}$ on X an i-th Chern class $c_i(\mathcal{E}) \in A^i(X)$ and satisfies the following properties:

C0 It holds that $c_0(\mathcal{E}) = 1$ and $c_i(\mathcal{E}) = 0$ for $i > \operatorname{rk} \mathcal{E}$.

C1 For an invertible sheaf $\mathcal{O}_X(D)$, we have that $c_1(\mathcal{O}_X(D)) = [D]$.

C2 For a morphism of smooth quasi-projective varieties $f : X \longrightarrow Y$ and any positive integer i, we have that $f^*(c_i(\mathcal{E})) = c_i(f^*(\mathcal{E}))$.

C3 If

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

is an exact sequence of vector bundles on X, then

$$c_t(\mathcal{E}) = c_t(\mathcal{E}')c_t(\mathcal{E}'')$$

in $A(X)[t]$.

C4 The mapping $\mathcal{E} \mapsto c_t(\mathcal{E})$ can be extended to a homomorphism $c_t : K_0(X) \longrightarrow 1 + \oplus_{i=1}^{\infty} A^i(X) \cdot t^i$.

Example 4.8. Let $f : X \longrightarrow \operatorname{Spec} k$ be the structural morphism and let $\mathcal{E} = \mathcal{O}_X^r$ be free of rank r. By C2, for any $i \geq 1$, we have that $c_i(\mathcal{O}_X^r) = f^* c_i(k^r) = 0$.

The proof of Theorem 4.7 uses in an essential way the so-called Splitting principle for the Chow ring and $\mathcal{E}$.

Theorem 4.9. (Splitting Principle) Fix a vector bundle $\mathcal{E}$ on X of rank r. There exists a smooth quasi-projective variety X' and a morphism $\pi : X' \longrightarrow X$ such that $\pi^* : A(X) \longrightarrow A(X')$ is injective, and $\mathcal{E}' = \pi^*\mathcal{E}$ splits. (That is, it has a filtration $\mathcal{E}' = \mathcal{E}'_0 \supset \mathcal{E}'_1 \supset \ldots \supset \mathcal{E}'_r = 0$ such that $\mathcal{L}_i = \mathcal{E}_{i-1}/\mathcal{E}_i$ is an invertible sheaf for $1 \leq i \leq r$.)

Proposition 4.10. There is a homomorphism $\operatorname{ch} : K_0(X) \longrightarrow A(X)_{\mathbf{Q}}$ which is uniquely determined by the following properties.

1. For any morphism $f : Y \longrightarrow X$ of smooth quasi-projective varieties, it holds that

$$f^* \circ \operatorname{ch} = \operatorname{ch} \circ f^*.$$

2. For any invertible sheaf $\mathcal{L}$ on X with $c_1(\mathcal{L}) = [D]$, it holds that

$$\operatorname{ch}(\operatorname{cl}(\mathcal{L})) = \operatorname{ch}(\mathcal{L}) = \sum_{i \geq 0} \frac{1}{i!}[D]^i.$$

3. The homomorphism ch is a morphism of rings. □

The uniqueness clearly follows from the splitting principle. In fact, its existence also. To show this, take a vector bundle $\mathcal{E}$ on X and choose a morphism $\pi : X' \longrightarrow X$ such that $\pi^* : A^\cdot(X) \longrightarrow A^\cdot(X')$ is injective and $\pi^*\mathcal{E}$ splits. Then, we consider the pull-back of $c_t(\mathcal{E})$. This equals

$$\pi^*(c_t(\mathcal{E})) = c_t(\pi^*\mathcal{E}) = c_t(\mathcal{L}_1) \cdot \ldots \cdot c_t(\mathcal{L}_r) = \prod_{i=1}^{r}(1 + c_1(\mathcal{L}_i)t) = \prod_{i=1}^{r}(1 + \alpha_i t).$$

We then define

$$\mathrm{ch}(\mathcal{E}) = \mathrm{ch}(\mathrm{cl}(\mathcal{E})) = \sum_{i=1}^{r} \exp(\alpha_i).$$

The α_i are called Chern roots. (The expression is well-defined and independent of the choices we make.)

The Todd class of a vector bundle $\mathcal{E}$ on X is an element of $A^\cdot(X) \otimes_{\mathbf{Z}} \mathbf{Q}$. It is defined as

$$\mathrm{td}(\mathcal{E}) = \prod_{i=1}^{r} \left(\frac{\alpha_i}{1 - \exp(-\alpha_i)} \right) \in A^\cdot(X)_{\mathbf{Q}}.$$

The Todd class of X is defined as the Todd class of its tangent sheaf $\mathcal{T}_X$.

Remark 4.11. The Chern character commutes with pull-backs. Its “lack” of commuting with the push-forward is measured by the Todd class.

5 Riemann-Roch for curves and fibered surfaces

A reference for this section is the original article of Borel and Serre [BorSer].

A curve is a 1-dimensional variety. A surface is a 2-dimensional variety.

Let X be a smooth projective curve over an algebraically closed field k. Note that $A^\cdot(X) = \mathbf{Z} \oplus \mathrm{Cl}(X)$, where $\mathrm{Cl}(X)$ is the divisor class group of X. The group structure on $\mathbf{Z} \oplus \mathrm{Cl}(X)$ is given by $(n, D) + (m, E) = (n + m, D + E)$ whereas the multiplication is given by $(n, D)(m, E) = (nm, mD + nE)$.

Proposition 5.1. We have an isomorphism of rings $K_0(X) \longrightarrow \mathbf{Z} \oplus \mathrm{Pic}(X)$ given by $\alpha \mapsto (\mathrm{rk}(\alpha), \det \alpha)$. □

Consequently, the Chern character $\mathrm{ch} : K_0(X) \longrightarrow A(X)$ given by

$$\mathrm{ch}(\alpha) = (\mathrm{rk}(\alpha), c_1(\det(\alpha)) = (\mathrm{rk}(\alpha), c_1(\alpha))$$

is an isomorphism of rings.

Let g be the genus of X. It is well-known that for every $D \in \mathrm{Cl}(X)$, we have that $\chi(X, \mathcal{O}_X(D)) = \deg D + 1 - g$.

The Todd class of X is given by $\mathrm{td}(X) = (1, -\frac{1}{2}c_1(\omega_X)) = (1, -\frac{1}{2}K_X)$ in $A(X)_{\mathbf{Q}}$. The *degree* on the Chow ring is the function $\deg : A(X) \longrightarrow \mathbf{Z}$ given by $\deg(n, D) = \deg(D)$, where $\deg D$ denotes the degree of a divisor.

Theorem 5.2. (Riemann-Roch) The following diagram of groups

$$\begin{CD} K_0(X) @>{\mathrm{ch}\cdot \mathrm{Td}_X}>> A(X)_{\mathbf{Q}} \\ @V{\chi(X,-)}VV @VV{\deg}V \\ \mathbf{Z} @>>> \mathbf{Q} \end{CD}$$

is commutative.

Proof. By the identification of $K^0(X)$ with $K_0(X)$, it suffices to show that, for any vector bundle $\mathcal{E}$ of rank r, it holds that

$$\chi(X, \mathcal{E}) = \deg(\mathrm{ch}(\mathcal{E})\,\mathrm{td}(X)) = \deg(c_1(\mathcal{E})) - \frac{1}{2} r \deg(K_X) = \deg \mathcal{E} + r(1-g).$$

Since $\chi(X, \mathcal{O}_X(D)) = \deg D + 1 - g$ for all $D \in \mathrm{Cl}(X)$, the statement holds for (the class of) a line bundle $\mathcal{L}$ in $K_0(X)$. To prove the Riemann-Roch theorem it suffices to do so for the class of a vector bundle $\mathcal{E}$ of rank $r > 1$. Suppose that we have a short exact sequence

$$0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}' \longrightarrow 0,$$

where $\mathcal{L}$ is a line bundle and $\mathcal{E}'$ is a vector bundle of rank $r-1$. Then, by the additivity of the Euler characteristic and the Riemann-Roch theorem for line bundles, the theorem follows by induction on r. Let us show that we always have such a short exact sequence. Choose $n \gg 0$ such that $\mathcal{E}(n)$ is generated by its global sections. Then, since $\dim X = 1$ and $\mathrm{rk}\,\mathcal{E}(n) > 1$, there is a global section s which is nowhere zero ([Har, Exercise II.8.2]). Note that $\mathcal{E}(n)/s\mathcal{O}_X$ is a vector bundle of rank $r-1$ and that we have a short exact sequence

$$0 \longrightarrow s\mathcal{O}_X \longrightarrow \mathcal{E}(n) \longrightarrow \mathcal{E}(n)/s\mathcal{O}_X \longrightarrow 0.$$

Now tensor with $\mathcal{O}_X(-n)$ to get the desired short exact sequence. $\square$

We turn to an interesting application of the above result.

Let C be a smooth projective curve. A smooth projective fibered surface over C is a non-singular integral separated 2-dimensional C-scheme which is flat and projective. We define $\mathrm{td}(X/C)$ as $\mathrm{td}(X) \cdot (f^* \mathrm{td}(C))^{-1} \in A \cdot (X)_{\mathbf{Q}}$.

Corollary 5.3. The following diagram

$$\begin{CD} K_0(X) @>{\mathrm{ch}\cdot \mathrm{td}(X)}>> A(X)_{\mathbf{Q}} \\ @V{\chi(X,-)}VV @VV{\deg}V \\ \mathbf{Z} @>>> \mathbf{Q} \end{CD}$$

is commutative if and only if, for any vector bundle $\mathcal{E}$ on X, we have that

$$\deg_C c_1(f_! \mathcal{E}) = \deg_X(\mathrm{ch}(\mathcal{E})\,\mathrm{td}(X/C)).$$

Proof. Firstly, the diagram being commutative amounts to saying that, for any vector bundle $\mathcal{E}$ on X, we have that

$$\chi(X, \mathcal{E}) = \deg_X(\mathrm{ch}(\mathcal{E})\,\mathrm{td}(X)).$$

Note that both sides are additive on short exact sequences. Thus, it suffices to show that, for any vector bundle $\mathcal{E}$ on X, we have that $\deg_C c_1(f_!\mathcal{E}) = \deg_X(\mathrm{ch}(\mathcal{E})\,\mathrm{td}(X/C))$ if and only if $\chi(X, \mathcal{E}) = \deg_X(\mathrm{ch}(\mathcal{E})\,\mathrm{td}(X))$. By the functoriality of K-theoretic push-forward and the above Riemann-Roch theorem for C, we have that

$$\chi(X, \mathcal{E}) \quad = \quad \chi(C, f_!\mathcal{E}) = \mathrm{rk}(f_!\mathcal{E})\deg(\mathrm{td}(C)) + \deg c_1(f_!\mathcal{E}).$$

If η is the generic point of X and X_η is the generic fibre of f, we have that $\mathrm{rk}\, f_!\mathcal{E} = \chi(X_\eta, \mathcal{E}_\eta)$. Invoking the Riemann-Roch theorem for the smooth projective curve X_η, the Corollary follows from

$$\mathrm{rk}\, f_!\mathcal{E} = \chi(X_\eta, \mathcal{E}_\eta) = r\deg_{X_\eta}(\mathrm{td}(X_\eta)) + \deg_{X_\eta} c_1(\mathcal{E}_\eta).$$ □

Remark 5.4. Combining the classical Riemann-Roch theorem for surfaces with Noether's formula, one can show that the above diagram commutes ([Har, Example A.4.1.2]). The equality $\deg_C c_1(f_!\alpha) = \deg_X(\mathrm{ch}(\alpha)\,\mathrm{td}(X/C))$ is an expression for the degree of the determinant of cohomology $\det f_!\alpha$. By the functoriality of push-forward in intersection theory, this equality can also be written as

$$\deg_C c_1(f_!\alpha) = \deg_C(f_*(\mathrm{ch}(\alpha)\,\mathrm{td}(X/C)).$$

One is tempted to think that the stronger equality of cycle classes

$$c_1(f_!\alpha) = f_*(\mathrm{ch}(\alpha)\,\mathrm{td}(X/C))_{(1)}$$

holds in $A^{\cdot}(C)_{\mathbf{Q}}$. This equality holds and follows from the Grothendieck-Riemann-Roch theorem for f:

the following diagram

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\mathrm{ch}\cdot\mathrm{td}(X)} & A(X)_{\mathbf{Q}} \\ \downarrow{\scriptstyle f_!} & & \downarrow{\scriptstyle f_*} \\ K_0(C) & \xrightarrow{\mathrm{ch}\cdot\mathrm{td}(C)} & A(C)_{\mathbf{Q}} \end{array}$$

is commutative.

It follows from the more general Grothendieck-Riemann-Roch theorem.

Theorem 5.5. (Grothendieck-Riemann-Roch) Let $f : X \longrightarrow Y$ be a proper morphism of smooth quasi-projective varieties over a field k. The following diagram

$$\begin{array}{ccc} K_0(X) & \xrightarrow{\mathrm{ch}\cdot\mathrm{td}(X)} & A(X)_{\mathbf{Q}} \\ \downarrow{\scriptstyle f_!} & & \downarrow{\scriptstyle f_*} \\ K_0(Y) & \xrightarrow{\mathrm{ch}\cdot\mathrm{td}(Y)} & A(Y)_{\mathbf{Q}} \end{array}$$

is commutative. □

References

[BorSer] A. Borel, J.P. Serre, *Le théorème de Riemann-Roch.* Bull. Soc. math. France, 86, 1985, 97-136.

[Ful1] W. Fulton, *Intersection Theory.* Springer-Verlag.

[Groth] A. Grothendieck, *La théorie des classes de Chern.* Bulletin de la S.M.F., tome 86 (1958)m, 137-154.

[Har] R. Hartshorne, *Algebraic geometry.* Springer Science 2006.

[Hirz] F. Hirzebruch, *Topological methods in Algebraic Geometry.* Springer-Verlag.

[Liu] Q. Liu, *Algebraic geometry and arithmetic curves.* Addison-Wesley.

[Man] Y. Manin, *Lecture notes on the K-functor in algebraic geometry.* 1969 Russ. Math. Surv. 24 1.

[Mur] J. Murre, *Algebraic cycles and algebraic aspects of cohomology and K-theory.* Lecture Notes in Mathematics, 1594 Algebraic Cycles and Hodge Theory, Torino, 1993, pp. 93-152.

[Ser] J.P. Serre, *Algèbre Locale.* Springer-Verlag.

[Weibel] C. Weibel, *The K-book: An introduction to algebraic K-theory.* Available online.