Université Paris 7

Habilitation à diriger des recherches

Rewriting methods in higher algebra

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18 juin 2019

1. Introduction

Rewriting in monoids, from Thue to Squier

1.1. The word problem for monoids

Definition [Thue 1914]

A monoid M has a decidable word problem if there exist

- a finite generating set X for M ($\exists\, \pi : X^* \twoheadrightarrow M$, X^* = free monoid)
- an algorithm that decides, for all $u, v \in X^*$, if $\pi(u) = \pi(v)$ or not

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Examples

$$\mathbb{N}^n \simeq \langle a_1, \ldots, a_n \,|\, (a_i a_j = a_j a_i)_{1 \leqslant i < j \leqslant n} \rangle^+$$

- $X = \{a_1, \ldots, a_n\}$
- Count the a_i's in u and v

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$$T \simeq \Big\langle\, a, b, c, d, e \,\Big|\, ac = ca, ad = da, bc = cb, bd = db, \\ eca = ce, edb = de, ccae = cca \,\Big\rangle^+$$

▶ Undecidable word problem [Tseitin 1958]

Basic rewriting notions [Thue 1914, Newman 1943, . . .]

$X = (X_0 \mid X_1)$ is a presentation (by generators and relations) of a monoid M

u reduces into v ($u \to v$)	if	u can be transformed into v by using relations of X_1 from left to right only

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Proposition

If X is a convergent (= terminating + confluent) presentation of M, then:

1. *Every $u \in X_0^*$ has a unique normal form $\widehat{u}$*
2. *The monoid M is isomorphic to $(\mathrm{Red}(X),\ (u,v) \mapsto \widehat{uv})$*

1.2. Rewriting theory

Example

$$\mathbb{N}^3 \simeq \langle a, b, c \mid ba \to ab, ca \to ac, cb \to bc \rangle^+$$

- Termination: $u \to v$ implies $\|u\| = \|v\|$ and $u \geqslant_{\text{lex}} v$
- Confluence: one critical branching (= min. overlap of two relations)

$$\begin{array}{ccc} & \nearrow & bca \\ \overline{c\underline{b}}\underline{a} & & \\ & \searrow & cab \end{array}$$

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Convergence $\iff$ *Termination* + *Confluence of critical branchings*

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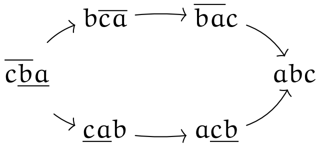

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Example (The standard presentation)

$$M \simeq \left\langle M \setminus \{1\} \,\middle|\, (a|b \to ab)_{a,b\neq 1} \right\rangle^{+}$$

$a|b$ = product in the free monoid M^* over M

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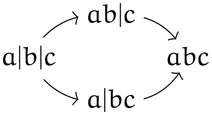

- ▶ Every monoid admits a convergent presentation

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Theorem ( )

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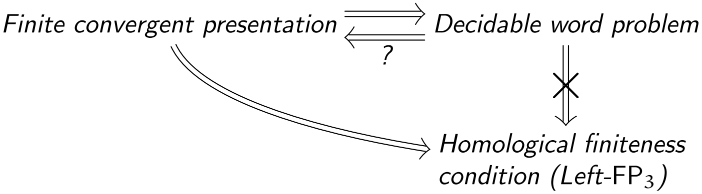

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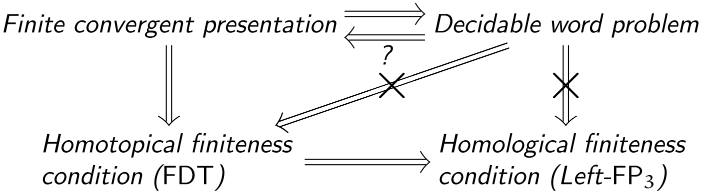

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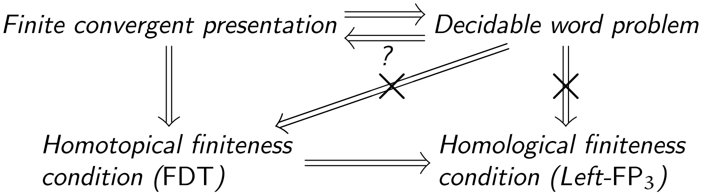

Remark

Finite convergent presentation $\Longrightarrow$ Homotopical finiteness condition

1.3. Rewriting vs. the word problem

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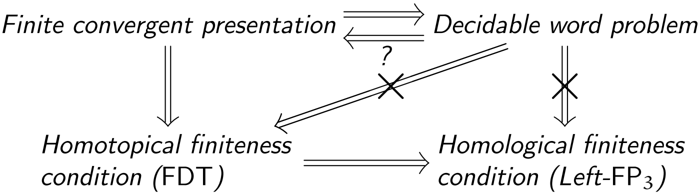

Remark

~~Finite~~ convergent presentation $\Longrightarrow$ Homotopical construction: coherent presentation

- Extract the essence of the proof: Squier's theorem
- Develop new rewriting methods in algebra

2. Squier's theorem

Coherence from convergence

2.1. Coherent presentations of monoids

Definition

Presentation X of a monoid $\quad\rightsquigarrow\quad$ X^* = free (strict) monoidal groupoid

- Objects: elements of X_0^*
- Morphisms:
  - freely generated by the one-step reductions and their inverses

$$vuw \underset{v\alpha^- w}{\overset{v\alpha w}{\rightleftarrows}} vu'w \qquad \text{for all } (u \xrightarrow{\alpha} u') \in X_1 \text{ and } v, w \in X_0^*$$

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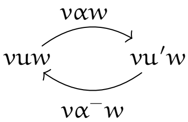

for all $(u \xrightarrow{\alpha} u') \in X_1$ and $v, w \in X_0^*$

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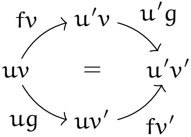

for all $f : u \to u'$ and $g : v \to v'$

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Definition

Let $\mathcal{G}$ be a (strict) monoidal groupoid

Sphere of $\mathcal{G}$: pair $\cdot \rightrightarrows \cdot$ of parallel morphisms of $\mathcal{G}$

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An extension Y of $\mathcal{G}$ is acyclic if

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Definition

Extended presentation of a monoid M: triple $X = (X_0 \,|\, X_1 \,|\, X_2)$ s.t.

$(X_0 \,|\, X_1)$ presentation of M $\qquad$ X_2 extension of $(X_0 \,|\, X_1)^*$

Elements of X_n are called n-cells of X

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Coherent presentation of M: extended presentation X of M s.t.

X_2 is acyclic

2.2. Squier's theorem

Theorem (Squier's theorem [G.-Malbos 2009, 2018])

If $X = (X_0 \,|\, X_1)$ *is a convergent presentation of* M*, then*

$$\mathrm{Sq}(X) = \left(X_0 \;\middle|\; X_1 \;\middle|\; \left(\begin{array}{ccc} & v & \\ {}^{f}\nearrow & & \searrow \\ u & \boxed{S_{f,g}} & \cdot \\ {}_{g}\searrow & & \nearrow \\ & w & \end{array} \right)_{(f,\,g) \text{ critical branching of } X} \right)$$

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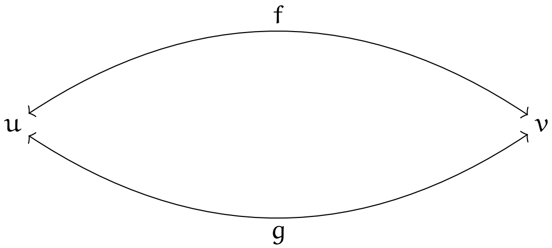
Corollary (Squier 1994)

If M *admits a finite convergent presentation, then* M *is "of finite derivation type" (FDT), i.e. it admits a* <u>*finite*</u> *coherent presentation*

2.2. Squier's theorem

Sketch of proof



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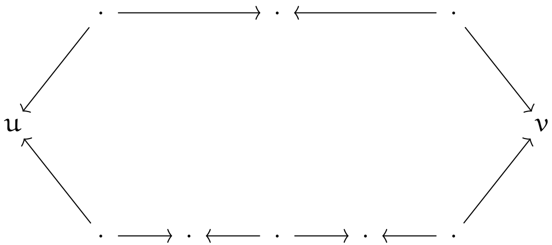

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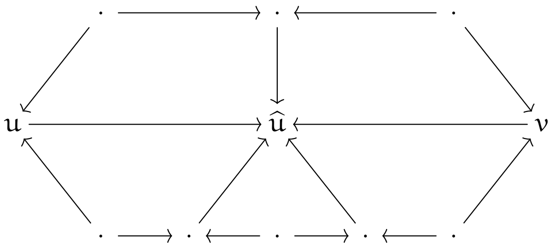

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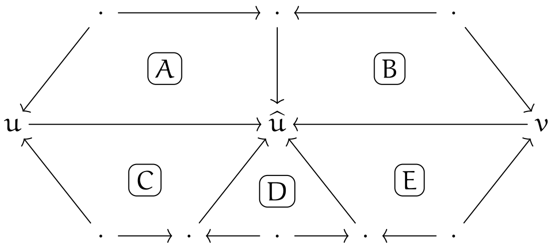

1. Consider a sphere (f, g) of X^*
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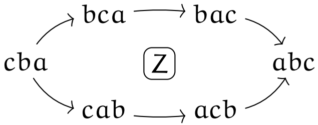
4. Tile each confluence diagram with 2-cells and equalities, by wellfounded induction and case analysis

2.3. Examples

Examples

$$\mathbb{N}^3 \simeq \langle a, b, c \mid ba \to ab, ca \to ac, cb \to bc \rangle^+$$

Convergent presentation, with one (confluent) critical branching Z:



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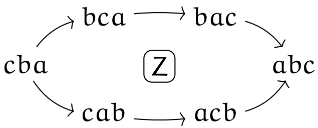

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$$M \simeq \langle a, b, c, d, d' \mid ab \to a, da \to ac, d'a \to ac \rangle^+$$

- M has a decidable word problem
- M is <u>not</u> FDT [Lafont-Prouté 1991]

- M admits no finite convergent presentation

2.3. Examples

Example (The standard coherent presentation)

$$M \simeq \left\langle M \setminus \{1\} \;\middle|\; \left(a|b \xrightarrow{\gamma_{ab}} ab \right)_{a,b \neq 1} \right\rangle^{+}$$

- Convergent presentation of M

▶ Coherent presentation of M:

$$\mathrm{Std}(M) = \left(M \setminus \{1\} \;\middle|\; \left(a|b \xrightarrow{\gamma_{ab}} ab \right)_{a,b \neq 1} \;\middle|\; \left(\begin{array}{ccc} & ab|c & \\ \nearrow & & \searrow \\ a|b|c & \boxed{\gamma_{abc}} & abc \\ \searrow & & \nearrow \\ & a|bc & \end{array} \right)_{a,b,c \neq 1} \right)$$

▶ Every monoid admits a coherent presentation

3. Interlude

Three applications of coherent presentations

3.1. Abelianisation

Theorem ( )

If X is a presentation of a monoid M, then

$$0 \longleftarrow \mathbb{Z} \longleftarrow \mathbb{Z}M \longleftarrow \mathbb{Z}M[X_0] \longleftarrow \mathbb{Z}M[X_1]$$

is an exact sequence of $\mathbb{Z}M$-modules, where:

$\mathbb{Z}M[X_i]$*: free* $\mathbb{Z}M$*-module over* X_i

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If M admits a finite convergent presentation, then M is left-FP_3 *(i.e. $\mathbb{Z}$ admits a partial resolution of length 3 by f.g. projective $\mathbb{Z}M$-modules)*

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Variations [G.-Malbos 2009, 2013]

- Other possible coefficients: right/bi $\mathbb{Z}M$-modules, natural systems
- X_2 also generates the module of identities among relations of X

3.2. Actions of monoids on categories

Definition and question [Deligne 1997]

- $\mathscr{A}ct(\mathrm{M}, \mathcal{C}) = \mathscr{M}on\mathscr{C}at_{\mathsf{ps}}(\mathrm{M}, \mathscr{E}nd(\mathcal{C}))$ pseudomonoidal functors
- $\mathscr{A}ct'(\widetilde{\mathrm{X}}, \mathcal{C}) = \mathscr{M}on\mathscr{C}at_{\mathsf{st}}(\widetilde{\mathrm{X}}, \mathscr{E}nd(\mathcal{C}))$ strict monoidal functors
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▶ Characterise the X's s.t. $\mathcal{A}ct(\mathrm{M}, \mathcal{C}) \approx \mathcal{A}ct'(\widetilde{\mathrm{X}}, \mathcal{C})$, e.g.

$$\mathcal{A}ct(\mathrm{M}, \mathcal{C}) \simeq \mathcal{A}ct'(\widetilde{\mathsf{Std}(\mathrm{M})}, \mathcal{C})$$

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Theorem (Gaussent-G.-Malbos 2015)

Let X be an extended presentation of a monoid M. TFAE:

1. *X is a coherent presentation of M*
2. *$\widetilde{\mathrm{X}}$ is a cofibrant approximation of M (for Lack's model structure)*
3. *$\mathcal{M}on\mathcal{C}at_{\mathsf{ps}}(\mathrm{M}, \mathcal{D}) \approx \mathcal{M}on\mathcal{C}at_{\mathsf{st}}(\widetilde{\mathrm{X}}, \mathcal{D})$ for every strict monoidal category $\mathcal{D}$, and this equivalence is natural in $\mathcal{D}$*

3.3. Categorical coherence theorems

Theorem

Rewriting and Squier's theorem generalise to n-categories [G.-Malbos 2009]

$\rightsquigarrow$ *Coherence theorems of (symmetric, braided) monoidal categories by rewriting [Huet 1985, G.-Malbos 2012]*

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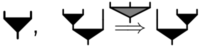
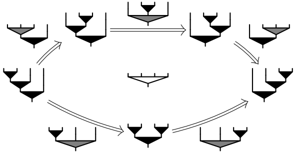
Example

AsCat: strict monoidal $(2,1)$-category

generated by |,  up to 

- Associative category = strict monoidal 2-functor AsCat $\to$ $\mathcal{C}at$

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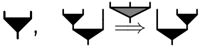
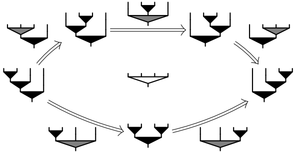
Example

AsCat: strict monoidal $(2,1)$-category

generated by |,  up to 

- Associative category = strict monoidal 2-functor AsCat $\to$ $\mathcal{C}at$
-  is a convergent presentation (of the PRO of semigroups)

▶  is acyclic

▶ In an associative category, all "structural" diagrams commute

4. Coherent presentations of Artin monoids

A concrete computation

4.1. Coxeter groups and Artin monoids

Definition

Coxeter group: group W with a presentation

$$W = \left\langle S \text{ finite} \,\middle|\, \left(a^2 = 1\right)_{a \in S}, \left(bab \cdots = aba \cdots\right)_{a < b \in S} \right\rangle$$

4.1. Coxeter groups and Artin monoids

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Artin monoid associated to a Coxeter group W:

$$B^+(W) = \langle S \text{ finite} \mid (bab\cdots = aba\cdots)_{a<b\in S} \rangle^+$$

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Examples

The permutation groups S_n, e.g. $S_3 = \langle a, b \mid a^2 = b^2 = 1, bab = aba \rangle$

or $S_4 = \langle a, b, c \mid a^2 = b^2 = c^2 = 1, bab = aba, cbc = bcb, ca = ac \rangle$

The braid monoids $B_n^+ = B^+(S_n)$, e.g. $B_3^+ = \langle a, b \mid bab = aba \rangle^+$

4.2. Computation methodology

Problem

The presentation of $B^+(W)$ is not confluent in general, e.g. if $bab = aba$

$$\overline{ba\underline{b}}\underline{ab} \rightarrow abaab$$

$$\overline{ba\underline{b}}\underline{ab} \rightarrow baaba$$

- Squier's theorem is not directly applicable

4.2. Computation methodology

Problem

The presentation of $\mathrm{B}^+(W)$ is not confluent in general, e.g. if $bab = aba$

$$\overline{bab}ab \to abaab$$

$$ba\underline{bab} \to baaba$$

- Squier's theorem is not directly applicable

Solution

1. Add generators and relations $\rightsquigarrow$ convergent presentation
2. Apply Squier's theorem $\rightsquigarrow$ coherent presentation
3. Use Tietze moves to eliminate cells $\rightsquigarrow$ smaller coherent presentation

Example (B_3^+)

$$\left(a, b \quad \middle| \, bab \xrightarrow{\alpha} aba \qquad \middle| \qquad \right)$$

4.2. Computation methodology

Example (B_3^+ with Kapur-Narendran presentation (1985))

$$\left(a, b, x \;\middle|\; bx \xrightarrow{\alpha} xa,\ ab \xrightarrow{\beta} x \;\middle|\; \right)$$

4.2. Computation methodology

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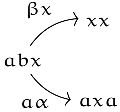

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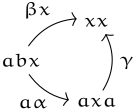

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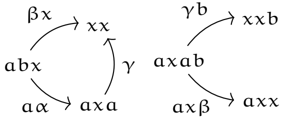

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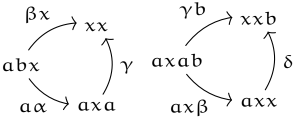

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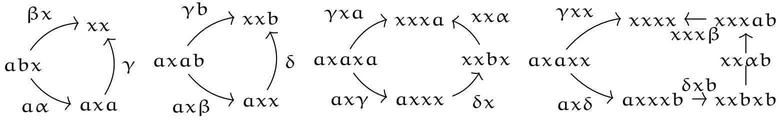

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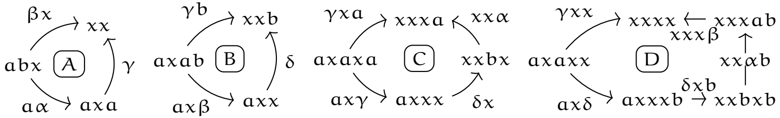
$$\left(a, b, x \;\middle|\; bx \xrightarrow{\alpha} xa, ab \xrightarrow{\beta} x, axa \xrightarrow{\gamma} xx, axx \xrightarrow{\delta} xxb \;\middle|\; A, B, C, D \right)$$



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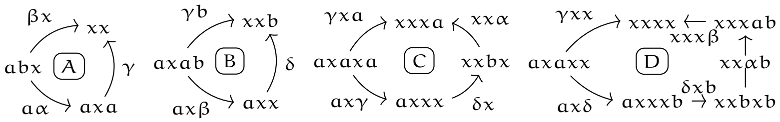

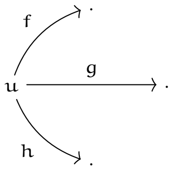

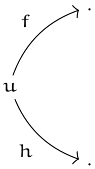

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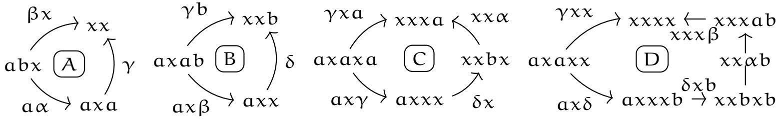

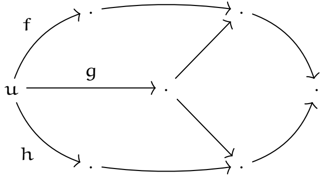

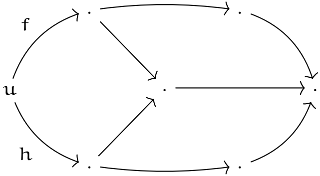

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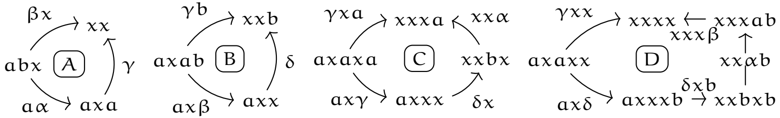

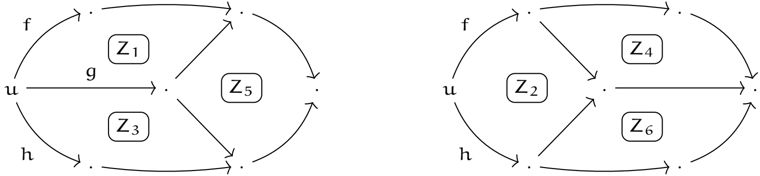

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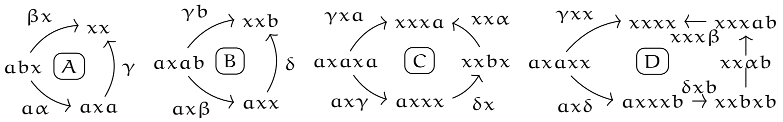

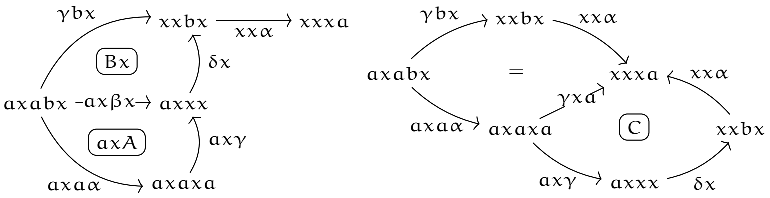

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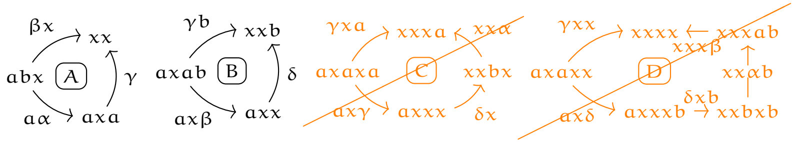

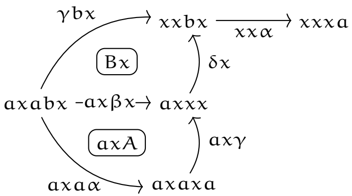

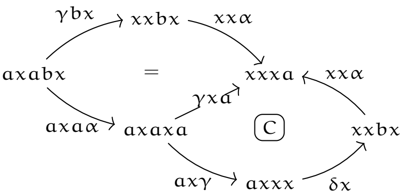

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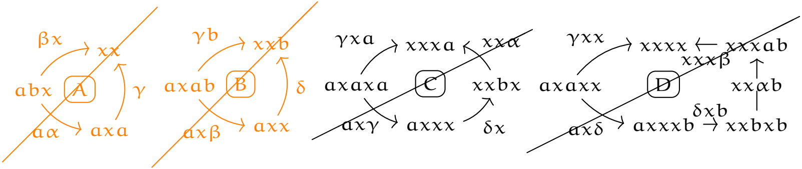

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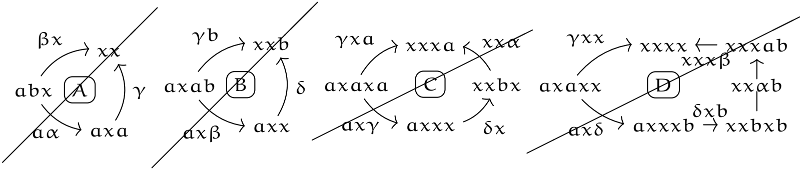

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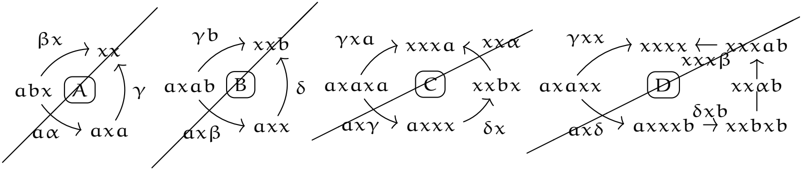

Lemma

The monoid B_3^+ *admits* $(a, b \,|\, bab \to aba \,|\, \emptyset)$ *as a coherent presentation*

Theorem (Gaussent-G.-Malbos 2015)

For every Coxeter group W, the Artin monoid $\mathrm{B}^+(W)$ admits the following coherent presentations:

1. $\mathsf{Gar}(W) = \left(W \setminus \{1\} \,\middle|\, \left(u|v \xrightarrow{\gamma_{u,v}} uv\right)_{u \frown v} \,\middle|\, \right)$

   where $u \frown v$ *means* $l(uv) = l(u) + l(v)$ *in* W

4.3. The Garside and Artin coherent presentations

Theorem (Gaussent-G.-Malbos 2015)

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   where $u \frown v$ *means* $l(uv) = l(u) + l(v)$ *in* W

2. $$\mathrm{Art}(W) = \left(S \;\middle|\; \left(bab\cdots \xrightarrow{Z_{a,b}} aba\cdots \right)_{\substack{a<b \in S \\ |W_{a,b}|<\infty}} \;\middle|\; \left(Z_{a,b,c} \right)_{\substack{a<b<c \in S \\ |W_{a,b,c}|<\infty}} \right)$$

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For every Coxeter group W, the Artin monoid $\mathrm{B}^+(W)$ admits the following coherent presentations:

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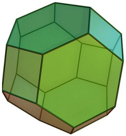
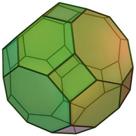
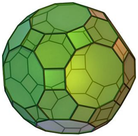
2. $$\mathrm{Art}(W) = \left(S \;\middle|\; \left(bab\cdots \xrightarrow{Z_{ab}} aba\cdots \right)_{\substack{a<b\in S \\ |W_{ab}|<\infty}} \;\middle|\; \left(Z_{abc} \right)_{\substack{a<b<c\in S \\ |W_{abc}|<\infty}} \right)$$

   where Z_{abc} is a (2D-projection of a) "Zamolodchikov" sphere that is entirely determined by the Coxeter type of W_{abc}

A_3 

B_3 

H_3 

5. Higher Squier theory

From coherent presentations to polygraphic resolutions

5.1. Polygraphic resolutions

Definition (Street 1976, 1987; Burroni 1993)

A 0-polygraph is a set X $\rightsquigarrow$ Free (strict) monoid(al 0-groupoid) X^*

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An n-polygraph is a family $X = (X_0 \,|\, \cdots \,|\, X_n)$ s.t.

$(X_0 \,|\, \cdots \,|\, X_{n-1})$: $(n-1)$-polygraph

X_n: extension of $(X_0 \,|\, \cdots \,|\, X_{n-1})^*$

$\rightsquigarrow$ Free monoidal n-groupoid X^*

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An ∞-polygraph is an increasing sequence $X = (X_0 \,|\, X_1 \,|\, \cdots)$ of n-polygraphs

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Definition (Street 1976, 1987; Burroni 1993)

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- $(X_0 \,|\, \cdots \,|\, X_{n-1})$: $(n-1)$-polygraph
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An ∞-polygraph is an increasing sequence $X = (X_0 \,|\, X_1 \,|\, \cdots)$ of n-polygraphs $\rightsquigarrow$ Free monoidal ∞-groupoid X^*

Definition (Métayer 2003)

A polygraphic resolution of a monoid M is an ∞-polygraph X such that

- $(X_0 \,|\, X_1)$ is a presentation of M
- for every $n \geqslant 2$, the extension X_n of $(X_0 \,|\, \cdots \,|\, X_{n-1})^*$ is acyclic

5.2. Squier's polygraphic resolution

Theorem (G.-Malbos 2012, G. 2019)

If X *is a convergent presentation of a monoid* M*, then* M *admits a polygraphic resolution* $\mathrm{Sq}(X)$ *whose* n*-cells are the families* $u_0|\cdots|u_n$ *of elements of* $\mathrm{Red}(X) \setminus \{1\}$ *s.t.* $u_0 \in X_0$ *and, for every* i,

1. $u_i u_{i+1} \notin \mathrm{Red}(X)$
2. *every proper left-factor of* $u_i u_{i+1}$ *is reduced*

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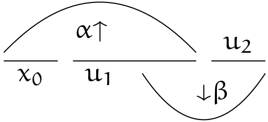
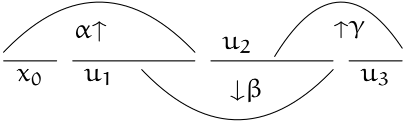
1. $u_i u_{i+1} \notin \mathsf{Red}(X)$
2. *every proper left-factor of* $u_i u_{i+1}$ *is reduced*

Example

If the relations of X are "interreduced", the cells of $\mathsf{Sq}(X)$ are

- the $x_0 \in X_0$
- the $x_0|u_1$ s.t. $x_0 u_1$ is the source of a relation
- the $x_0|u_1|\cdots|u_n$ s.t. $x_0 u_1 \cdots u_n$ is the source of a critical n-branching





5.2. Squier's polygraphic resolution

Squier theory for associative algebras

1. A rewriting theory for associative algebras
   - Strictly more general than Gröbner bases and PBW bases, e.g. $(x, y, z \mid xyz \to x^3 + y^3 + z^3)$ is convergent
2. Polygraphic resolutions in $\infty\mathcal{G}pd(\mathcal{A}lg)$ (instead of $\infty\mathcal{G}pd(\mathcal{M}on)$)

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2. Polygraphic resolutions in $\infty\mathcal{Gpd}(\mathcal{Alg})$ (instead of $\infty\mathcal{Gpd}(\mathcal{Mon})$)

Theorem (G.-Hoffbeck-Malbos 2017)

Let X be a convergent presentation of an associative algebra A. Then A admits a polygraphic resolution $\mathrm{Sq}(X)$ in $\infty\mathcal{Gpd}(\mathcal{Alg})$, whose n-cells are the critical n-branchings of X

5.2. Squier's polygraphic resolution

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Theorem (G.-Hoffbeck-Malbos 2017)

Let X be a convergent presentation of an associative algebra A. Then A admits a polygraphic resolution $\mathsf{Sq}(X)$ *in $\infty\mathcal{G}pd(\mathcal{A}lg)$, whose n-cells are the critical n-branchings of X*

Ongoing and planned work

Generalise ∞-categories, polygraphs and the model structure to

- $\infty\mathcal{G}pd(\mathcal{M}on(\mathcal{C}))$, where $\mathcal{C}$ is some monoidal category [with M. Fiore]
- $\infty\mathcal{G}pd(\mathcal{M}od(\mathcal{A}))$, where $\mathcal{A}$ is a projective sketch [with D. Ara]

5.3. Polygraphic resolution of Artin monoids

Conjecture

For every Coxeter group W, the Artin monoid $\mathrm{B}^+(W)$ admits the following polygraphic resolutions:

1. $\mathsf{Gar}(W) = \left\{ \left(\gamma_{u_0\cdots u_n}\right)_{u_0 \cdots u_n} \right\}$ $\qquad \gamma_{u_0\cdots u_n} = n$-*cube*
   - *Generalises to monoids with a Garside family*

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2. $\mathsf{Art}(W) = \left\{ \left(\mathsf{Z}_I\right)_{I \subseteq S \text{ s.t. } |W_I| < \infty} \right\}$ $\qquad \mathsf{Z}_I =$ *Zamolodchikov* $|I|$*-cell of* W_I

   - *Related to the Salvetti complex of* $\mathrm{B}(W)$
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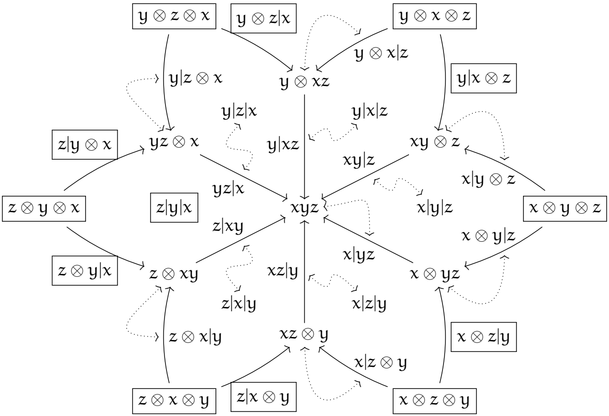
Ongoing and planned work

- Resolutions $\mathsf{Gar}(W)$ and $\mathsf{Art}(W)$ of $\mathbb{k}\mathrm{B}^+(W)$ in $\mathrm{dg}\mathscr{A}lg$ [with M. Picantin]
- Polygraphic versions [Postdoc A. Hadzihasanovic, PhD A. Đjurić]

Merci



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