COHERENT PRESENTATIONS OF MONOIDS WITH A RIGHT-NOETHERIAN GARSIDE FAMILY

PIERRE-LOUIS CURIEN, ALEN ĐURIĆ, AND YVES GUIRAUD

Abstract. This paper shows how to construct coherent presentations (presentations by generators, relations and relations among relations) of monoids admitting a right-noetherian Garside family. Thereby, it resolves the question of finding a unifying generalisation of the following two distinct extensions of construction of coherent presentations for Artin-Tits monoids of spherical type: to general Artin-Tits monoids, and to Garside monoids. The result is applied to some monoids which are neither Artin-Tits nor Garside.

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1. Introduction

1.1. **Coherent presentations of monoids.** A monoid can be presented by a generating set and a set of relations between words over the generating set. A coherent presentation of a monoid consists of a set of generators, a set of generating relations, and a set of generating relations among relations, having the property that, for every pair of parallel sequences of relations, there is a relation among relations between those two sequences.

Coherent presentations generalise 2-syzygies for presentations of groups. They form the first dimensions of polygraphic resolutions of monoids, from which abelian resolutions can be deduced. For motivation and context of the notion of coherent presentations, we refer the reader to [13]. In particular, it has been proved in [13] that Deligne's characterisation [9] of the weak actions of an Artin-Tits monoid $B^+(W)$ of spherical type on categories is equivalent to constructing a certain coherent presentation, denoted $\mathrm{Gar}_3(W)$ in [13], of $B^+(W)$. This construction has been extended in [13], using methods from rewriting theory, in two disjoint directions: to general Artin Tits monoids, and to Garside monoids. Coherent presentations are also studied in [10], under the name 3-presentations.

1.2. **Rewriting methods.** Generating relations, when considered directed from left to right (i.e. as ordered pairs), provide rewriting rules. A presentation is called terminating if there is no infinite rewriting sequence; it is called confluent if any two distinct rewriting sequences starting from the same word can be completed in such a way that they eventually lead to a common result; it is convergent if it is both terminating and confluent. A homotopical completion-reduction procedure, developed in [13], enriches a terminating presentation to a coherent one. The main element is Squier's theorem, which allows one to simply compute generators of the relations among relations for a convergent presentation. This procedure has three stages. Firstly, a Knuth-Bendix completion procedure enriches a terminating presentation to a convergent one by adding a (not necessarily finite) number of relations. Secondly, a Squier completion procedure adjoins relations among relations, thus providing a coherent presentation of the monoid admitting the starting presentation. Thirdly, a homotopical reduction procedure removes redundant relations. These homotopical transformations of presentations having certain properties are illustrated by the following diagram and recollected in Section 3.

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\begin{array}{c}\text{coherent}\\\text{reduced}\end{array} & \begin{array}{c}\text{coherent}\\\text{convergent}\end{array} \arrow[l, "\text{homotopical}", "\text{reduction}"'] \\
\text{terminating} \arrow[u, "\begin{array}{r}\text{homotopical}\\\text{completion-reduction}\end{array}"] \arrow[ur, "\begin{array}{c}\text{homotopical}\\\text{completion}\end{array}" description] \arrow[r, "\text{Knuth-Bendix}", "\text{completion}"'] & \text{convergent} \arrow[u, "\begin{array}{l}\text{Squier}\\\text{completion}\end{array}"']
\end{tikzcd} \tag{1.1}$$

Let us also illustrate the second stage by giving a preview of Example 3.2.3. Consider the following convergent presentation of the Klein bottle monoid:

$$\left\langle a, b \,\middle|\, bab \xrightarrow{\alpha} a, baa \xrightarrow{\beta} aab \right\rangle.$$

There are exactly two critical branchings, i.e. minimal overlaps of the rewriting steps: $\{\alpha ab, ba\alpha\}$ and $\{\alpha aa, ba\beta\}$. Both branchings are confluent. A Squier completion procedure adds the generators A and B of the relations among relations. Here are the shapes of A and B:

$$\begin{tikzcd}
babab \arrow[rr, Rightarrow, "\alpha ab"] \arrow[dr, Rightarrow, "ba\alpha"'] & \arrow[d, phantom, "A"] & aab \\
& baa \arrow[ur, Rightarrow, "\beta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
babaa \arrow[rrr, Rightarrow, bend left, "\alpha aa"] \arrow[dr, Rightarrow, "ba\beta"'] & & & aaa \\
& baaab \arrow[r, Rightarrow, "\beta ab"'] \arrow[r, phantom, shift left=6, "B"] & aabab \arrow[ur, Rightarrow, "aa\alpha"'] &
\end{tikzcd}.$$

In [13], Gaussent, the third author and Malbos have performed a homotopical completion-reduction procedure to compute coherent presentations of two disjoint generalisations of Artin-Tits monoids of spherical type: general Artin-Tits monoids, and Garside monoids. We recall those two generalisations in Subsection 3.5 as Examples 3.5.1 and 3.5.2, respectively. In [17], the third author, Malbos and Mimram have computed coherent presentations of plactic and Chinese monoids by applying a homotopical completion-reduction procedure.

1.3. **Garside families.** A Garside family in a monoid is a generating family, not minimal in general, but ensuring some desirable properties. Namely, the notion of a Garside family [5] is a result of successive generalisations to wider classes of monoids of a particular type of normal form, first implicitly hinted in braid monoids by Garside [12] in 1969, known as the greedy normal form. In particular, it generalises Artin-Tits monoids and Garside monoids. The greedy normal form is easily computed as it has very nice locality properties. These notions are recalled in Section 4.

Garside [12] investigated arithmetic properties of braid groups. He solved the word problem and the conjugacy problem in braid groups by introducing braid monoids. Among other things, he proved that the braid monoid B_n^+ is left-cancellative, and that any two elements of B_n^+ admit a least common multiple. He also introduced the Garside element (he called it the fundamental word) of a braid monoid.

Garside's observations for braid monoids were generalised to Artin-Tits monoids of spherical type by Brieskorn and Saito [1], and by Deligne who later explicitly gave Garside's presentation for Artin-Tits monoids of spherical type in [9]. Michel [19] extended this presentation to all Artin-Tits monoids.

The greedy normal form was later generalised to Artin-Tits monoids, based on Garside's observations (see [4, Introduction] for references). Dehornoy and Paris [8] introduced Garside monoids in order to abstract properties which establish the existence of the greedy normal form. Dehornoy, Digne and Michel [5] further generalised Garside monoids to categories admitting Garside families (as recalled for monoids in Subsection 4.2 here). A thorough development of the notion of a Garside family can be found in the book [4]. Dehornoy and the third author [7] introduced monoids admitting quadratic normalisations, thereby generalising monoids admitting Garside families. We refer the reader to the survey [3] for an overview of the successive extensions of the greedy normal form from braid monoids to monoids admitting left-weighted quadratic normalisations.

1.4. **Contributions.** The objective of the present paper is to unify the two above-mentioned results of [13] in the same generalisation. Namely, we apply a homotopical completion-reduction procedure to compute coherent presentations of a certain class of monoids admitting a Garside family. Our present contribution has the following two main steps.

(1) First, we use the fact that every left-cancellative monoid M containing no nontrivial invertible element and for every Garside family S in M, there is a presentation, here denoted $\mathrm{Gar}_2\,(S)$, having $S\setminus\{1\}$ as generating set, with generating relations α of the form $s|t = st$, for $s, t \in S \setminus \{1\}$ with $st \in S$ (Proposition 4.2.7, adapted from [7]). We observe (Example 4.2.8) that Garside's presentation $\mathrm{Gar}_2\,(W)$ of an Artin-Tits monoid $B^+\,(W)$ is a special case, with S being the Coxeter group W. Similarly (Example 4.2.9), Garside's presentation $\mathrm{Gar}_2\,(M)$ of a Garside monoid M is another special case of $\mathrm{Gar}_2\,(S)$.

(2) Then, starting from $\mathrm{Gar}_2\,(S)$, we embark on extending [13, Theorem 3.1.3] (which we recall in Example 3.5.1) to a wider class of monoids, including left-cancellative noetherian monoids containing no nontrivial invertible element, admitting a Garside family. Working in a more general setting, we encounter additional critical branchings which cannot occur in the case of Artin-Tits or Garside monoids due to specific properties not shared by Garside families in general. Therefore, we construct new generating relations among relations. Conveniently, we then remove all the additional relations using the homotopical reduction procedure.

This results in Theorem 5.1.4, our main result, of which we give here a weaker, but simpler version (our Corollary 5.5.1).

Theorem. *Assume that M is a left-cancellative noetherian monoid containing no nontrivial invertible element, and $S \subseteq M$ is a Garside family containing* 1. *Then M admits the coherent presentation* $\mathrm{Gar}_3\,(S)$ *which extends* $\mathrm{Gar}_2\,(S)$ *with the following set of generating relations among relations:*

$$\begin{tikzcd}
 & uv|w \arrow[dr, Rightarrow, "\alpha_{uv,w}"] & \\
u|v|w \arrow[ur, Rightarrow, "\alpha_{u,v}|w"] \arrow[dr, Rightarrow, "u|\alpha_{v,w}"'] \arrow[rr, phantom, "A_{u,v,w}"] & & uvw \\
 & u|vw \arrow[ur, Rightarrow, "\alpha_{u,vw}"'] &
\end{tikzcd},$$

for all $u, v, w \in S \setminus \{1\}$ such that $uv, vw, uvw \in S$.

Note that $A_{u,v,w}$ can be read as a relation ensuring associativity. We shall reach $\mathrm{Gar}_3\,(S)$ by applying a homotopical completion-reduction procedure to the presentation $\mathrm{Gar}_2\,(S)$.

In Section 6, the result is used to compute coherent presentations of some monoids which are neither Artin-Tits nor Garside, and to construct a finite coherent presentation of the Artin-Tits monoid of type $\widetilde{A}_2$, taking a finite generating set. In some cases, homotopical reduction can be carried further: as a matter of fact, in Subsection 6.3, we prove that Artin's presentation of the Artin-Tits monoid of type $\widetilde{A}_2$ is coherent (with the empty set of generating relations among relations).

We mainly consider monoids because that is where our applications lie, but the approach presented here can be extended to categories.

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2. Presentations of monoids by polygraphs

In this section, we briefly recall the notions concerning polygraphic presentations of monoids (technical elaboration whereof can be found in [13]). Basic terminology is given in Subsection 2.1. Some basic notions of polygraphic rewriting theory are recollected in Subsection 2.2. Subsection 2.3 recalls the notion of coherent presentation.

Throughout the present article, 2-categories and 3-categories are always assumed to be strict (see e.g. [16, Section 2]). In diagrams, distinct arrows are used to denote k-cells for low k: $\to$, $\Rightarrow$, $\Rrightarrow$ for k equal to 1, 2 and 3, respectively.

2.1. **Presentations by 2-polygraphs.** Polygraphs encompass words, rewriting rules, and homotopical properties of the rewriting systems in the same globular object. They provide a generalisation of a presentation of a monoid by generators and relations to the higher categories which are free up to codimension 1.

A polygraph is a higher-dimensional generalisation of a graph. Recall that a (directed) graph is a pair (X_0, X_1) of sets, together with two maps, called source and target, from X_1 to X_0. A 0-polygraph (X_0) is a set, a **1-polygraph** (X_0, X_1) is a graph. The free category generated by a 1-polygraph (X_0, X_1) is denoted by X_1^*. A **2-polygraph** is a triple $X = (X_0, X_1, X_2)$, where (X_0, X_1) is a 1-polygraph and X_2 is a set of **1-spheres**, i.e. pairs of parallel paths, in X_1^*.

For a 2-polygraph X, the **category presented by** X, denoted $\overline{X} = X_1^*/X_2$, is obtained by factoring out generating 2-cells, regarded as relations among 1-cells of X_1^*. For a monoid M, a **presentation of** M is a 2-polygraph X such that M is isomorphic to $\overline{X}$. In this case, which we are mainly interested in, X_0 is a singleton so any pair of paths in X_1^* forms a 1-sphere, and X_1^* is the free monoid generated by the set X_1. Elements of X_k are called **generating k-cells**.

Example 2.1.1 (The standard presentation)**.** Let M be a monoid. The **standard presentation** of M is the 2-polygraph $\mathrm{Std}_2(M)$ consisting of:

- one generating 0-cell x;
- a generating 1-cell $\widehat{u}$ for every element u of M;
- a generating 2-cell $\gamma_{u,v} : \widehat{u}\widehat{v} \Rightarrow \widehat{uv}$ for every pair of elements u and v of M;
- one generating 2-cell $\iota_x : 1_x \Rightarrow \widehat{1_x}$.

2.2. **Rewriting properties of 2-polygraphs.** Let us adopt some basic terminology from string rewriting. If S is a set, S^* denotes the free monoid over S. Elements of S and S^* are respectively called letters and **words**. We write $u|v$ for the concatenation of two words u and v, sometimes omitting the separation symbol when that does not cause ambiguity. Let M be a monoid generated by a set S. A **normal form** for M with respect to S is a set-theoretic section of the evaluation map (canonical projection) $\mathrm{ev} : S^* \to M$. In other words, a normal form maps elements of M to distinguished representative words. A word $s_1|\cdots|s_p$ is said to be a **decomposition** of an element f of M if the equality $s_1 \cdots s_p = f$ holds in M.

Assume that a 2-polygraph X is a presentation of a monoid M. Generating 2-cells of X are called **rewriting rules**. The **free 2-category over** X, denoted $X_2^* = X_1^*[X_2]$, is obtained by adjoining to X_1^* all the formal compositions of elements of X_2, treated as formal 2-cells. Standard notions from rewriting theory naturally translate into the framework of polygraphs. A **rewriting step** of a 2-polygraph X is a 2-cell of the free category X_2^* which contains a single generating 2-cell of X, here considered as a transformation of its source into its target. So, a rewriting step has a shape

$$\begin{tikzcd} \bullet \arrow[r, "w"] & \bullet \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\alpha"] & \bullet \arrow[r, "w'"] & \bullet \end{tikzcd} ,$$

where $\alpha : u \Rightarrow v$ is a generating 2-cell of X, and w and w' are 1-cells of X_2^*, and the 0-cell is denoted by $\bullet$.

Let u and v be 1-cells of X_2^*. It is said that u **rewrites** to v if there is a finite composable sequence of rewriting steps with source u and target v. A 1-cell u is **reduced** if there is no rewriting step whose source is u.

Let X be a 2-polygraph. A **termination order** on X is a well-founded order relation $\leq$ on parallel 1-cells of X_2^* enjoying the following properties:

- the compositions by 1-cells of X_2^* are strictly monotone in both arguments, i.e. $\leq$ is compatible with the composition of 1-cells;
- for every generating 2-cell α of X, the strict inequality $s(\alpha) > t(\alpha)$ holds.

A 2-polygraph X is **terminating** if it has no infinite sequence of rewriting steps. Admitting a termination order is equivalent to being terminating (in a terminating polygraph, a termination order is obtained by putting $u > v$ for 1-cells u and v if u rewrites to v).

A **branching** of a 2-polygraph X is an unordered pair $\{\alpha, \beta\}$ of sequences of rewriting steps of X_2^* having the same source, called the source of branching. If α and β are rewriting steps, a branching $\{\alpha, \beta\}$ is called **local**. A local branching is **trivial** if it has one of the following two shapes: $\{\alpha, \alpha\}$, or $\{\alpha v, u\beta\}$ for $u = s(\alpha)$ and $v = s(\beta)$. Local branchings can be compared by the order $\preccurlyeq$ generated by the relations $\{\alpha, \beta\} \preccurlyeq \{u\alpha v, u\beta v\}$ given for every local branching $\{\alpha, \beta\}$ and all possible 1-cells u and v of X_2^*. A minimal nontrivial local branching is called **critical**. A branching $\{\alpha, \beta\}$ is confluent if α and β can be completed into sequences having the same target. A 2-polygraph X is **confluent** (resp. locally confluent, resp. critically confluent) if all its branchings (resp. local branchings, resp. critical branchings) are confluent. If X is terminating and confluent, it is called **convergent**. A convergent 2-polygraph X is called a **convergent presentation** of any category isomorphic to $\overline{X}$. In that case, for every 1-cell u of X^*, there is a unique reduced word, denoted by $\widehat{u}$, to which u rewrites.

Two basic results of rewriting theory concerning confluence, called Newman's lemma [20, Theorem 3] and the critical branchings theorem respectively, are also valid for polygraphs.

Theorem 2.2.1 ([16, Theorem 3.1.6])**.** *Let X be a 2-polygraph.*

(1) *If X is terminating, then X is confluent if, and only if, it is locally confluent.*
(2) *X is locally confluent if, and only if, it is critically confluent.*

As a consequence of Theorem 2.2.1, a 2-polygraph is convergent if, and only if, it is terminating and its critical branchings are confluent.

Example 2.2.2. Consider the free abelian monoid:

$$\mathbb{N}^3 = \left\langle a, b, c \,\middle|\, ba \overset{\alpha}{\Rightarrow} ab, cb \overset{\beta}{\Rightarrow} bc, ca \overset{\gamma}{\Rightarrow} ac \right\rangle. \tag{2.1}$$

This presentation (2.1) admits the following termination order: comparing the lengths of words, then applying lexicographic order, induced by $a < b < c$, if words have the same length. Hence, it is terminating.

Let us illustrate confluence of (2.1) on the unique critical branching $\{\beta a, c\alpha\}$:

$$\begin{tikzcd}
 & bca \arrow[r, Rightarrow, "b\gamma"] & bac \arrow[dr, Rightarrow, "\alpha c"] & \\
cba \arrow[ur, Rightarrow, "\beta a"] \arrow[dr, Rightarrow, "c\alpha"'] & & & abc \\
 & cab \arrow[r, Rightarrow, "\gamma b"'] & acb \arrow[ur, Rightarrow, "a\beta"'] &
\end{tikzcd} .$$

Thus, the presentation (2.1) is convergent, by Theorem 2.2.1.

2.3. **Coherent presentations.** A 2-category (resp. 3-category) is called a $(2,1)$-category (resp. a $(3,1)$-category) if its 2-cells (resp. 2-cells and 3-cells) are invertible. For a 2-polygraph X, the **free** $(2,1)$**-category over** X, denoted $X_2^\top = X_1^*(X_2)$, is constructed by adjoining to X_1^* all the formal compositions of elements of X_2 and formal inverses of elements of X_2, and then factoring out the compositions of elements with their corresponding inverses. A $(3,1)$**-polygraph** is a quadruple $X = (X_0, X_1, X_2, X_3)$, where (X_0, X_1, X_2) is a 2-polygraph and X_3 is a set of **2-spheres**, i.e. pairs of parallel paths of 2-cells, in $X_2^\top$. For a $(3,1)$-polygraph X, the **free** $(3,1)$**-category over** X, denoted $X_3^\top = X_2^\top(X_3)$, is constructed by adjoining to $X_2^\top$ all the formal compositions of elements of X_3 and formal inverses of elements of X_3, and then factoring out the compositions of elements with their corresponding inverses. A $(3,1)$-polygraph is called convergent if its underlying 2-polygraph is. The **category presented by a** $(3,1)$**-polygraph** X is again $\overline{X}$, the category presented by its underlying 2-polygraph. An **extended presentation of a monoid** M is a $(3,1)$-polygraph X such that M is isomorphic to $\overline{X}$.

Definition 2.3.1. A **coherent presentation of** a monoid M is an extended presentation (X_0, X_1, X_2, X_3) of M such that factoring out elements of X_3, leaves only trivial 2-spheres (where the parallel paths are equal).

Example 2.3.2 (The standard coherent presentation)**.** Let us extend $\mathrm{Std}_2(M)$ from Example 2.1.1 with the following 3-cells

$$
\begin{tikzcd}
 & \widehat{uv}\widehat{w} \arrow[dr, Rightarrow, "\gamma_{uv,w}"] \arrow[d, phantom, "\Downarrow A_{u,v,w}"] & \\
\widehat{u}\widehat{v}\widehat{w} \arrow[ur, Rightarrow, "\gamma_{u,v}\widehat{w}"] \arrow[dr, Rightarrow, "\widehat{u}\gamma_{v,w}"'] & {} & \widehat{uvw} \\
 & \widehat{u}\widehat{vw} \arrow[ur, Rightarrow, "\gamma_{u,vw}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \widehat{1_x}\widehat{u} \arrow[dr, Rightarrow, "\gamma_{1_x,u}"] \arrow[d, phantom, "\Downarrow L_u"] & \\
\widehat{u} \arrow[ur, Rightarrow, "\iota_x\widehat{u}"] \arrow[rr, equal] & {} & \widehat{u}
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \widehat{u}\widehat{1_y} \arrow[dr, Rightarrow, "\gamma_{u,1_y}"] \arrow[d, phantom, "\Downarrow R_u"] & \\
\widehat{u} \arrow[ur, Rightarrow, "\widehat{u}\iota_y"] \arrow[rr, equal] & {} & \widehat{u}
\end{tikzcd}
$$

for every triple u, v, w of elements of M. The resulting $(3,1)$-polygraph, denoted by $\mathrm{Std}_3(M)$, is called the **standard coherent presentation** of M (see [14, Subsection 3.3.3] for the explanation why $\mathrm{Std}_3(M)$ is, indeed, a coherent presentation).

3. Homotopical transformations of polygraphs

This section elaborates the diagram (1.1), by recalling the notion of homotopical completion-reduction, introduced in [13]. Subsection 3.1 recollects the Knuth-Bendix completion procedure which transforms a terminating 2-polygraph into a convergent one. Subsection 3.2 recalls the Squier completion procedure which upgrades a convergent 2-polygraph to a convergent coherent $(3,1)$-polygraph. In Subsection 3.3, we report on the homotopical reduction procedure which turns a coherent $(3,1)$-polygraph into a coherent one having fewer generating cells. Finally, Subsection 3.4 describes a particular method for obtaining a homotopical reduction in case when the starting coherent $(3,1)$-polygraph is also convergent.

3.1. **Knuth-Bendix completion.** Starting with a terminating 2-polygraph X, equipped with a total termination order $\leq$, the Knuth-Bendix completion procedure adjoins generating 2-cells aiming to produce a convergent 2-polygraph, which presents a category presented by X. It works by iteratively examining all the critical branchings and adjoining a new generating 2-cell whenever the branching is not already confluent. Namely, for a critical branching $\{\alpha, \beta\}$, if $\widehat{t(\alpha)} > \widehat{t(\beta)}$ (resp. $\widehat{t(\beta)} > \widehat{t(\alpha)}$), a generating 2-cell $\gamma : \widehat{t(\alpha)} \Rightarrow \widehat{t(\beta)}$ (resp. $\gamma : \widehat{t(\beta)} \Rightarrow \widehat{t(\alpha)}$) is adjoined, thus forcing the confluence of the branching:

$$\begin{tikzcd} & t(\alpha) \arrow[r, Rightarrow] & \widehat{t(\alpha)} \arrow[dd, Rightarrow, dashed, "\gamma"] \\ * \arrow[ur, Rightarrow, "\alpha"] \arrow[dr, Rightarrow, "\beta"'] & & \\ & t(\beta) \arrow[r, Rightarrow] & \widehat{t(\beta)} \end{tikzcd} .$$

If new critical branchings are created by adjoining additional generating 2-cells, confluence of such critical branchings is examined. For details, see [16, p. 3.2.1]. This procedure is not guaranteed to terminate. In fact, its termination depends on the chosen termination order (see [11, Example 6.3.1]). If it does terminate, the result is a convergent 2-polygraph. Otherwise, it produces an increasing sequence of 2-polygraphs, and the result is the union of this sequence. Either way, the result is called a **Knuth-Bendix completion** of X. Note that different orders of examining critical branchings may result in different 2-polygraphs.

Theorem 3.1.1 ([16, Theorem 3.2.2]). *Assume that X is a 2-polygraph, equipped with a total termination order, presenting a monoid M. Then every Knuth-Bendix completion of X is a convergent presentation of M.*

Remark 3.1.2. The Knuth-Bendix completion procedure, as described above, requires not only termination, but also the presence of a total termination order, to be able to orient the generating 2-cells which are added, and to be able to maintain the termination during the completion. There is an alternative approach. Namely, we can orient the newly added generating 2-cells "by hand", according to our inspiration, and verify after each addition in an ad hoc manner whether we maintain a terminating presentation, without having defined a total order at the beginning (we shall do this in the proof of Proposition 5.3.1). Therefore, we can invoke Theorem 3.1.1 even if we do not provide a total order, as long as we are able to ensure termination after each addition of a generating 2-cell (we shall do this in the proof of Corollary 5.3.3).

3.2. **Squier completion.** A **family of generating confluences** of a convergent 2-polygraph X is a set of 2-spheres, treated as formal 3-cells, in $X_2^{\top}$ containing, for every critical branching $\{\alpha, \beta\}$ of X, exactly one 3-cell A:

$$\begin{tikzcd} & * \arrow[dr, Rightarrow, "\alpha'"] \arrow[dd, Rrightarrow, dashed, "A"] & \\ * \arrow[ur, Rightarrow, "\alpha"] \arrow[dr, Rightarrow, "\beta"'] & & * \\ & * \arrow[ur, Rightarrow, "\beta'"'] & \end{tikzcd} ,$$

where α' and β' are completing α and β, respectively, into sequences having the same target (such α' and β' exist by the assumption of confluence).

A **Squier completion** of a convergent 2-polygraph X is a $(3,1)$-polygraph with X as underlying 2-polygraph, whose generating 3-cells form a family of generating confluences of X. The following result is due to Squier; we state a version in terms of polygraphs and higher-dimensional categories proved in [16].

Theorem 3.2.1 ([16, Theorem 4.3.2]). *If X is a convergent presentation of a monoid M, then every Squier completion of X is a convergent coherent presentation of M.*

Theorem 3.2.1 is extended to higher-dimensional polygraphs in [15, Proposition 4.3.4].

Let X be a terminating 2-polygraph equipped with a total termination order $\leq$. A homotopical completion of X is a Squier completion of a Knuth-Bendix completion of X. We have seen that a

Knuth-Bendix completion procedure enriches a terminating 2-polygraph to a convergent one, and that the Squier completion of a convergent 2-polygraph X is a coherent presentation of $\overline{X}$. Those two transformations can be performed consecutively. They can also be performed simultaneously (see [13, p. 2.2.4]). The result is called a **homotopical completion** of X. Theorem 3.2.1 has the following consequence.

Theorem 3.2.2. *Assume that a 2-polygraph X is a terminating presentation of a monoid M. Then, every homotopical completion of X is a coherent convergent presentation of M.*

Example 3.2.3 (Klein bottle monoid)**.** We consider the Klein bottle monoid K^+, as defined in [4, Subsection I.3.2]. It has the following presentation:

$$\langle a, b \mid bab = a \rangle . \tag{3.1}$$

The name comes from the fact that K^+ is the submonoid generated by a and b of the fundamental group of the Klein bottle generated by a and b subject to relation $bab = a$. Every element of K^+ admits a unique expression of the form $a^p b^q$ for $p, q \geq 0$ or $a^p b^q a$ for $p \geq 0$ and $q \geq 1$. That form is called canonical.

Let us apply a homotopical completion procedure to the presentation (3.1). We have the generating 1-cells a and b, and a single generating 2-cell $\alpha : bab \Rightarrow a$. Let us adopt the following termination order: comparing the lengths of words, then applying lexicographic order, induced by $a < b$, if words have the same length. For instance, $b < aa < ab$. The only critical branching is $\{\alpha ab, ba\alpha\}$, with source $babab$. The homotopical completion procedure adjoins the generating 2-cell $\beta : baa \Rightarrow aab$, and the generating 3-cell A for coherence. The generating 2-cell β causes only one new critical branching, namely $\{\alpha aa, ba\beta\}$ with source $babaa$, which is confluent, hence only the generating 3-cell B is adjoined. Diagrammatically, the generating 3-cells have the shapes as follows:

$$\begin{tikzcd}
babab \arrow[rr, Rightarrow, "\alpha ab"] \arrow[dr, Rightarrow, "ba\alpha"'] & \arrow[d, phantom, "A"] & aab \\
& baa \arrow[ur, Rightarrow, "\beta"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
babaa \arrow[rrr, Rightarrow, bend left, "\alpha aa"] \arrow[dr, Rightarrow, "ba\beta"'] & \arrow[r, phantom, "B"] & & aaa \\
& baaab \arrow[r, Rightarrow, "\beta ab"'] & aabab \arrow[ur, Rightarrow, "aa\alpha"'] &
\end{tikzcd} .$$

By Theorem 3.2.2, we have thus obtained a convergent coherent presentation of the Klein bottle monoid, consisting of two generating 1-cells, two generating 2-cells, and two generating 3-cells:

$$\left(a, b \,\middle|\, bab \overset{\alpha}{\Rightarrow} a, baa \overset{\beta}{\Rightarrow} aab \,\middle|\, A, B \right) .$$

Remark 3.2.4. For convenience, we mostly leave implicit the orientation of the 3-cells in the diagrams. We only label the corresponding area with the name of a 3-cell. The convention is that the source and the target of a 3-cell are always the upper and the lower paths, respectively, of the sphere bounding the area.

3.3. **Homotopical reduction.** A coherent presentation obtained by the homotopical completion procedure is not necessarily minimal, in the sense that it may contain superfluous cells. The homotopical reduction procedure aims to remove such superfluous cells by performing a series of elementary collapses, analogous to that used by Brown in [2]. We refer the reader to [13, Subsection 2.3] for a technical elaboration.

An **elementary Nielsen transformation** on a $(3,1)$-polygraph X is any of the following operations:

- replacement of a 2-cell or a 3-cell with its formal inverse;

- replacement of a 3-cell $A : \alpha \Rrightarrow \beta$ with

$$\begin{tikzcd}[column sep=small]
 & * \arrow[rr, Rightarrow, "\alpha"] \arrow[dd, phantom, ""] & & * \arrow[dr, Rightarrow, "\chi'"] & \\
* \arrow[ur, Rightarrow, "\chi"] \arrow[dr, Rightarrow, "\chi"'] & & \Downarrow \widetilde{A} & & * \\
 & * \arrow[rr, Rightarrow, "\beta"'] & & * \arrow[ur, Rightarrow, "\chi'"'] &
\end{tikzcd},$$

where χ and χ' are 2-cells of $X_3^\top$.

Elementary Nielsen transformations preserve presented 1-categories, equivalence of presented $(2,1)$-categories and homotopy type of $(3,1)$-polygraphs (see [13, p. 2.1.4]). In particular, they transform a coherent presentation of a monoid M into another coherent presentation of M. A **Nielsen transformation** is a composition of elementary ones. In a homotopical completion-reduction procedure, Nielsen transformations are performed implicitly for convenience.

Let X be a $(3,1)$-polygraph. A generating 2-cell (resp. 3-cell, resp. 3-sphere) α of X is called **collapsible** if it meets the following two requirements:

- the target of α is a generating 1-cell (resp. 2-cell, resp. 3-cell) of X,
- the source of α is a 1-cell (resp. 2-cell, resp. 3-cell) of the free $(3,1)$-category over $X \setminus \{t(\alpha)\}$.

For a $(3,1)$-polygraph $X = (X_0, X_1, X_2, X_3)$, a **collapsible part** of X is a triple $\Gamma = (\Gamma_2, \Gamma_3, \Gamma_4)$, wherein Γ_2, Γ_3, Γ_4 respectively denote families of generating 2-cells of X, generating 3-cells of X, 3-spheres of $X_3^\top$, such that the following requirements are met:

- every γ of every Γ_k is collapsible (possibly up to a Nielsen transformation);
- no γ of Γ_k is the target of an element of Γ_{k+1};
- there exist well-founded order relations on X_1, X_2 and X_3 such that, for every γ in every Γ_k, the target of γ is strictly greater than every generating $(k-1)$-cell that occurs in the source of γ.

The result of the **homotopical reduction of X with respect to Γ** is the $(3,1)$-polygraph which we denote X/Γ, whose generating cells are

$$X/\Gamma = \left(X_0, X_1 \setminus t(\Gamma_2), X_2 \setminus t(\Gamma_3), X_3 \setminus t(\Gamma_4)\right).$$

Sources and targets are given by $\pi_\Gamma \circ s$ and $\pi_\Gamma \circ t$, where π_Γ is the 3-functor from $X^\top$ to $(X/\Gamma)^\top$ given by the recursive formula

$$\pi_\Gamma(x) = \begin{cases} \pi_\Gamma(s(\gamma)) & \text{if } x = t(\gamma) \text{ for } \gamma \text{ in } \Gamma \\ 1_{\pi_\Gamma(s(x))} & \text{if } x \text{ in } \Gamma \\ x & \text{otherwise.} \end{cases}$$

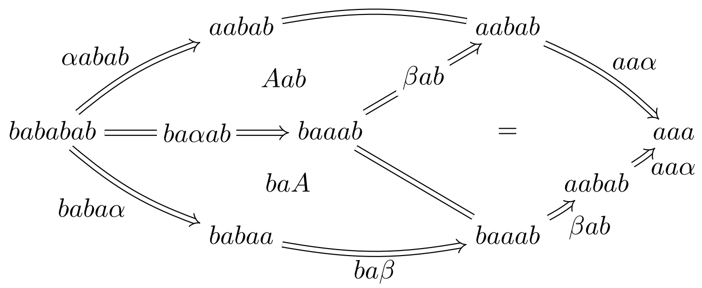
Such a transformation is called the homotopical reduction procedure.

Let X be a terminating 2-polygraph, with a termination order $\leq$. A **homotopical completion-reduction** of X is a $(3,1)$-polygraph, obtained as a homotopical reduction, with respect to a collapsible part, of a homotopical completion of X. Theorem 3.2.1 implies the following result.

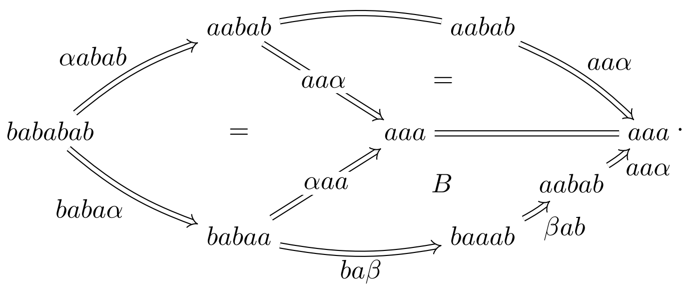
Theorem 3.3.1. *Assume that X is a terminating 2-polygraph presenting a monoid M. Then, every homotopical completion-reduction of X is a coherent presentation of M.*

3.4. **Special case of reduction.** We have just recalled the definition of a generic collapsible part of a $(3,1)$-polygraph X. For the applications considered here, however, it is practical to also recall a particular technique, described in [13, p. 3.2], to construct a collapsible part in the case when X is convergent and coherent. A **local triple branching** is an unordered triple $\{\alpha, \beta, \gamma\}$ of rewriting steps having a common source. A local triple branching is **trivial** if two of its components are equal or if one of its components forms branchings of the type $\{\alpha v, u\beta\}$, for $u = s(\alpha)$ and $v = s(\beta)$, with the other two. In a manner analogous to the case of local branchings, local triple branchings can be ordered by "inclusion", and a minimal nontrivial local triple branching is called **critical**. A **generating triple confluence** of X is a particular kind of 3-sphere Φ constructed using a critical triple branching. Referring the reader to [13, Subsection 3.2] for elaboration of the technique, we illustrate it by means of an example.

Example 3.4.1. Let us perform a homotopical reduction procedure on the homotopical completion of the Klein bottle monoid, computed in Example 3.2.3. We construct a collapsible part $\Gamma = (\Gamma_2, \Gamma_3, \Gamma_4)$. There is only one critical triple branching, namely $\{\alpha abab, ba\alpha ab, baba\alpha\}$. It yields a generating triple confluence, denoted Φ, whose boundary consists of the following two parts (we display the 3-cells A and B differently now, to make the generating triple confluence more evident):



and



Hence the component Γ_4 of the collapsible part contains the 3-sphere Φ which has the 3-cell B as target (recall that we implicitly perform a higher Nielsen transformation when needed). Hence the component Γ_4 of the collapsible part contains the 3-sphere Φ which has the 3-cell B as target. By the definition of a collapsible part, we also need to provide a well-founded order relation on the set of generating 3-cells, such that, for every 3-sphere (X, Y) in Γ_4, the target Y is strictly greater than every generating 3-cell that occurs in the source X. So, we put $B > A$. Proceeding as described in [13, Subsection 3.2], we examine the remaining 3-cells and construct the component Γ_3 out of those 3-cells whose boundary contains a generating 2-cell occurring only once in the boundary. There is only one 3-cell left, namely A, and the 2-cell β appears only once in the boundary of A. So, Γ_3 contains A, and we order the set of generating 2-cells by setting $\beta > \alpha$. The component Γ_2 is empty because there is no 2-cell whose source or target consists of a single generating 1-cell appearing only once.

Thus, after performing a homotopical reduction procedure with respect to the collapsible part $(\emptyset, \Gamma_3, \Gamma_4)$, we are left with the presentation

$$\left(a, b \,\middle|\, bab \overset{\alpha}{\Rightarrow} a \,\middle|\, \emptyset\right)$$

which is thus coherent by Theorem 3.3.1. Note that having a coherent presentation X with the empty set of generating 3-cells means that any two parallel rewriting paths represent the same 2-cell in $X_3^{\top}$.

3.5. **Application to Artin-Tits and Garside monoids.** In this subsection, we recollect two instances of a homotopical completion-reduction procedure, illustrating the results of [13, Section 3]. We shall recall these examples in Subsection 5.1, as the theorems of [13, Section 3] are special cases of our main result.

First, let us adopt a terminology concerning divisibility in monoids. A monoid M is **left-cancellative** (resp. **right-cancellative**) if for all f, g and g' of M, the equality $fg = fg'$ (resp. $gf = g'f$) implies the equality $g = g'$. A monoid is **cancellative** if it is both left-cancellative and right-cancellative.

An element f of a monoid M is said to be a **left divisor** of $g \in M$, and g is said to be a **right multiple** of f, denoted by $f \preceq g$, if there is an element $f' \in M$ such that $ff' = g$. If, additionally, f' is not invertible, then divisibility is called **proper**. We say that f is a proper left divisor of g, written as $f \prec g$, if $f \preceq g$ and $g \npreceq f$. If M is left-cancellative, then the element f' is uniquely defined and called **the right complement of f in g**.

For an element h of a left-cancellative monoid M and a subfamily S of M, we say that h is a **left-gcd** (resp. **right-lcm**) of S if $h \preceq s$ (resp. $s \preceq h$) holds for all $s \in S$ and if every element of M which is a left divisor (resp. right multiple) of all $s \in S$ is also a left divisor (resp. right multiple) of h.

A (proper) right divisor, a left multiple, a left complement, a left-lcm and a right-gcd are defined similarly.

We say that a left-cancellative monoid M **admits conditional right-lcms** if any two elements having a common right multiple have a right-lcm.

Example 3.5.1. Let W be a Coxeter group (see e.g. [13, Section 3]), and $B^+(W)$ the corresponding Artin-Tits monoid. Garside's presentation of the $B^+(W)$, seen as a 2-polygraph and denoted by $\mathrm{Gar}_2(W)$, has a single generating 0-cell, elements of $W \setminus \{1\}$ as generating 1-cells, and a generating 2-cell

$$\alpha_{u,v} : u|v \Rightarrow uv$$

for all $u, v \in W \setminus \{1\}$ such that $\ell(uv) = \ell(u) + \ell(v)$ holds, where $\ell(u)$ denotes the common length of all reduced expressions of u. Let $\mathrm{Gar}_3(W)$ denote the extended presentation of $B^+(W)$ obtained by adjoining to $\mathrm{Gar}_2(W)$ a generating 3-cell

$$\begin{tikzcd}
 & uv|w \arrow[dr, Rightarrow, "\alpha_{uv,w}"] & \\
u|v|w \arrow[ur, Rightarrow, "\alpha_{u,v}|w"] \arrow[dr, Rightarrow, "u|\alpha_{v,w}"'] & A_{u,v,w} & uvw \\
 & u|vw \arrow[ur, Rightarrow, "\alpha_{u,vw}"'] &
\end{tikzcd}$$

for all u, v and w of $W \setminus \{1\}$ such that $\ell(uv) = \ell(u) + \ell(v)$ and $\ell(vw) = \ell(v) + \ell(w)$ and $\ell(uvw) = \ell(u) + \ell(v) + \ell(w)$ hold. By [13, Theorem 3.1.3], $\mathrm{Gar}_3(W)$ is a homotopical completion-reduction of $\mathrm{Gar}_2(W)$ so, by Theorem 3.3.1, it is a coherent presentation of $B^+(W)$.

Example 3.5.2. Recall that a **Garside monoid** (see [4, Definition I.2.1]) is a pair (M, Δ) such that the following conditions hold:

(1) M is a cancellative monoid;
(2) there is a map $\lambda : M \to \mathbb{N}$ such that $\lambda(fg) \geq \lambda(f) + \lambda(g)$ and $\lambda(f) = 0 \implies f = 1$;
(3) every two elements have a left-gcd and a right-gcd and a left-lcm and a right-lcm;

(4) there is element Δ, called the Garside element, such that the left and the right divisors of Δ coincide, and they generate M;
(5) the family of all divisors of Δ is finite.

We write $f \wedge g$ for the left-gcd of f and g. For a (left) divisor f of Δ, we write $\partial(f)$ for the right complement of f in Δ.

Garside's presentation of a Garside monoid M is the 2-polygraph $\mathrm{Gar}_2(M)$, having divisors of Δ, other than 1, as generating 1-cells and a generating 2-cell $\alpha_{u,v} : u|v \Rightarrow uv$ whenever the condition $\partial(u) \wedge v = v$ is satisfied. To be able to define generating 3-cells, we need to generalise this condition, in a suitable way, to three elements. Let us first observe that the condition $\partial(u) \wedge v = v$ is equivalent to saying that v is a left divisor of $\partial(u)$. In other words, there is w in M such that $vw = \partial(u)$. By definition of $\partial(u)$, this means that $uvw = \Delta$, so uv is a divisor of Δ. This reformulation allows an extension of the given condition to a greater number of elements. Let $\mathrm{Gar}_3(M)$ denote the extended presentation of M obtained by adjoining to $\mathrm{Gar}_2(M)$ a generating 3-cell

$$\begin{tikzcd}
 & uv|w \arrow[dr, Rightarrow, "\alpha_{uv,w}"] & \\
u|v|w \arrow[ur, Rightarrow, "\alpha_{u,v}|w"] \arrow[dr, Rightarrow, "u|\alpha_{v,w}"'] \arrow[rr, phantom, "A_{u,v,w}"] & & uvw \\
 & u|vw \arrow[ur, Rightarrow, "\alpha_{u,vw}"'] &
\end{tikzcd}$$

for all u, v and w divisors of Δ, not equal to 1, such that uv, vw and uvw are divisors of Δ. By Theorem [13, Theorem 3.3.3], $\mathrm{Gar}_3(M)$ is a homotopical completion-reduction of $\mathrm{Gar}_2(M)$ so, by Theorem 3.3.1, it is a coherent presentation of M.

4. Garside families

This section briefly recollects the basic notions and results concerning Garside families (for technical elaboration, see the book [4]).

4.1. **Right-mcms.** Let M be a left-cancellative monoid, and S a subfamily of M. The left divisibility relation $\preceq$ is a preorder of elements; it is an order if, and only if, M has no nontrivial invertible element.

A subfamily S of a left-cancellative monoid M is **closed under right comultiple** if every common right multiple of two elements f and g of S (if there is any) is a right multiple of a common right multiple of f and g that lies in S.

For f and g in a monoid M, a minimal common right multiple, or **right-mcm**, of f and g if is a right multiple h of f and g, such that no proper left divisor of h is a common right multiple of f and g. A monoid M **admits right-mcms** if, for all f and g of M, every common right multiple of f and g is a right multiple of some right-mcm of f and g. Observe that in a monoid admitting conditional right-lcms, the notions of a right-mcm and right-lcm coincide. Let us state a rather basic observation about right-mcm in a left-cancellative monoid, which we use in one step of the main proof in Subsection 5.3. The following lemma is similar to [18, Lemma 11.24], which deals with lcms whereas here it suffices to consider mcms (under weaker assumptions).

Lemma 4.1.1. *Assume that M is a left-cancellative monoid. If v' is a right-mcm of v_1 and v_2 in M, then uv' is a right-mcm of uv_1 and uv_2 for every u in M.*

Following [4, Propositions II.2.28 and II.2.29], a left-cancellative monoid M is said to be **left-noetherian** (resp. **right-noetherian**) if for every g in M, every increasing sequence of right (resp. left) divisors of g with respect to proper right divisibility (resp. left divisibility) is finite. A left-cancellative monoid M is **noetherian** if it is both left-noetherian and right-noetherian.

Example 4.1.2. Proper division, left or right, strictly reduces the length of an element of an Artin-Tits monoid. Therefore, no element admits an infinite number of divisors, so Artin-Tits monoids are noetherian.

Garside monoids are noetherian by definition (thanks to the map $\lambda : M \to \mathbb{N}$).

4.2. **Notion of a Garside family.** In this subsection, we recollect the definition and some basic properties of the all-important notion of a Garside family which provides a way of extending the notion of a greedy decomposition beyond Garside monoids.

Given a subfamily S of a left-cancellative monoid M, an M-word $g_1|\cdots|g_q$ is said to be S-**greedy** if for all $i < q$,

$$\forall h \in S, \forall f \in M, (h \preceq f g_i g_{i+1} \implies h \preceq f g_i).$$

In other words, if the diagram

$$\begin{tikzcd}
\bullet \arrow[r, "h"] \arrow[d, "f"'] & \bullet \arrow[d, dashed] \arrow[dr, bend left] & \\
\bullet \arrow[r, "g_i"] & \bullet \arrow[r, "g_{i+1}"] & \bullet
\end{tikzcd}$$

commutes without the dashed arrow, then there exists a dashed arrow making the square on the left commute. The arc joining g_i and g_{i+1} denotes greediness. By definition, a word of length zero or one is S-greedy for any subfamily S.

Given a subfamily S of M, an M-word $g_1|\cdots|g_q$ is said to be S-**normal** if it is S-greedy and if, moreover, $g_1, \ldots, g_q$ all lie in S. An S-normal word $g_1|\cdots|g_q$ is **strict** if $g_q \neq 1$. Observe that the existence of an S-normal form implies the existence of a strict one.

Note that, by the very definition of being greedy, a word is normal if, and only if, its length-two factors are. More is true: the procedure of transforming a word into its normal form consists of transforming its length-two factors (we refer the reader to [7] for elaboration).

In general, an S-**normal** decomposition of an element g of M is not unique. Nevertheless, the number of non-invertible letters in all S-normal decompositions of g is the same (see [5, Proposition 2.11] or [4, Proposition III.1.25] for exposition). If M has no nontrivial invertible element, then every g in M admits at most one strict S-normal decomposition. Given a subfamily S of a left-cancellative monoid M, and an element g of M admitting at least one S-normal decomposition, one defines the S-**length** of an element $g \in M$ to be the common number of non-invertible letters in all S-normal decompositions of g.

A subfamily S of a left-cancellative monoid M is called a **Garside family** in M if every element of M admits an S-normal decomposition. Since every left-cancellative monoid M is a Garside family in itself (for every g in M, simply take a length-one word g as a M-normal decomposition of g), we are interested only in proper (meaning other than M itself) Garside families. Observe that, if M has no nontrivial invertible element and S is a Garside family in M, then every element of M admits a unique strict S-normal decomposition.

Example 4.2.1. Every Artin-Tits monoid admits a finite Garside family. In the case of an Artin-Tits monoid of spherical type, a finite Garside family is given by the corresponding Coxeter group. In the particular case of a braid monoid, the family of all simple braids is a Garside family.

The Coxeter group W which corresponds to a general Artin-Tits monoid $B^+(W)$ is a possibly infinite Garside family, but $B^+(W)$ admits a finite Garside family in any case (see [6]).

Any Garside monoid (M, Δ) has a finite Garside family given by the family of all divisors o Δ (see [5, Proposition 2.18] or [4, Proposition III.1.43]).

The following proposition gives a simple characterisation of a Garside family.

Proposition 4.2.2 ([5, Proposition 3.1] or [4, Proposition III.1.39]). *A subfamily S of a monoid M containing no nontrivial invertible element is a Garside family if, and only if, the following conjunction holds: S generates M and every element of S^2 admits an S-normal decomposition.*

Let us recall another characterisation of Garside family, one direction whereof we invoke in Subsection 5.3. More characterisations of Garside families can be found in [5, Subsection 3.2] or in [4, Subsection IV.1.2].

Proposition 4.2.3 ([5, Proposition 3.9]). *A family S of a left-cancellative monoid M containing no nontrivial invertible element is a Garside family if, and only if, the following conditions are satisfied: S generates M, it is closed under right comultiple and right divisor, and every non-invertible element of S^2 admits a $\prec$-maximal left divisor in S.*

We recall another result to be used in Subsection 5.3.

Lemma 4.2.4 ([4, Lemma IV.2.24]). *Assume that M is a left-cancellative monoid that contains no nontrivial invertible element and admits right-mcms. Then for every subfamily S of M, the following are equivalent.*

- *The family S is closed under right comultiple.*
- *The family S is* ***closed under right-mcm****, i.e. if f and g lie in S, then so does every right-mcm of f and g.*

Given a Garside family S in a left-cancellative monoid M with no nontrivial invertible element, the **normalisation map** $N^S : S^* \to S^*$ is the map which assigns to each $w \in S^* \setminus \{1\}$ the strict S-normal decomposition of the element of M represented by w; and $N^S(1) = 1$. The following result provides an important property of S-normal decomposition.

Lemma 4.2.5 ([7, Lemma 6.9]). *Assume that M is a left-cancellative monoid having no nontrivial invertible element, and S is a Garside family in M. For every word $w \in S^*$, the leftmost letter of w left-divides the leftmost letter of $N^S(w)$.*

Proof. Let $N^S(w) = s_1|\cdots|s_q$. Since w and $s_1|s_2\cdots s_q$ evaluate to the same element of M, the leftmost letter of w left-divides $s_1(s_2\cdots s_q)$. By *[5, Lemma 2.12]*, for a subfamily S of a left-cancellative monoid M, if an M-word $g_1|\cdots|g_q$ is S-greedy, then $g_1|g_2\cdots g_q$ is S-greedy, as well. Hence, the length-two M-word $s_1|s_2\cdots s_q$ is S-greedy. □

A normalisation map satisfying the conclusion of Lemma 4.2.5, but limited to S-words of length two, is called **left-weighted** in [7, Subsection 6.2], and Lemma 4.2.5 is a (quite straightforward) generalisation of [7, Lemma 6.9] to all S-words.
A Garside family yields a presentation in the following sense.

Proposition 4.2.6 ([7, Proposition 6.17] or [14, Corollary 6.6.4]). *Assume that M is a left-cancellative monoid containing no nontrivial invertible element, and $S \subseteq M$ is a Garside family. Then M admits, as a convergent presentation, the 2-polygraph $(\{\bullet\}, S\setminus\{1\}, X_S)$, where $\{\bullet\}$ denotes a singleton and X_S is the set of generating 2-cells of the form*

$$s|t \Rightarrow N^S(s|t) \tag{4.1}$$

for all s and t in $S\setminus\{1\}$ such that $s|t$ is not S-normal. In particular, every Artin-Tits monoid admits a finite convergent presentation.

A Garside family also induces a "smaller" presentation, beside the one provided by Proposition 4.2.6, which will be instrumental in deriving our main result in the next section. The following proposition is adapted from [7, Propositions 6.10 and 6.15].

Proposition 4.2.7. *Assume that M is a left-cancellative monoid containing no nontrivial invertible element, and $S \subseteq M$ is a Garside family containing 1. Then M admits, as a presentation, the 2-polygraph $\mathrm{Gar}_2(S)$ which contains a single generating 0-cell, one generating 1-cell for every element of $S \setminus \{1\}$, and one generating 2-cell of the form*

$$s|t \Rightarrow st, \tag{4.2}$$

for all s and t in $S \setminus \{1\}$ whose product st in M lies in S.

Proof. Proposition 4.2.6 grants a presentation of M in terms of S by the relations 4.1. Let us show that the relations (4.2) are included in the relations (4.1). If s and t in $S \setminus \{1\}$ are such that st lies in $S \setminus \{1\}$, then the strict S-decomposition of $s|t$ is st, hence $N^S(s|t) = st$. Otherwise, $st = 1$ holds and yields $N^S(s|t) = 1$. In both cases, the strict S-normal decomposition of $s|t$ is st. Hence, the relations (4.2) are included in the relations (4.1).

Conversely, let us show that each relation (4.1), with s and t in $S \setminus \{1\}$, follows from a finite number of relations (4.2). Assume that s and t lie in $S \setminus \{1\}$ and let $s'|t' := N^S(s|t)$. If $t' = 1$ holds, it implies $s' = st$, which is a (4.2) relation, so the result is true in this case. Otherwise, Lemma 4.2.5 implies that there exists r in M, satisfying $sr = s'$, which is a (4.2) relation. Being a right divisor of $s' \in S$, the element r also lies in S by Proposition 4.2.3. Multiplying the equality $sr = s'$ by t' on the right yields $srt' = s't' = st$. Then the left cancellation property of M implies $rt' = t$, which is a (4.2) relation. Since the relation $s|t = s|r|t' = s'|t'$ follows from the (4.2) relations $s|r = s'$ and $r|t' = t$, the result is true in this case, too. $\square$

We call the 2-polygraph $\mathrm{Gar}_2(S)$ the **Garside's presentation** of M, with respect to the Garside family S. We study it in the next section. Here, let us just observe that it extends the Garside's presentation of Artin-Tits monoids, recalled in Subsection 3.5.

Example 4.2.8. Garside's presentation $\mathrm{Gar}_2(W)$ of an Artin-Tits monoid $B^+(W)$ is an instance of a Garside's presentation $\mathrm{Gar}_2(S)$ with respect to a Garside family S. Indeed, the Artin-Tits monoid $B^+(W)$ is a cancellative monoid (see [1]) with no nontrivial invertible element, and the Coxeter group W is a Garside family containing 1; hence $B^+(W)$ meets all the requirements of Proposition 4.2.7 which, for this particular input of W for S, produces precisely Garside's presentation $\mathrm{Gar}_2(W)$.

Example 4.2.9. Garside's presentation $\mathrm{Gar}_2(M)$ of a Garside monoid M is another instance of a Garside's presentation with respect to Garside family S. Namely, M is cancellative, by definition. Note that the property (2) of Garside monoid implies that it has no nontrivial invertible element. All the divisors of Δ form a Garside family. If we take this Garside family for S, Proposition 4.2.7 yields Garside's presentation $\mathrm{Gar}_2(M)$.

5. Coherent presentations from Garside families

Having recalled necessary notions and results in previous sections, in this section we aim to state and prove Theorem 5.1.4 which provides a unifying generalisation of theorems recalled in Examples 3.5.1 and 3.5.2.

5.1. **Main statement and sketch of proof.**

In this subsection, we adapt some notation from [13] and set a convenient noetherianity condition. Then we state our main result.

Let M be a monoid generated by a set S containing 1. We define the notations $u \frown v$ and $u \,{}^{\times} v$, as follows. Given two elements u and v of $S \setminus \{1\}$, we write:

$$u \frown v \iff uv \in S,$$
$$u \,{}^{\times} v \iff uv \notin S.$$

The notation extends to a greater number of elements. For three elements $u, v, w \in S$, we write $u \frown v \frown w$ if both conditions $uv \in S$ and $vw \in S$ hold. The condition $u \frown v \frown w$ splits into two mutually exclusive subcases:

$$
\begin{aligned}
\overset{\frown}{u \frown v \frown w} \quad &\Longleftrightarrow \quad \Big(u \frown v \frown w \quad \text{and} \quad uvw \in S \Big), \\
\overset{\times}{u \frown v \frown w} \quad &\Longleftrightarrow \quad \Big(u \frown v \frown w \quad \text{and} \quad uvw \notin S \Big).
\end{aligned}
$$

We formally redefine symbols Gar_2 and Gar_3 in our general context as follows. The 2-polygraph $\mathrm{Gar}_2(S)$ contains: a single generating 0-cell; one generating 1-cell for every element of $S \setminus \{1\}$; one generating 2-cell of the form

$$\alpha_{u,v} : u|v \Rightarrow uv,$$

for all u and v in $S \setminus \{1\}$ such that $u \frown v$ holds. Here, $u|v$ denotes product in S^*, whereas uv denotes product in M. The $(3,1)$-polygraph $\mathrm{Gar}_3(S)$ is consisting of the 2-polygraph $\mathrm{Gar}_2(S)$ and the generating 3-cells of the form

$$
\begin{tikzcd}
 & uv|w \arrow[dr, Rightarrow, "\alpha_{uv,w}"] & \\
u|v|w \arrow[ur, Rightarrow, "\alpha_{u,v}|w"] \arrow[dr, Rightarrow, "u|\alpha_{v,w}"'] \arrow[rr, phantom, "A_{u,v,w}"] & & uvw \\
 & u|vw \arrow[ur, Rightarrow, "\alpha_{u,vw}"'] &
\end{tikzcd}
$$

for all u, v and w in $S \setminus \{1\}$ such that $\overset{\frown}{u \frown v \frown w}$.

Remark 5.1.1. Note that the 2-polygraph $\mathrm{Gar}_2(S)$ is not a presentation of M, in general. Consequently, since $\mathrm{Gar}_3(S)$ is an extended presentation of a monoid presented by $\mathrm{Gar}_2(S)$, it is not necessarily an extended presentation of M. Proposition 4.2.7 gives sufficient conditions for $\mathrm{Gar}_2(S)$ to be a presentation of M, thus making $\mathrm{Gar}_3(S)$ an extended presentation of M.

To formulate our main result, we need a restriction of right noetherianity to a Garside family.

Definition 5.1.2. Given a Garside family S in a left-cancellative monoid M, we say that S is **right-noetherian** if for every g in S, every increasing sequence of proper left divisors in S of g with respect to proper left divisibility is finite.

Example 5.1.3. Every Garside family in a right-noetherian left-cancellative monoid M is right-noetherian.

Now, we state the main result.

Theorem 5.1.4. *Assume that M is a left-cancellative monoid containing no nontrivial invertible element, and admitting a right-noetherian Garside family S containing* 1. *If M admits right-mcms, then M admits the $(3,1)$-polygraph $\mathrm{Gar}_3(S)$ as a coherent presentation.*

Before we proceed to prove the theorem, let us show that it gives a common generalisation of the two distinct directions of extension, given in [13], of Deligne's result [9, Theorem 1.5].

Corollary 5.1.5 ([13, Theorem 3.1.3])**.** *For every Coxeter group W, the Artin-Tits monoid $B^+(W)$ admits $\mathrm{Gar}_3(W)$ as a coherent presentation.*

Proof. Let us restrict the conditions $u \frown v$ and $\overset{\frown}{u \frown v \frown w}$, defined in the beginning of the current subsection, to the case of the Artin-Tits monoid $B^+(W)$, with the Coxeter group W as Garside family S. Observe that, for $u, v \in W \setminus \{1\}$, the condition $u \frown v$, i.e. $uv \in W$, boils down to the condition $\ell(uv) = \ell(u) + \ell(v)$ given in Example 3.5.1 (see Matsumoto's lemma, e.g. [4, Corollary IX.1.11]). Accordingly, the condition $\overset{\frown}{u \frown v \frown w}$ becomes the conjunction of $u \frown v$ and $v \frown w$ and $\ell(uvw) = \ell(u) + \ell(v) + \ell(w)$. Recall that Artin-Tits monoids and Garside monoids are cancellative and noetherian (Example 4.1.2), and that they contain no nontrivial invertible element. Consequently, Theorem 5.1.4 specialises to [13, Theorem 3.1.3] when a monoid considered is Artin-Tits with Coxeter group as a Garside family. $\square$

Similarly, one shows that Theorem 5.1.4 specialises to [13, Theorem 3.3.3] when a monoid considered is Garside with S being the set of divisors of the Garside element.

Corollary 5.1.6 ([13, Theorem 3.3.3]). *Every Garside monoid M admits $\mathrm{Gar}_3(M)$ as a coherent presentation.*

Proof. If we restrict the conditions $u \frown v$ and $\overset{\frown}{u \frown v \frown w}$ to the case of a Garside monoid (M, Δ), with divisors of Δ as Garside family S, then we get precisely our equivalent reformulation, given in Example 3.5.2, of the conditions stated in [13, Subsection 3.3]. Literally, the condition $\overset{\frown}{u \frown v \frown w}$ then says that uv is an element of the set of divisors of Δ. Garside monoids are cancellative by definition. Note that the property (2) of a Garside monoid implies noetherianity as well as the fact that there are no nontrivial invertible elements. $\square$

The following diagram summarises key steps of the proof and 3.5.2 and thus motivates the next three subsections (which together contain the proof).

$$\begin{tikzcd}
\begin{matrix}\mathrm{Gar}_3(S)\\ \text{coherent, reduced}\end{matrix} & & \begin{matrix}\underline{\mathrm{Gar}}_3(S)\\ \text{coherent, convergent}\end{matrix} \arrow[ll, "\text{homotopical}"', "\text{reduction}"] \\
\begin{matrix}\mathrm{Gar}_2(S)\\ \text{terminating}\end{matrix} \arrow[u, "\begin{matrix}\text{homotopical}\\ \text{completion-reduction}\end{matrix}"] \arrow[rr, "\text{Knuth-Bendix}", "\text{completion}"'] & & \begin{matrix}\underline{\mathrm{Gar}}_2(S)\\ \text{convergent}\end{matrix} \arrow[u, "\begin{matrix}\text{Squier}\\ \text{completion}\end{matrix}"']
\end{tikzcd}$$

In Subsection 5.2, starting with the Garside's presentation $\mathrm{Gar}_2(S)$ of M, we add the generating 2-cells β which results in a terminating presentation $\underline{\mathrm{Gar}}_2(S)$. This is, in fact, a convergent presentation, namely a Knuth-Bendix completion of $\mathrm{Gar}_2(S)$, but we do not prove it until Subsection 5.3. Nevertheless, this hindsight prompts us to begin Subsection 5.2 with a formal definition of the 2-polygraph $\underline{\mathrm{Gar}}_2(S)$.

In Subsection 5.3, first we formally compute a Squier completion of the polygraph $\underline{\mathrm{Gar}}_2(S)$, under certain assumptions on the monoid. We denote the resulting $(3,1)$-polygraph by $\underline{\mathrm{Gar}}_3(S)$. Then we show that this construction applies to a terminating presentation $\underline{\mathrm{Gar}}_2(S)$ of M and produces a coherent convergent presentation $\underline{\mathrm{Gar}}_3(S)$.

Finally, in Subsection 5.4, we compute a homotopical reduction of $\underline{\mathrm{Gar}}_3(S)$ to obtain the $(3,1)$-polygraph $\mathrm{Gar}_3(S)$ as a coherent presentation of M.

5.2. **Attaining termination.** In this subsection, we ensure that a certain presentation, denoted $\underline{\mathrm{Gar}}_2(S)$, is terminating. This presentation arises naturally as a result of applying the Knuth-Bendix completion to the Garside's presentation $\mathrm{Gar}_2(S)$. Hence the motivation for the formal definition of the 2-polygraph $\underline{\mathrm{Gar}}_2(S)$.

Let M be a monoid generated by a set S containing 1. Observe that the 2-polygraph $\mathrm{Gar}_2(S)$ has exactly one critical branching for all u, v and w of $S \setminus \{1\}$ such that $\overset{\frown}{u \frown v \frown w}$ holds:

$$uv|w \xLeftarrow{\alpha_{u,v}|w} u|v|w \xRightarrow{u|\alpha_{v,w}} u|vw.$$

If the subcase $\overset{\frown}{u \frown v \frown w}$ holds, then the branching is already confluent. Otherwise $u \frown \overset{\times}{v} \frown w$ holds, and the branching requires a new generating 2-cell to reach confluence, so the generating 2-cell $\beta_{u,v,w} : u|vw \Rightarrow uv|w$ is adjoined. We write $\underline{\mathrm{Gar}}_2\,(S)$ for the 2-polygraph which contains a single generating 0-cell, one generating 1-cell for every element of $S \setminus \{1\}$, the generating 2-cells

$$\alpha_{u,v} : u|v \Rightarrow uv, \qquad\qquad u, v \in S \setminus \{1\}\,, \quad u \frown v,$$

$$\beta_{u,v,w} : u|vw \Rightarrow uv|w, \qquad\qquad u, v, w \in S \setminus \{1\}\,, \quad u \frown \overset{\times}{v} \frown w.$$

To show that the 2-polygraph $\underline{\mathrm{Gar}}_2\,(S)$, under certain conditions, is a Knuth-Bendix completion of the 2-polygraph $\mathrm{Gar}_2\,(S)$, we need to ensure two things: a way to maintain a terminating presentation in the sense of Remark 3.1.2, and a demonstration that all new critical branchings caused by the generating 2-cells β are confluent. These are respectively given by Proposition 5.2.1, and the proof of Proposition 5.3.1.

For an element u of S^*, where S is a set, we use the following notations: $\ell\,(u)$ is the S-length of u, $\mathrm{h}\,(u)$ is the leftmost letter of u, and $\mathrm{t}\,(u)$ is the word obtained by removing the letter $\mathrm{h}\,(u)$ from u.

Proposition 5.2.1. *Assume that M is a left-cancellative monoid containing no nontrivial invertible element, admitting a right-noetherian Garside family S containing* 1. *Then the* 2*-polygraph $\underline{\mathrm{Gar}}_2\,(S)$ is terminating.*

Proof. Let us first adopt some notation. For a generating 2-cell χ, a χ-step is a rewriting step in which the generating 2-cell involved is χ, and χ_i is a χ-step

$$\begin{tikzcd} \bullet \arrow[r, "w"] & \bullet \arrow[r, bend left, "u"] \arrow[r, bend right, "v"'] \arrow[r, phantom, "\Downarrow\chi"] & \bullet \arrow[r, "w'"] & \bullet \end{tikzcd} ,$$

where w has length $i-1$. If $i_1|i_2|\cdots$ is an infinite sequence of positive integers, we denote the path $\cdots \circ \chi_{i_2} \circ \chi_{i_1}$ by $\chi_{i_1|i_2|\cdots}$.

Suppose that there is an infinite rewriting path. Note that an α-step strictly reduces the $(S \setminus \{1\})$-length of a word, so there can be only finitely many of the generating 2-cells α in any rewriting path. Hence, there is no loss in generality if we consider only β-steps. Namely, we can simply consider an infinite path after the last α-step is applied and we are left with an infinite path containing only β-steps. So assume that there is an infinite rewriting path of β-steps. Let $\beta_{i_1|i_2|\cdots}$ be such a path having source u of minimal $(S \setminus \{1\})$-length. Note that $\ell\,(u)$ is at least two.

Note that the minimality assumption about $\ell\,(u)$ implies that the position 1 occurs infinitely many times in $i_1|i_2|\cdots$. Namely, if the position 1 occurred only finitely many times in $i_1|i_2|\cdots$, then $\beta_{i_{k+1}-1|i_{k+2}-1|\cdots}$ would be an infinite path starting from $\mathrm{t}\left(\beta_{i_1|i_2|\cdots|i_k}\,(u)\right)$ of $(S \setminus \{1\})$-length $\ell\,(u)-1$, where $i_k = 1$ is the last occurrence of 1 in the sequence $i_1|i_2|\cdots$. That would contradict the minimality assumption about $\ell\,(u)$. We write $i_{c_1}|i_{c_2}|\cdots$ for the constant subsequence of the sequence $i_1|i_2|\cdots$ taking all the members whose value is 1. In other words, c_1 is the least j such that $i_j = 1$; and for all n, we have that c_{n+1} is the least j such that conditions $j > c_n$ and $i_j = 1$ hold.

Let $u^{(n)}$ denote the nth word in the path $\beta_{i_1|i_2|\cdots}$, that is the source of the step β_{i_n}. Note that the leftmost letter of the word is modified by a step β_{i_n} if, and only if, i_n equals 1. In this case, the modification is such that the current leftmost letter $\mathrm{h}\left(u^{(n)}\right)$ is a proper left divisor of the next leftmost letter $\mathrm{h}\left(u^{(n+1)}\right)$, and the corresponding complement lies in S by the definition of the generating 2-cells β. In formal terms,

$$\mathrm{h}\left(u^{(n+1)}\right) = \begin{cases} \mathrm{h}\left(u^{(n)}\right) & \text{if } i_{n+1} \neq 1, \\ \mathrm{h}\left(u^{(n)}\right) f_n \text{ for some } f_n \in S & \text{if } i_{n+1} = 1. \end{cases} \tag{5.1}$$

Let s denote the leftmost letter of the S-normal form of u. Observe that all the words in the path $\beta_{i_1|i_2|\ldots}$ have the same evaluation in M and that, consequently, the equality $N^S(u) = N^S\left(u^{(n)}\right)$ holds for all n by the definition of N^S. By Lemma 4.2.5, we have that $\mathrm{h}\left(u^{(n)}\right)$ left-divides s for all n.

Consider the sequence

$$\left(\mathrm{h}\left(u^{(c_n)}\right)\right)_{n=1}^{\infty} \tag{5.2}$$

of elements of S that divide g. Observe that, by (5.1), we have $\mathrm{h}\left(u^{(c_{n+1})}\right) = \mathrm{h}\left(u^{(c_n)}\right) f_{c_n}$. The existence of the sequence (5.2) contradicts the fact that S is right-noetherian. We conclude that the 2-polygraph $\underline{\mathrm{Gar}}_2(S)$ is terminating. □

5.3. **Homotopical completion of Garside's presentation.** In this subsection, we enrich Garside's presentation to reach a coherent convergent presentation. First (Proposition 5.3.1) we compute, purely formally, the homotopical completion of a terminating presentation of a monoid satisfying certain conditions, but not presumed to have a proper Garside family. Then we show, in Corollary 5.3.4, that this provides a coherent convergent presentation of a left-cancellative monoid containing no nontrivial invertible element, admitting right-mcms and a right-noetherian Garside family containing 1.

Proposition 5.3.1. *Assume that M is a left-cancellative monoid admitting right-mcms, and S is a subfamily of M closed under right-mcm and right divisor. Assume that the 2-polygraph $\underline{\mathrm{Gar}}_2(S)$ is a terminating presentation of M. Then M admits, as a coherent convergent presentation, the $(3,1)$-polygraph $\underline{\mathrm{Gar}}_3(S)$ which extends $\underline{\mathrm{Gar}}_2(S)$ with the following twelve families of generating 3-cells, indexed by all the possible elements of $S \setminus \{1\}$:*

$$\begin{tikzcd}
 & uv|w \arrow[dr, Rightarrow, "\alpha_{uv,w}"] & \\
u|v|w \arrow[ur, Rightarrow, "\alpha_{u,v}|w"] \arrow[dr, Rightarrow, "u|\alpha_{v,w}"'] \arrow[rr, phantom, "A_{u,v,w}"] & & uvw \\
 & u|vw \arrow[ur, Rightarrow, "\alpha_{u,vw}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
u|v|w \arrow[rr, Rightarrow, "\alpha_{u,v}|w"] \arrow[dr, Rightarrow, "u|\alpha_{v,w}"'] & \arrow[d, phantom, "B_{u,v,w}"] & uv|w \\
 & u|vw \arrow[ur, Rightarrow, "\beta u, v, w"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & uv|wx \arrow[dr, Rightarrow, "\beta_{uv,w,x}"] & \\
u|v|wx \arrow[ur, Rightarrow, "\alpha_{u,v}|wx"] \arrow[dr, Rightarrow, "u|\beta_{v,w,x}"'] \arrow[rr, phantom, "C_{u,v,w,x}"] & & uvw|x \\
 & u|vw|x \arrow[ur, Rightarrow, "\alpha_{u,vw}|x"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
u|v|wx \arrow[rrr, Rightarrow, "\alpha_{u,v}|wx"] \arrow[dr, Rightarrow, "u|\beta_{v,w,x}"'] & \arrow[r, phantom, "D_{u,v,w,x}"] & & uv|wx \\
 & u|vw|x \arrow[r, Rightarrow, "\beta_{u,v,w}|x"'] & uv|w|x \arrow[ur, Rightarrow, "uv|\alpha_{w,x}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & uv|w|x \arrow[dr, Rightarrow, "uv|\alpha_{w,x}"] & \\
u|vw|x \arrow[ur, Rightarrow, "\beta_{u,v,w}|x"] \arrow[dr, Rightarrow, "u|\alpha_{vw,x}"'] \arrow[rr, phantom, "E_{u,v,w,x}"] & & uv|wx \\
 & u|vwx \arrow[ur, Rightarrow, "\beta_{u,v,wx}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & uv|w|x \arrow[r, Rightarrow, "uv|\alpha_{w,x}"] & uv|wx \arrow[dr, Rightarrow, "\alpha_{uv,wx}"] & \\
u|vw|x \arrow[ur, Rightarrow, "\beta_{u,v,w}|x"] \arrow[drr, Rightarrow, "u|\alpha_{vw,x}"'] & \arrow[r, phantom, "E'_{u,v,w,x}"] & & uvwx \\
 & & u|vwx \arrow[ur, Rightarrow, "\alpha_{u,vwx}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & & uv|w|xy \arrow[rrd, Rightarrow, "uv|\alpha_{w,xy}"] & & \\
u|vw|xy \arrow[rru, Rightarrow, "\beta_{u,v,w}|xy"] \arrow[rd, Rightarrow, "u|\beta_{vw,x,y}"'] & & F_{u,v,w,x,y} & & uv|wxy \\
 & u|vwx|y \arrow[rr, Rightarrow, "\beta_{u,v,wx}|y"'] & & uv|wx|y \arrow[ru, Rightarrow, "uv|\alpha_{wx,y}"'] &
\end{tikzcd}$$

$$\begin{tikzcd}
 & uv|w|xy \arrow[rr, Rightarrow, "uv|\alpha_{w,xy}"] & & uv|wxy \arrow[rd, Rightarrow, "\beta_{uv,wx,y}"] & \\
u|vw|xy \arrow[ru, Rightarrow, "\beta_{u,v,w}|xy"] \arrow[rrd, Rightarrow, "u|\beta_{vw,x,y}"'] & & F'_{u,v,w,x,y} & & uvwx|y \\
 & & u|vwx|y \arrow[rru, Rightarrow, "\alpha_{u,vwx}|y"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & uv|w|xy \arrow[rd, Rightarrow, "uv|\beta_{w,x,y}"] & \\
u|vw|xy \arrow[ru, Rightarrow, "\beta_{u,v,w}|xy"] \arrow[rd, Rightarrow, "u|\beta_{vw,x,y}"'] & G_{u,v,w,x,y} & uv|wx|y \\
 & u|vwx|y \arrow[ru, Rightarrow, "\beta_{u,v,wx}|y"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & uv|w|xy \arrow[rr, Rightarrow, "uv|\beta_{w,x,y}"] & & uv|wx|y \arrow[rd, Rightarrow, "\alpha_{uv,wx}|y"] & \\
u|vw|xy \arrow[ru, Rightarrow, "\beta_{u,v,w}|xy"] \arrow[rrd, Rightarrow, "u|\beta_{vw,x,y}"'] & & G'_{u,v,w,x,y} & & uvwx|y \\
 & & u|vwx|y \arrow[rru, Rightarrow, "\alpha_{u,vwx}|y"'] & &
\end{tikzcd}$$

$$\begin{tikzcd}
 & uv|xy \arrow[rd, Rightarrow, "\beta_{uv,x,y}"] & \\
u|vxy \arrow[ru, Rightarrow, "\beta_{u,v,xy}"] \arrow[rr, Rightarrow, "\beta_{u,vx,y}"'] & H_{u,v,x,y} & uvx|y
\end{tikzcd}
\qquad
\begin{tikzcd}
 & uv_1|w_1 = uv_1|x_1y \arrow[rd, Rightarrow, "\beta_{uv_1,x_1,y}"] & \\
u|v_1w_1 = u|v_2w_2 \arrow[ru, Rightarrow, "\beta_{u,v_1,w_1}"] \arrow[rd, Rightarrow, "\beta_{u,v_2,w_2}"'] & I_{u,v_1,w_1,v_2,w_2} & uv_1x_1|y = uv_2x_2|y \\
 & uv_2|w_2 = uv_2|x_2y \arrow[ru, Rightarrow, "\beta_{uv_2,x_2,y}"'] &
\end{tikzcd}$$

The meanings of the 1-cells (i.e. words) x_1, x_2, y and x, y which appear respectively in the definitions of the generating 3-cells I and H, are as follows. Since v_1 and v_2 have the common right multiple $v_1w_1 = v_2w_2$, they also have a right-mcm. The words x_1 and x_2 are the right complements of v_1 and v_2, respectively, in their right-mcm. The word y is the right complement of $v_1x_1 = v_2x_2$ in $v_1w_1 = v_2w_2$. If either x_1 or x_2 is equal to 1, then the other one is simply denoted by x (in the generating 3-cell H).

The structure of the following proof closely resembles that of the proof of [13, Proposition 3.2.1], but we need to devise more general arguments to assure favourable properties in a more general context.

Proof. Termination of the 2-polygraph $\underline{\mathrm{Gar}}_2(S)$ is assumed, so we can perform a relaxed version of the Knuth-Bendix completion procedure, as described in Remark 3.1.2, simultaneously with the Squier completion procedure. It will turn out that all critical branchings are confluent, and hence that only a Squier completion will be actually computed, i.e. no further 2-generating cells will be added.

Let us first consider critical branchings consisting only of the generating 2-cells α. There is only one such critical branching for all u, v and w of $S \setminus \{1\}$ such that $u \frown v \frown w$ holds:

$$uv|w \overset{\alpha_{u,v}|w}{\Longleftarrow} u|v|w \overset{u|\alpha_{v,w}}{\Longrightarrow} u|vw.$$

If the subcase $\overbrace{u \frown v \frown w}$ holds, the branching is already confluent, so the homotopical completion procedure adjoins only the generating 3-cell $A_{u,v,w}$. If $\overset{\times}{\overbrace{u \frown v \frown w}}$ holds, the branching is again confluent, so the generating 3-cell $B_{u,v,w}$ is adjoined.

Let us now consider critical branchings containing the generating 2-cell β. The sources of 2-cells forming such a branching can either overlap on one element of $S \setminus \{1\}$ or be equal, as the lengths in $(S \setminus \{1\})^*$ of the sources of the generating 2-cells α and β equal two. We consider the two cases accordingly.

For the first case, the proof of [13, Proposition 3.2.1] applies here to a great extent. The source of a branching has length three, as a word in $(W \setminus \{1\})^*$. One of the 2-cells which form a branching, rewrites the leftmost two generating 1-cells of the source, and the other one rewrites the rightmost two. There are three distinct forms of such branchings:

$$uv|wx \xLeftarrow{\alpha_{u,v}|wx} u|v|wx \xRightarrow{u|\beta_{v,w,x}} u|vw|x,$$

$$uv|w|x \xLeftarrow{\beta_{u,v,w}|x} u|vw|x \xRightarrow{u|\alpha_{vw,x}} u|vwx,$$

$$uv|w|xy \xLeftarrow{\beta_{u,v,w}|xy} u|vw|xy \xRightarrow{u|\beta_{vw,x,y}} u|vwx|y.$$

The first form is defined under the condition $u\ v\ w\ x$, which splits into two mutually exclusive possibilities $u\ v\ w\ x$ and $u\ v\ w\ x$, which respectively yield the generating 3-cells $C_{u,v,w,x}$ and $D_{u,v,w,x}$ by the homotopical completion procedure. The second form is defined under the condition $u\ v\ w\ x$ which splits into $u\ v\ w\ x$ and $u\ v\ w\ x$, which respectively produce the generating 3-cells $E_{u,v,w,x}$ and $E'_{u,v,w,x}$. The third form is defined under the conditions $u\ v\ w\ x\ y$. This situation splits into two mutually exclusive possibilities $u\ v\ w\ x\ y$ and $u\ v\ w\ x\ y$. The former possibility further splits into $u\ v\ w\ x\ y$ and $u\ v\ w\ x\ y$ which respectively yield the generating 3-cells $F_{u,v,w,x,y}$ and $F'_{u,v,w,x,y}$; the latter splits into $u\ v\ w\ x\ y$ and $u\ v\ w\ x\ y$ yielding the generating 3-cells $G_{u,v,w,x,y}$ and $G'_{u,v,w,x,y}$, respectively.

We have thus considered the first case. The second case is going to be considered in greater detail because this is where new justifications are needed. Assume that the two 2-cells which generate a critical branching, have the same source. One of those two 2-cells has to be a β (otherwise, the branching is trivial). Therefore, the source has to have a form $u|v_1w_1$ satisfying the condition $u\ v_1\ w_1$. Since 2-cells α are not defined under this condition, the other 2-cell also has to be a β. The only way for the generating 2-cells β with the same source $u|v_1w_1$ to form a critical branching is for v_1w_1 to have another decomposition $v_1w_1 = v_2w_2$ such that $u\ v_2\ w_2$. Then the branching is as follows:

$$uv_1|w_1 \xLeftarrow{\beta_{u,v_1,w_1}} u|v_1w_1 = u|v_2w_2 \xRightarrow{\beta_{u,v_2,w_2}} uv_2|w_2.$$

Let us invoke the assumed property of M admitting right-mcms. Since v_1 and v_2 have a common right multiple, namely $v_1w_1 = v_2w_2$, they also have a right-mcm, say v'. Since S is closed under right-mcm by assumption, v' lies in S. By the left cancellation property which grants the uniqueness of right complements, we define x_1 and x_2 as the right complements in v' of v_1 and v_2, respectively. Since S is closed under right divisor, x_1 and x_2 are elements of S. We also define y as the right complement of v' in $v_1w_1 = v_2w_2$. Note that y is in S as a right divisor of v_1w_1 which is in S. Uniqueness of the right complements of v_1 and v_2 in v_1w_1 and

v_2w_2, respectively, yields $w_1 = x_1y$ and $w_2 = x_2y$. To sum up, the diagram

$$\begin{tikzcd}
 & \bullet \arrow[dr, "x_1"] \arrow[drr, bend left, "w_1"] & & \\
\bullet \arrow[ur, "v_1"] \arrow[rr, "v'"] \arrow[dr, "v_2"'] & & \bullet \arrow[r, "y"] & \bullet \\
 & \bullet \arrow[ur, "x_2"'] \arrow[urr, bend right, "w_2"'] & &
\end{tikzcd}$$

commutes, where $\bullet$ denotes the unique 0-cell. Furthermore, the equality $w_k = x_ky$, the fact that v' lies in S, and the condition $v_k \frown w_k$ together imply $\overset{\frown}{v_k \frown x_k \frown y}$ for $k \in \{1, 2\}$.

We have only showed that x_1, x_2 and y are elements of S. Let us verify that all the generating 1-cells involved are, indeed, elements of $S \setminus \{1\}$. First we demonstrate that y cannot be equal to 1. Assume the opposite. Then the condition $u \frown \overset{\times}{v_1} \frown w_1$ reduces to $u \frown \overset{\times}{v_1} \frown x_1$. On the other hand, uv' is a right-mcm of uv_1 and uv_2 by Lemma 4.1.1. Since S is closed under right-mcm, uv' lies in S, which contradicts the condition $u \frown \overset{\times}{v_1} \frown x_1$. Thus, we deduce that y is not equal to 1.

Note that if x_1 and x_2 were both equal to 1, the branching $\{\beta_{u,v_1,w_1}, \beta_{u,v_2,w_2}\}$ would be trivial. So, at most one of the 1-cells x_1 and x_2 can be equal to 1. If $x_2 = 1$, the generating 3-cell $H_{u,v,x,y}$ is constructed with $v \coloneqq v_1$ and $x \coloneqq x_1$. Similarly, for $x_1 = 1$, the generating 3-cell $H_{u,v,x,y}$ is constructed with $v \coloneqq v_2$ and $x \coloneqq x_2$. Finally, if neither x_1 nor x_2 is equal to 1, the generating 3-cell I_{u,v_1,w_1,v_2,w_2} is adjoined.

By Theorem 3.2.2, the constructed $(3,1)$-polygraph $\underline{\mathrm{Gar}}_3(S)$ is a coherent convergent presentation of M. $\square$

Remark 5.3.2. Observe that Proposition 5.3.1 gives three new families of generating 3-cells (namely, E', F' and G') that were not a part of the [13, Proposition 3.2.1], an analogous result for Artin-Tits monoids. The reason for this is that the Garside families considered in [13] for Artin-Tits monoids and Garside monoids are closed under left and right divisors, while a family S in Proposition 5.3.1 is only closed under right divisor (like a Garside family in general). Consequently, certain conjunctions of conditions, discussed in the proof of Proposition 5.3.1, could not be satisfied in the setting of Artin-Tits monoids. For instance, here we consider the possibility $uvwx \in S$ under the condition $uvw \notin S$, among others, to construct the generating 3-cell E'. In an Artin-Tits monoid, on the other hand, $uvwx \in \sigma(W)$ would imply $uvw \in \sigma(W)$ due to closure under left divisor.

We can now deduce that the 2-polygraph $\underline{\mathrm{Gar}}_2(S)$ is a Knuth-Bendix completion of the Garside's presentation $\mathrm{Gar}_2(S)$, as hinted in Subsection 5.2.

Corollary 5.3.3. *Assume that M is a left-cancellative monoid containing no nontrivial invertible element, and admitting a right-noetherian Garside family S containing* 1. *Then the* 2*-polygraph $\underline{\mathrm{Gar}}_2(S)$ is a convergent presentation of M.*

Proof. Proposition 4.2.7 grants that the 2-polygraph $\mathrm{Gar}_2(S)$ is a presentation of M. Since the generating 2-cells α strictly decrease the S-length, the 2-polygraph $\mathrm{Gar}_2(S)$ is terminating. Thanks to Proposition 5.2.1, we can compute its Knuth-Bendix completion in a manner described in Remark 3.1.2. As shown in Subsection 5.2, the generating 2-cells β are added.

Note that Proposition 4.2.3 and Lemma 4.2.4, together with the assumptions that S contains 1 and that M contains no nontrivial invertible element, yield the property of S being closed under right-mcm. By Proposition 4.2.3, S is closed under right divisor. With all these conditions satisfied, the proof of Proposition 5.3.1 applies in a straightforward fashion. In particular, it shows that all new critical branchings caused by the generating 2-cells β are confluent. Thus, the 2-polygraph $\underline{\mathrm{Gar}}_2\,(S)$ is a Knuth-Bendix completion of the Garside's presentation $\mathrm{Gar}_2\,(S)$, which yields the desired conclusion by Theorem 3.1.1 and Remark 3.1.2. $\square$

Observe that Proposition 5.2.1, together with Proposition 4.2.7, immediately implies that the 2-polygraph $\underline{\mathrm{Gar}}_2\,(S)$ is a terminating presentation of M. On the other hand, the fact that $\underline{\mathrm{Gar}}_2\,(S)$ is also a convergent presentation of M was reachable only after Proposition 5.3.1 when we made sure that no additional generating 2-cells were required to obtain confluence.

Corollary 5.3.4. *Assume that M is a left-cancellative monoid containing no nontrivial invertible element, and admitting a right-noetherian Garside family S containing* 1. *If M admits right-mcms, then M admits the* $(3,1)$*-polygraph* $\underline{\mathrm{Gar}}_3\,(S)$*, defined in Proposition 5.3.1, as a coherent convergent presentation.*

Proof. Corollary 5.3.3 grants that $\underline{\mathrm{Gar}}_2\,(S)$ is a terminating presentation of M. As shown in the proof of Corollary 5.3.3, all the requirements are met for applying Proposition 5.3.1, which completes the proof. $\square$

5.4. **Homotopical reduction of Garside's presentation.** The homotopical reduction procedure from [13, p. 3.2.2] applies verbatim to the coherent convergent presentation provided by Proposition 5.3.1 (and echoed by Corollary 5.3.4), with respect to a collapsible part Γ obtained as follows. The component Γ_4 of Γ contains seven generating triple confluences whose targets are the families C, ..., I of generating 3-cells, with the order $I > H > \cdots > C$. For the sake of illustration, we recall one such generating triple confluence in the case $u\ v\ w\ x$ (we refer the reader to [13, p. 3.2.2] for the other six generating triple confluences). Its boundary consists of the following two parts:

$$
\begin{tikzcd}
 & uv|w|x \arrow[rr, Rightarrow, "\alpha_{uv,w}|x"] \arrow[d, phantom, "A_{u,v,w}|x"] & & uvw|x \\
u|v|w|x \arrow[ur, Rightarrow, "\alpha_{u,v}|w|x"] \arrow[rr, Rightarrow, "u|\alpha_{v,w}|x"] \arrow[dr, Rightarrow, "u|v|\alpha_{w,x}"'] & & u|vw|x \arrow[ur, Rightarrow, "\alpha_{u,vw}|x"] \arrow[dr, Rightarrow, "u|\alpha_{vw,x}"] \arrow[r, phantom, "B_{u,vw,x}"] & {} \\
 & u|v|wx \arrow[rr, Rightarrow, "u|\alpha_{v,wx}"'] \arrow[u, phantom, "u|A_{v,w,x}"] & & u|vwx \arrow[uu, Rightarrow, bend right, "\beta_{u,vw,x}"']
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
 & uv|w|x \arrow[rr, Rightarrow, "\alpha_{uv,w}|x"] \arrow[dr, Rightarrow, "uv|\alpha_{w,x}"] \arrow[r, phantom, "B_{uv,w,x}"] & {} & uvw|x \\
u|v|w|x \arrow[ur, Rightarrow, "\alpha_{u,v}|w|x"] \arrow[dr, Rightarrow, "u|v|\alpha_{w,x}"'] & = & uv|wx \arrow[ur, Rightarrow, "\beta_{uv,w,x}"'] \arrow[r, phantom, "H_{u,v,w,x}"] & {} \\
 & u|v|wx \arrow[rr, Rightarrow, "u|\alpha_{v,wx}"'] \arrow[ur, Rightarrow, "\alpha_{u,v}|wx"] \arrow[r, phantom, "B_{u,v,wx}"] & {} & u|vwx \arrow[ul, Rightarrow, "\beta_{u,v,wx}"'] \arrow[uu, Rightarrow, bend right, "\beta_{u,vw,x}"']
\end{tikzcd}
.
$$

The target of this particular generating triple confluence is the generating 3-cell $H_{u,v,w,x}$.

Note, however, that does not suffice to eliminate any of the generating 3-cells $E'_{u,v,w,x}$, $F'_{u,v,w,x,y}$ and $G'_{u,v,w,x,y}$ since these particular families of generating 3-cells do not even occur in [13, Section 3] (recall Remark 5.3.2). So, we have yet to eliminate these cells here. To this end, we consider the following generating triple confluences in the $(3, 1)$-polygraph $\underline{\mathrm{Gar}}_3\,(S)$.

The boundary of our first 3-sphere of interest consists of

$$\begin{tikzcd}
& uv|w|x \arrow[rr, equal, bend left] \arrow[d, phantom, "B|x"] & & uv|w|x \arrow[dr, Rightarrow, "uv|\alpha"] \arrow[dd, phantom, "E'"] & & \\
& {} & & & uv|wx \arrow[dr, Rightarrow, "\alpha"] & \\
u|v|w|x \arrow[uur, Rightarrow, "\alpha|w|x"] \arrow[rr, Rightarrow, "u|\alpha|x"] \arrow[dr, Rightarrow, "u|v|\alpha"'] & {} & u|vw|x \arrow[uur, Rightarrow, "\beta|x"] \arrow[dr, Rightarrow, "u|\alpha"] \arrow[dl, phantom, "u|A"] & & & uvwx \\
& u|v|wx \arrow[rr, Rightarrow, bend right, "u|\alpha"'] & & u|vwx \arrow[urr, Rightarrow, "\alpha"'] & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& uv|w|x \arrow[dr, Rightarrow, "uv|\alpha"] \arrow[dd, phantom, "="] & & & \\
u|v|w|x \arrow[ur, Rightarrow, "\alpha|w|x"] \arrow[dr, Rightarrow, "u|v|\alpha"'] & & uv|wx \arrow[rr, Rightarrow, "\alpha"] \arrow[dr, phantom, "A"] & & uvwx \\
& u|v|wx \arrow[ur, Rightarrow, "\alpha|wx"] \arrow[rr, Rightarrow, bend right, "u|\alpha"'] & & u|vwx \arrow[ur, Rightarrow, "\alpha"'] &
\end{tikzcd} .$$

The target is the generating 3-cell $E'_{u,v,w,x}$.

The second generating triple confluence which we are going to use has the boundary consisting of

$$\begin{tikzcd}
& uv|w|xy \arrow[rr, equal, bend left] \arrow[d, phantom, "B|xy"] & & uv|w|xy \arrow[dr, Rightarrow, "uv|\alpha"] \arrow[dd, phantom, "F'"] & & \\
& {} & & & uv|wxy \arrow[dr, Rightarrow, "\beta"] & \\
u|v|w|xy \arrow[uur, Rightarrow, "\alpha|w|xy"] \arrow[rr, Rightarrow, "u|\alpha|xy"] \arrow[dr, Rightarrow, "u|v|\alpha"'] & {} & u|vw|xy \arrow[uur, Rightarrow, "\beta|xy"] \arrow[dr, Rightarrow, "u|\beta"] \arrow[dl, phantom, "u|B"] \arrow[d, phantom, "u|H"] & & & uvwx|y \\
& u|v|wxy \arrow[ur, Rightarrow, "u|\beta"] \arrow[rr, Rightarrow, bend right, "u|\beta"'] & {} & u|vwx|y \arrow[urr, Rightarrow, "\alpha|y"'] & &
\end{tikzcd}$$

and

$$\begin{tikzcd}
& uv|w|xy \arrow[dr, Rightarrow, "uv|\alpha"] \arrow[dd, phantom, "="] & & & \\
u|v|w|xy \arrow[ur, Rightarrow, "\alpha|w|xy"] \arrow[dr, Rightarrow, "u|v|\alpha"'] & & uv|wxy \arrow[rr, Rightarrow, "\beta"] \arrow[dr, phantom, "C"] & & uvwx|y \\
& u|v|wxy \arrow[ur, Rightarrow, "\alpha|wxy"] \arrow[rr, Rightarrow, bend right, "u|\beta"'] & & u|vwx|y \arrow[ur, Rightarrow, "\alpha|y"'] &
\end{tikzcd} .$$

The target is the generating 3-cell $F'_{u,v,w,x,y}$.

Finally, we construct the 3-sphere whose boundary has the following parts:

$$
\begin{tikzcd}
 & uv|w|xy \arrow[rr, equal, bend left] & & uv|w|xy \arrow[dr, Rightarrow, "uv|\beta"] & & \\
 & B|xy & & G' & uv|wx|y \arrow[dr, Rightarrow, "\alpha|y"] & \\
u|v|w|xy \arrow[ur, Rightarrow, "\alpha|w|xy"] \arrow[rr, Rightarrow, "u|\alpha|xy"] \arrow[dr, Rightarrow, "u|v|\beta"'] & & u|vw|xy \arrow[uur, Rightarrow, "\beta|xy"] \arrow[dr, Rightarrow, "u|\beta"] & & & uvwx|y \\
 & u|v|wx|y \arrow[rr, Rightarrow, "u|\alpha|y"'] & u|C & u|vwx|y \arrow[urr, Rightarrow, "\alpha|y"'] & &
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
 & uv|w|xy \arrow[dr, Rightarrow, "uv|\beta"] & & & \\
u|v|w|xy \arrow[ur, Rightarrow, "\alpha|w|xy"] \arrow[dr, Rightarrow, "u|v|\beta"'] & = & uv|wx|y \arrow[rr, Rightarrow, "\alpha|y"] & A|y & uvwx|y \\
 & u|v|wx|y \arrow[ur, Rightarrow, "\alpha|wx|y"] \arrow[rr, Rightarrow, "u|\alpha|y"'] & & u|vwx|y \arrow[ur, Rightarrow, "\alpha|y"'] &
\end{tikzcd} .
$$

The target is the generating 3-cell $G'_{u,v,w,x,y}$.

So we extend the above mentioned component Γ_4 of the collapsible part (inherited from [13, p. 3.2.2]) with these three freshly constructed 3-spheres. We also extend the order relation on generating 3-cells to $G' > F' > E' > I > H > \cdots > C$. The component Γ_3 of the collapsible part contains the family B of generating 3-cells having the generating 2-cells β as targets, with the order $\beta > \alpha$.

The homotopical reduction of the resulting $(3,1)$-polygraph of Proposition 5.3.1, with respect to the collapsible part Γ, is precisely $\mathrm{Gar}_3\,(S)$. By Theorem 3.3.1, we conclude that $\mathrm{Gar}_3\,(S)$ is a coherent presentation of M. Through Corollary 5.3.4, the proof of Theorem 5.1.4 is hereby completed.

5.5. **Noetherianity.** Let us state an immediate corollary of Theorem 5.1.4, having somewhat simpler (although more restrictive) requirements.

Corollary 5.5.1. *Assume that M is a left-cancellative noetherian monoid containing no non-trivial invertible element, and $S \subseteq M$ is a Garside family containing 1. Then M admits the $(3,1)$-polygraph $\mathrm{Gar}_3\,(S)$ as a coherent presentation.*

Proof. Since M is right-noetherian, so is S. By [4, Proposition II.2.40], every left-cancellative left-noetherian monoid admits right-mcms, so M admits right-mcms. Hence, all the conditions of Theorem 5.1.4 are satisfied. □

The next section demonstrates advantages of using our results in applications. The following example, however, shows that taking a Garside family as a generating set is not always the most practical way to get a coherent presentation.

Example 5.5.2. We revisit the Klein bottle monoid K^+ from (Examples 3.2.3 and 3.4.1). One of the infinitely many Garside families in K^+, none of which is finite (see [4, Example IV.2.35]), is the set of left divisors of a^2, which we denote by S. Let us check if the conditions of Theorem 5.1.4 are satisfied. Note that K^+ is cancellative as it is embeddable in a group. The presentation (3.1) contains no relation of the form $u = v$ with exactly one the words u and v being empty, hence K^+ has no nontrivial invertible element. Note that the left divisibility relation of K^+ is a linear order ([4, Figure I.6]), which is a lot more than necessary for admitting conditional right-lcms (consequently, right-mcms, too). However, the sequence $(ab^n)_{n=1}^{\infty}$ shows that S is not right-noetherian. Even worse, S contains an infinite path of the generating 2-cells β, as defined in Proposition 5.2.1:

$$b|a^2 \mapsto b^2|aba \mapsto b^3|ab^2a \mapsto \cdots \mapsto b^q|ab^{q-1}a \mapsto \cdots$$

Even if we took another Garside family, we would not be successful, as witnessed by [4, Example IV.2.35]. Therefore, neither Theorem 5.1.4 nor its proof is applicable to K^+.

If one found a way to use a Garside family as a generating set, they would have an infinite number of 1-cells. On the other hand, by directly performing the homotopical completion-reduction procedure in Examples 3.2.3 and 3.4.1, we have demonstrated that the presentation (3.1), which has two generating 1-cells and one generating 2-cell, is coherent. Therefore, for this particular example, the direct application of the homotopical completion-reduction procedure is a preferable way of reaching a coherent presentation.

6. Applications of Theorem 5.1.4

In this section, we consider applications of Theorem 5.1.4 to certain monoids. In Subsections 6.1 and 6.2, we apply it to monoids which are neither Artin-Tits nor Garside. In Subsection 6.3, we compute a finite coherent presentation of an Artin-Tits monoid $B^+(W)$ that is not of spherical type, with a finite Garside family F (hence, $F \neq W$).

6.1. **The free abelian monoid over an infinite basis.** Consider the free abelian monoid $\mathbb{N}^{(I)}$ of all I-indexed sequences of nonnegative integers with finite support. Note that $\mathbb{N}^{(I)}$ is not necessarily of finite type, hence it is neither Artin-Tits nor Garside. Define

$$S_I = \left\{ g \in \mathbb{N}^{(I)} \,\middle|\, \forall k \in I, g(k) \in \{0,1\} \right\}.$$

Observe that S_I is a Garside family in $\mathbb{N}^{(I)}$ (say, by applying Proposition 4.2.2). The following properties follow from the fact that the definition of the product on $\mathbb{N}^{(I)}$ is based on the pointwise addition of nonnegative integers: $\mathbb{N}^{(I)}$ is a cancellative monoid, it has no nontrivial invertible elements, and it admits conditional right-lcms. Since every element of $\mathbb{N}^{(I)}$ has only finitely many divisors, $\mathbb{N}^{(I)}$ is noetherian. So, all the conditions of Theorem 5.1.4 are satisfied.

Let us describe the cells of the coherent presentation of $\mathbb{N}^{(I)}$ granted by Theorem 5.1.4. The generating 2-cells are relations $\alpha_{u,v} : u|v \Rightarrow uv$ for $u, v \in S_I \setminus \{1\}$ with $uv \in S_I$, which in this particular context means that u and v have disjoint supports. A generating 3-cell $A_{u,v,w}$ is adjoined for any $u, v, w \in S_I \setminus \{1\}$ which have pairwise disjoint supports.

As expected, for $I = \{1, 2, \ldots, n\}$, we recover Garside's presentation of the Artin-Tits monoid $\mathbb{N}^n$ recalled in Example 3.5.1, as well as Garside's presentation of the Garside monoid $\mathbb{N}^n$ recalled in Example 3.5.2.

6.2. **Infinite braids.** Denote by B_∞^+ the monoid of all positive braids on infinitely many strands indexed by positive integers, as defined in [4, Subsection I.3.1]. It is shown that B_∞^+ is not of finite type, therefore it is neither Artin-Tits nor Garside. Put

$$S_\infty = \bigcup_{n \geq 1} \{\text{the family of all divisors of } \Delta_n\},$$

where Δ_n denotes the half-turn braid on n strands. In other words, S_∞ consists of all simple braids for all $n \geq 1$. This is made precise in [4, Subsection I.3.1]. Basically, B_n^+ is identified with its image in B^+_{n+1} under the homomorphism induced by the identity map on $\{\sigma_1, \ldots, \sigma_n\}$. In that sense, B^+_∞ is seen as the union of all braid monoids B^+_n. By Proposition 4.2.2, S_∞ is a Garside family in B^+_∞. Cancellation properties, and having no nontrivial invertible elements are preserved from braid monoids because the respective definitions do not depend on n. The monoid is noetherian for the same reason as Artin-Tits monoids (Example 4.1.2). So, we can apply Theorem 5.1.4 to construct a coherent presentation.

The generating 2-cells are relations $\alpha_{u,v} : u|v \Rightarrow uv$ for $u, v \in S_\infty \setminus \{1\}$ whenever $uv \in S_\infty$, which in this example means that uv is a simple braid. A generating 3-cell $A_{u,v,w}$ is adjoined for any $u, v, w \in S_\infty \setminus \{1\}$ with $uv \in S_\infty$, $vw \in S_\infty$, and $uvw \in S_\infty$, which here means that uv, vw and uvw are simple braids. So, formally, each cell is constructed exactly like in the coherent presentation provided by [13, Theorem 3.1.3] for a (finite) braid monoid, regarded as an Artin-Tits monoid, which comes as no surprise because Theorem 5.1.4 is a formal generalisation of [13, Theorem 3.1.3].

6.3. **Artin-Tits monoids that are not of spherical type.** For an Artin-Tits monoid $B^+(W)$ of spherical type, [13, Theorem 3.1.3] provides a finite coherent presentation having $W \setminus \{1\}$ as a generating set. On the other hand, if a Coxeter group W is infinite, [13, Theorem 3.1.3] still provides a coherent presentation but an infinite one. Recall that every Artin-Tits monoid admits a finite Garside family (we refer the reader to [6] for elaboration), regardless of whether the monoid is of spherical type or not. An advantage of having Theorem 5.1.4 at our disposal is that we can take a finite Garside family for a generating set in computing a coherent presentation (whereas with [13, Theorem 3.1.3], one has to take the corresponding Coxeter group).

Let us consider the Artin-Tits monoid of type $\widetilde{A}_2$, i.e. the monoid presented by

$$\langle \sigma_1, \sigma_2, \sigma_3 | \sigma_1\sigma_2\sigma_1 = \sigma_2\sigma_1\sigma_2, \sigma_2\sigma_3\sigma_2 = \sigma_3\sigma_2\sigma_3, \sigma_3\sigma_1\sigma_3 = \sigma_1\sigma_3\sigma_1 \rangle^+ . \tag{6.1}$$

By [6, Table 1 and Proposition 5.1], the smallest Garside family F in this monoid consists of the sixteen right divisors of the elements $\sigma_3\sigma_1\sigma_2\sigma_1$, $\sigma_1\sigma_2\sigma_3\sigma_2$, and $\sigma_2\sigma_3\sigma_1\sigma_3$. Namely,

$$F = \{1, \sigma_1, \sigma_2, \sigma_3, \sigma_1\sigma_2, \sigma_2\sigma_1, \sigma_2\sigma_3, \sigma_3\sigma_2, \sigma_3\sigma_1, \sigma_1\sigma_3, \\ \sigma_1\sigma_2\sigma_1, \sigma_2\sigma_3\sigma_2, \sigma_3\sigma_1\sigma_3, \sigma_3\sigma_1\sigma_2\sigma_1, \sigma_1\sigma_2\sigma_3\sigma_2, \sigma_2\sigma_3\sigma_1\sigma_3\}.$$

The Cayley graph of F can be seen in [6, Figure 1].

As noted in Remark 5.1.5, all the conditions of Theorem 5.1.4 are satisfied. Following Theorem 5.1.4, we construct a generating 2-cell $u|v \Rightarrow uv$ for $u, v \in F \setminus \{1\}$ with $uv \in F$. Thus we obtain three pairs of generating 2-cells of the form

$$\alpha_{\sigma_i,\sigma_j} : \sigma_i|\sigma_j \Rightarrow \sigma_i\sigma_j \qquad \alpha_{\sigma_j,\sigma_i} : \sigma_j|\sigma_i \Rightarrow \sigma_j\sigma_i,$$

three pairs of generating 2-cells of the form

$$\alpha_{\sigma_i,\sigma_j\sigma_i} : \sigma_i|\sigma_j\sigma_i \Rightarrow \sigma_i\sigma_j\sigma_i \qquad \alpha_{\sigma_j,\sigma_i\sigma_j} : \sigma_j|\sigma_i\sigma_j \Rightarrow \sigma_i\sigma_j\sigma_i,$$

three pairs of generating 2-cells of the form

$$\alpha_{\sigma_i\sigma_j,\sigma_i} : \sigma_i\sigma_j|\sigma_i \Rightarrow \sigma_i\sigma_j\sigma_i \qquad \alpha_{\sigma_j\sigma_i,\sigma_j} : \sigma_j\sigma_i|\sigma_j \Rightarrow \sigma_i\sigma_j\sigma_i,$$

three generating 2-cells of the form

$$\alpha_{\sigma_k,\sigma_i\sigma_j\sigma_i} : \sigma_k|\sigma_i\sigma_j\sigma_i \Rightarrow \sigma_k\sigma_i\sigma_j\sigma_i,$$

and three pairs of generating 2-cells of the form

$$\alpha_{\sigma_k\sigma_i,\sigma_j\sigma_i} : \sigma_k\sigma_i|\sigma_j\sigma_i \Rightarrow \sigma_k\sigma_i\sigma_j\sigma_i \qquad \alpha_{\sigma_k\sigma_j,\sigma_i\sigma_j} : \sigma_k\sigma_j|\sigma_i\sigma_j \Rightarrow \sigma_k\sigma_i\sigma_j\sigma_i,$$

with $i, j, k \in \{1, 2, 3\}$ and $j = i + 1$ and $k = j + 1$ modulo 3.

We proceed to construct the generating 3-cells $A_{u,v,w}$ for $u, v, w \in F \setminus \{1\}$ with $uv \in F$, $vw \in F$, and $uvw \in F$. We obtain pairs of generating 3-cells of the form

$$\begin{tikzcd}[column sep=small] & \sigma_i\sigma_j|\sigma_i \arrow[dr, Rightarrow, "\alpha_{\sigma_i\sigma_j,\sigma_i}"] & \\ \sigma_i|\sigma_j|\sigma_i \arrow[ur, Rightarrow, "\alpha_{\sigma_i,\sigma_j}|\sigma_i"] \arrow[dr, Rightarrow, "\sigma_i|\alpha_{\sigma_j,\sigma_i}"'] & A_{\sigma_i,\sigma_j,\sigma_i} & \sigma_i\sigma_j\sigma_i \\ & \sigma_i|\sigma_j\sigma_i \arrow[ur, Rightarrow, "\alpha_{\sigma_i,\sigma_j\sigma_i}"'] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & \sigma_j\sigma_i|\sigma_j \arrow[dr, Rightarrow, "\alpha_{\sigma_j\sigma_i,\sigma_j}"] & \\ \sigma_j|\sigma_i|\sigma_j \arrow[ur, Rightarrow, "\alpha_{\sigma_j,\sigma_i}|\sigma_j"] \arrow[dr, Rightarrow, "\sigma_j|\alpha_{\sigma_i,\sigma_j}"'] & A_{\sigma_j,\sigma_i,\sigma_j} & \sigma_i\sigma_j\sigma_i \\ & \sigma_j|\sigma_i\sigma_j \arrow[ur, Rightarrow, "\alpha_{\sigma_j,\sigma_i\sigma_j}"'] & \end{tikzcd}$$

or of the form

$$\begin{tikzcd}[column sep=small] & \sigma_k\sigma_i|\sigma_j\sigma_i \arrow[dr, Rightarrow, "\alpha_{\sigma_k\sigma_i,\sigma_j\sigma_i}"] & \\ \sigma_k|\sigma_i|\sigma_j\sigma_i \arrow[ur, Rightarrow, "\alpha_{\sigma_k,\sigma_i}|\sigma_j\sigma_i"] \arrow[dr, Rightarrow, "\sigma_k|\alpha_{\sigma_i,\sigma_j\sigma_i}"'] & A_{\sigma_k,\sigma_i,\sigma_j\sigma_i} & \sigma_k\sigma_i\sigma_j\sigma_i \\ & \sigma_k|\sigma_i\sigma_j\sigma_i \arrow[ur, Rightarrow, "\alpha_{\sigma_k,\sigma_i\sigma_j\sigma_i}"'] & \end{tikzcd} \qquad \begin{tikzcd}[column sep=small] & \sigma_k\sigma_j|\sigma_i\sigma_j \arrow[dr, Rightarrow, "\alpha_{\sigma_k\sigma_j,\sigma_i\sigma_j}"] & \\ \sigma_k|\sigma_j|\sigma_i\sigma_j \arrow[ur, Rightarrow, "\alpha_{\sigma_k,\sigma_j}|\sigma_i\sigma_j"] \arrow[dr, Rightarrow, "\sigma_k|\alpha_{\sigma_j,\sigma_i\sigma_j}"'] & A_{\sigma_k,\sigma_j,\sigma_i\sigma_j} & \sigma_k\sigma_i\sigma_j\sigma_i \\ & \sigma_k|\sigma_i\sigma_j\sigma_i \arrow[ur, Rightarrow, "\alpha_{\sigma_k,\sigma_i\sigma_j\sigma_i}"'] & \end{tikzcd}$$

with i, j and k as above.

We have thus computed the finite coherent presentation of the Artin-Tits monoid of type $\widetilde{A}_2$, which consists of fifteen generating 1-cells, twenty-seven generating 2-cells, and twelve generating 3-cells.

Like in [13], one can further perform a homotopical reduction procedure. Here, the resulting $(3,1)$-polygraph contains: a single generating 0-cell; the generating 1-cells σ_1, σ_2, σ_3; the generating 2-cells $\alpha_{\sigma_2,\sigma_1\sigma_2}$, $\alpha_{\sigma_3,\sigma_2\sigma_3}$, $\alpha_{\sigma_1,\sigma_3\sigma_1}$; and no generating 3-cells. As a side result, we have thus shown that Artin's presentation of the Artin-Tits monoid of type $\widetilde{A}_2$, with the empty set of generating 3-cells, is coherent.

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(Pierre-Louis Curien) Université Paris Cité, CNRS, Inria, IRIF, F-75013, Paris, France
Email address: `curien@irif.fr`

(Alen Đurić) Université Paris Cité, CNRS, Inria, IRIF, F-75013, Paris, France
Email address: `alen.djuric@protonmail.com`

(Yves Guiraud) Université Paris Cité and Sorbonne Université, CNRS, INRIA, IMJ-PRG, F-75013, Paris, France
Email address: `yves.guiraud@imj-prg.fr`