PERFECTOID RINGS AS THOM SPECTRA

ZHOUHANG MAO

Abstract. The Hopkins-Mahowald theorem realizes the Eilenberg-Maclane spectra $H\mathbb{F}_p$ as Thom spectra for all primes $p\in\mathbb{N}_{>0}$. In this article, we record a known proof of a generalization of Hopkins-Mahowald theorem, realizing Hk as Thom spectra for perfect rings k, and we provide a further generalization by realizing HR as Thom spectra for perfectoid rings R. We also discuss even further generalizations to prisms (A, I) and indicate how to adapt our proofs to Breuil-Kisin case.

Table of contents

Date: November 24, 2020

Note: This article has been written using GNU TeXmacs; see www.texmacs.org.

1. Introduction

In this article, since most of our results are p-typical, we fix a prime $p \in \mathbb{N}_{>0}$. We first describe the classical Hopkins-Mahowald theorem. We know that $\mathbb{F}_p \cong \mathbb{Z}_p/p$, that is to say, $\mathbb{F}_p$ is the free $\mathbb{Z}_p$-algebra in which $p=0$. For some reasons, we need to extend this kind of results to a category of "less linear" algebras in which the addition is not commutative or even associative on the nose, but only up to coherent homotopy. To be more precise, we need to understand whether the (Eilenberg-Maclane) ring spectrum $H\mathbb{F}_p$ is still the free object in the category of $\mathbb{S}_p^\wedge$-algebras satisfying certain associativity and commutativity with $p=0$? The classical Hopkins-Mahowald theorem answers this affirmatively: they are the free object in the category of $\mathbb{E}_2$-$\mathbb{S}_p^\wedge$-algebras with $p=0$. There are two ways to describe "free $\mathbb{E}_2$-algebras with $p=0$". In this article, we will mainly adopt the description via Thom spectra. We will go to another, more direct and natural but technically more burdened description in Section 7. We start with formal definitions of Thom spectra with informal illustrations and refer to [AB19] for further discussions.

Definition 1.1. Given a ring spectrum R, we define the ∞-category $\mathrm{BGL}_1(R)$ to be the full subcategory of $\mathrm{LMod}_R^{\simeq}$ spanned by left R-module spectra equivalent to R, where we denote by $\mathcal{C}^{\simeq}$ the maximal groupoid associated to an ∞-category $\mathcal{C}$.

Remark 1.2. The ∞-category $\mathrm{BGL}_1(R)$ is in fact an ∞-groupoid, and if we further suppose that R is an $\mathbb{E}_{n+1}$-ring spectrum, then $\mathrm{BGL}_1(R)$ inherits an $\mathbb{E}_n$-monoidal structure from LMod_R.

We admit the following result, which could be understand as an analogue of the fact that $\pi_1(BG)=G$ for any discrete group G:

Proposition 1.3. $\pi_1(\mathrm{BGL}_1(R))=\mathrm{GL}_1(\pi_0 R)$ *for any ring spectra R. Concretely, an invertible element $a\in\pi_0 R$ corresponds to a multiplication map $m_a: R\to R$ in* $\mathrm{BGL}_1(R)$.

Remark 1.4. In fact, $\mathrm{BGL}_1(R)$ is a delooping of the group of invertible elements in R.

Now we recall the definition of Thom spectra:

Definition 1.5. Given a ring spectrum R, a space X and a map $f\colon X\to\mathrm{BGL}_1(R)$, *the Thom spectrum Mf associated to f* is the colimit of the composition

$$X\to\mathrm{BGL}_1(R)\to\mathrm{LMod}_R$$

We note that by definition of colimits, we can understand the colimit as a kind of "free objects satisfying several equations". We will choose a special space X to encode the $\mathbb{E}_2$-commutativity (understood as a generalized version of classical associativity, a collection of equations) and a map $f: X\to\mathrm{BGL}_1(R)$ to encode the "equation" $p=0$.

Remark 1.6. As a special case of [Lur09, Proposition 4.1.2.6], any homotopy equivalence of Kan complexes is cofinal, therefore the formation of the colimit does not depend on the choice of models of the space X.

Remark 1.7. In this article, we only consider the case that R is a connective $\mathbb{E}_\infty$-ring spectrum. As a consequence, we can replace LMod_R by Mod_R and the Thom spectrum Mf is connective.

Remark 1.8. In Definition 1.5, if X is endowed with an $\mathbb{E}_n$-algebra structure, and f is assumed to be $\mathbb{E}_n$-monoidal, then the Thom spectrum Mf naturally inherits an $\mathbb{E}_n$-R-algebra structure. In this case, we will call Mf the $\mathbb{E}_n$-Thom spectrum associated to f.

In the classical Hopkins-Mahowald theorem, we will choose $X = \Omega^2 S^3$, the free $\mathbb{E}_2$-group in the ∞-category $\mathcal{S}$ of spaces.

Remark 1.9. As a special case, $\pi_1(\mathrm{BGL}_1(\mathbb{S}_p^\wedge)) = \mathrm{GL}_1(\mathbb{Z}_p) = \{a \in \mathbb{Z}_p \mid a \bmod p \neq 0\}$. The invertible element $1 - p\,u$ in $\mathbb{Z}_p$ gives rise to a map $S^1 \to \mathrm{BGL}_1(\mathbb{S}_p^\wedge)$ where $u \in \mathrm{GL}_1(\mathbb{Z}_p)$ is an invertible element in $\mathbb{Z}_p$. Since the p-adic sphere spectrum $\mathbb{S}_p^\wedge$ is an $\mathbb{E}_\infty$-ring spectrum, by Remark 1.2 this map extends to a double loop map $\Omega^2 S^3 \simeq \Omega^2 \Sigma^2 S^1 \to \mathrm{BGL}_1(\mathbb{S}_p^\wedge)$, which we denote by $f_{\mathbb{F}_p, pu}$.

We note that the choice of $1 - p\,u$ essentially imposes an equation $1 - p\,u = 1$. This could be seen by the fact that taking the colimit along $f_{\mathbb{F}_p, pu}$ is essentially taking the homotopy orbits of the $\Omega^2 S^3$-action, which is somehow "multiplying by" $1 - p\,u$.

Remark 1.10. In the first drafts of this article, we simply took $u = 1$. Later, we realized that it might be easier to introduce u to fix a gap in commutative algebra for technical reasons.

Now we formulate the classical Hopkins-Mahowald theorem (cf. [AB19, Theorem 5.1], where $u = 1$, but the proof works for the general case. See also [KN, Theorem A.1]):

Theorem 1.11. (Hopkins-Mahowald) *The Eilenberg-Maclane spectrum $H\mathbb{F}_p$ is the $\mathbb{E}_2$-Thom spectrum associated to the map $f_{\mathbb{F}_p, pu} : \Omega^2 S^3 \to \mathrm{BGL}_1(\mathbb{S}_p^\wedge)$.*

This arouses a natural question: what other discrete rings are Thom spectra in a similar fashion? The first guess will come from the observation that $\mathbb{Z}_p \cong W(\mathbb{F}_p)$, so it would be natural to ask whether we have similar results for perfect $\mathbb{F}_p$-algebras?

In this article, our main purpose is to show that this is the case for perfectoid rings (which is inspired by computational results of topological Hochschild homology of perfectoid rings in [BMS19]), and consequently, for perfect $\mathbb{F}_p$-algebras. In order to do so, we need the concept of spherical Witt vectors $W^+(k)$ for perfect $\mathbb{F}_p$-algebras k, which we will recall in section 2. For the moment, we will take advantage of the fact that $\pi_0(W^+(k)) = W(k)$ where $W(k)$ is the ring of (classical) Witt vectors. One example is that $W^+(\mathbb{F}_p) \simeq \mathbb{S}_p^\wedge$.

Remark 1.12. Given a perfectoid ring R, denote by ξ a generator of the kernel of Fontaine's pro-infinitesimal thickening $\theta : W(R^\flat) \to R$, which we will review in section 4. As in Remark 1.9, the invertible element in $W(R^\flat)$, $1 - \xi \in \mathrm{GL}_1(W(R^\flat)) = \pi_1(\mathrm{BGL}_1(W^+(R^\flat)))$ gives rise to a map $S^1 \to \mathrm{BGL}_1(W^+(R^\flat))$ which extends to a double loop map $f_{R,\xi} : \Omega^2 S^3 \to \mathrm{BGL}_1(W^+(R^\flat))$.

Theorem 1.13. (Main Theorem) *The Eilenberg-Maclane spectrum HR is the $\mathbb{E}_2$-Thom spectrum associated to the map $f_{R,\xi}$ for any perfectoid ring R.*

Fontaine's pro-infinitesimal thickening θ is in fact surjective. Note that $R \cong W(R^\flat)/\xi$, and our result is amount to say that the ring spectrum HR is a free $\mathbb{E}_2$-$W^+(R^\flat)$-algebra with $\xi = 0$.

Remark 1.14. When R is a perfect $\mathbb{F}_p$-algebra, we can take $\xi = p\,u$ where $u \in \mathrm{GL}_1(R)$ is an invertible element in R, and we note that $R^\flat = R$. Especially, when $R = \mathbb{F}_p$, $f_{R,pu}$ coincides with $f_{\mathbb{F}_p,pu}$, hence Theorem 1.13 generalizes Theorem 1.11.

Remark 1.15. The composite map $W^+(R^\flat) \xrightarrow{\tau_{\leq 0}} HW(R^\flat) \xrightarrow{H\theta} HR$ should be understood as a spherical analogue of Fontaine's map $\theta: W(R^\flat) \to R$. We will establish a universal property, Proposition 4.18, similar to Fontaine's, Proposition 4.16, which might be of independent interest.

The motivation to realize $H\mathbb{F}_p$ as a free $\mathbb{E}_2$-algebra with $p = 0$ is that it describes a direct "generation-relation" like description with respect to the (p-completed) sphere spectrum $\mathbb{S}_p^\wedge$. Similarly, realization of HR as a free $\mathbb{E}_2$-$W^+(R)$-algebra with $\xi = 0$ enables us to relate HR more directly to the ring $W^+(R^\flat)$ of spherical Witt vectors, which allows us to deduce "topological" results about these rings. For example, as a consequence, we can compute the topological Hochschild homology $\mathrm{THH}(HR)$ (of a perfectoid ring R) as an $\mathbb{E}_1$-ring spectrum and deduce Bökstedt's periodicity. By [KN, Proposition 4.7], as in the proof of Theorem 4.1 there, we have

Proposition 1.16. *The (relative) topological Hochschild homology* $\mathrm{THH}(HR/W^+(R^\flat)) \simeq HR \otimes \Omega S^3$ *as* $\mathbb{E}_1$-$W^+(R^\flat)$*-algebras for any perfectoid ring* R.

The proof is somehow technical, but essentially it is similar to the classical computation of the Hochschild homology $\mathrm{HH}(R/W(R^\flat))$, via resolving R by $W(R^\flat)$-CDGAs. We refer to first paragraphs of the proof of [HN19, Theorem 1.3.2] for this classical case. As a consequence of Proposition 1.16, we have (see subsection 5.5):

Proposition 1.17. *The (absolute) topological Hochschild homology* $\mathrm{THH}(HR)_p^\wedge \simeq HR \otimes \Omega S^3$ *as* $\mathbb{E}_1$*-ring spectra.*

By known results on the homology of ΩS^3 (a classical reference is [Bot82]), we deduce Bökstedt's periodicity for perfectoid rings (cf. [BMS19, Theorem 6.1]).

Corollary 1.18. (Bökstedt's periodicity) $\pi_*(\mathrm{THH}(HR)_p^\wedge) \cong R[u]$ *where* u *is any generator of* $\pi_2(\mathrm{THH}(HR)_p^\wedge)$ *as a* $\pi_0(\mathrm{THH}(HR)_p^\wedge)$*-module.*

In fact, our question was motivated by Bökstedt's periodicity for perfectoid rings: we wanted to understand why Bökstedt's periodicity holds.

Further generalizations of Theorem 1.13 to prisms, the concept introduced in [BS19], seem plausible. However, we are only capable to reach another special case of prisms motivated by Breuil-Kisin cohomology, parallel to the perfectoid case, proposed by Matthew Morrow:

Theorem 1.19. *Let* A *be complete discrete valuation ring of mixed characteristic with residue field* k *being perfect of characteristic* p*. Then the Eilenberg-Maclane spectrum* HA *is the* $\mathbb{E}_2$*-Thom spectrum associated to a map* $f_E: \Omega^2 S^3 \to \mathrm{BGL}_1(W^+(k)[[u]])$.

Inspired by [KN19, Section 9], we will also provide a version of Hopkins-Mahowald theorem for complete regular local rings:

Theorem 1.20. *Let* $(A, \mathfrak{m})$ *be a complete regular local ring of mixed characteristic with residue field* k *being perfect of characteristic* p*. Let* $(a_1, \ldots, a_n) \subseteq \mathfrak{m}$ *be a regular sequence which generates the maximal ideal* $\mathfrak{m}$*. Then the Eilenberg-Maclane spectrum* HA *is the* $\mathbb{E}_2$*-Thom spectrum associated to a map* $f_A: \Omega^2 S^3 \to \mathrm{BGL}_1(W^+(k)[[u_1, \ldots, u_n]])$.

In this article, we will first review spherical Witt vectors. We then record a known proof of perfect rings being Thom spectrum, the special case of Theorem 1.13 for perfect rings, which we learn from Sanath Devalapurkar, but the proof is also well-known to experts such as Achim Krause and Thomas Nikolaus, see [KN19]. This result is needed in the proof of the general case of Theorem 1.13. Then we start with recalling the definition and some basic properties of perfectoid rings, and prove Theorem 1.13. As far as we know, although this is known to several experts, the proof is not found in the literature. We will finally discuss further generalizations to prisms in Section 6, and especially Hopkins-Mahowald theorem for Breuil-Kisin cases, which seems also to be known by experts (see [KN19, Remark 3.4]). We take an opportunity to write down those proofs. The author thanks Matthew Morrow for various suggestions during the construction of this article.

Warning 1.21. For spectra M, N, we will denote the smash product of M, N by $M \otimes N$. Let R be an $\mathbb{E}_1$-ring (spectrum), M a right R-module (spectrum) and N a left R-module (spectrum), we will denote the relative tensor product by $M \otimes_R N$. In order to avoid possible ambiguities, for discrete rings A, right A-modules P and left A-modules Q, we will denote the ordinary (algebraic) tensor product by $\mathrm{Tor}_0^A(P, Q)$ (instead of $P \otimes_A Q$). It is important that in general the Eilenberg-Maclane spectrum $H\,\mathrm{Tor}_0^A(P, Q)$ do not coincide with the relative tensor product $HP \otimes_{HA} HQ$ of spectra. Rather, the relative tensor $HP \otimes_{HA} HQ$ coincides with the Eilenberg-Maclane spectrum $H(P \otimes_A^{\mathbb{L}} Q)$ of the derived tensor product. Since the concept of the derived tensor product does not play a great role in this article, we will not use the notation $\otimes_A^{\mathbb{L}}$, and we will uniformly preserve the notation $\otimes$ for smash products and relative tensor products of spectra.

Notation 1.22. *In this article, we mainly adopt notations in [Lur17], [Lur18a] and [Lur18b]. In particular, we will let* LMod_R *denote the* ∞*-category of an* $\mathbb{E}_1$*-ring* R*, let* Mod_R *denote the symmetric monoidal* ∞*-category of an* $\mathbb{E}_\infty$*-ring* R *and let* $\mathrm{Alg}_R^{\mathbb{E}_n}$ *denote the* ∞*-category of* $\mathbb{E}_n$*-*R*-algebras for an* $\mathbb{E}_\infty$*-ring* R *and a positive integer* $n \in \mathbb{N}_{>0}$*. In particular, we will denote* $\mathrm{Alg}_R^{\mathbb{E}_\infty}$ *by* CAlg_R*, referred to as the* ∞*-category of commutative* R*-algebras. On the other hand, we will denote* $\mathrm{Mod}_R^\heartsuit$ *the* ∞*-category of discrete* R*-modules, and* $\mathrm{CAlg}_R^\heartsuit$ *the* ∞*-category of discrete commutative* R*-algebras.*

2. Recollection of spherical Witt vectors

In this section, we will review the definition and some basic properties of spherical Witt vectors. We quote some definitions and propositions from [Lur18a, Section 5.2].

Definition 2.1. ([Lur18a, Definition 5.2.1]) Let A be a connective $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 A$ be a finitely generated ideal, and set $A_0 = \pi_0(A)/I$. Suppose that we are given a commutative diagram of connective $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & \sigma & \downarrow \\ HA_0 & \xrightarrow{f_0} & HB_0 \end{array}$$

where B_0 is a discrete commutative ring. We will say that σ *exhibits* f *as an* A*-thickening of* f_0 if the following conditions are satisfied:

a) The $\mathbb{E}_\infty$-ring B is I-complete as an A-module;

b) The diagram σ induces an isomorphism of commutative rings $\pi_0(B)/I\pi_0(B) \to B_0$;

c) Let R be any connective $\mathbb{E}_\infty$-algebra over A which is I-complete. Then the canonical map

$$\mathrm{Map}_{\mathrm{CAlg}_A}(B, R) \to \mathrm{Hom}_{\mathrm{CAlg}^\heartsuit_{A_0}}(B_0, \pi_0(R)/I\pi_0(R))$$

is a homotopy equivalence. In particular, the mapping space $\mathrm{Map}_{\mathrm{CAlg}_A}(B, R)$ is discrete up to homotopy equivalence, that is, each connected component is contractible.

Remark 2.2. (Uniqueness, [Lur18a, Remark 5.2.2]) Let A be a connective $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 A$ be a finitely generated ideal, and set $A_0 = \pi_0(A)/I$. Suppose that we are given a homomorphism of commutative rings $f_0: A_0 \to B_0$. It follows immediately from the definition that if there exists a diagram σ:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & \sigma & \downarrow \\ HA_0 & \xrightarrow{f_0} & HB_0 \end{array}$$

which exhibits f as an A-thickening of f_0, then the morphism f (and the diagram σ) is uniquely determined up to equivalence.

Remark 2.3. ([Lur18a, Remark 5.2.4]) Suppose that we are given a commutative diagram of commutative $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & & \downarrow \\ A' & \xrightarrow{f'} & B' \\ \downarrow & & \downarrow \\ HA_0 & \xrightarrow{f_0} & HB_0 \end{array}$$

Assume that A_0, B_0 are discrete rings and the left vertical maps induce surjective ring morphisms $\pi_0 A \to \pi_0 A' \to A_0$ whose composition has kernel $I \subseteq \pi_0 A$. Suppose that the outer rectangle exhibits f as an A-thickening of f_0 and that the upper square exhibits B' as an I-completion of $B \otimes_A A'$. Then the lower square exhibits f' as an A'-thickening of f_0.

Theorem 2.4. ([Lur18a, Theorem 5.2.5]) *Let A be a connective $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 A$ be a finitely generated ideal, and set $A_0 = \pi_0(A)/I$. Suppose that A_0 is an $\mathbb{F}_p$-algebra such that HA_0 is almost perfect as an A-module and that the Frobenius map $\varphi_{A_0}: A_0 \to A_0$ is flat. Let $f: A_0 \to B_0$ be a morphism of commutative $\mathbb{F}_p$-algebras which is relatively perfect, then there exists a diagram*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & \sigma & \downarrow \\ HA_0 & \xrightarrow{f_0} & HB_0 \end{array}$$

which exhibits f as an A-thickening of f_0. Moreover, σ is a pushout square.

Example 2.5. (Classical Witt vectors, [Lur18a, Example 5.2.6]) In the statement of Theorem 2.4 take $A = H\mathbb{Z}_p$ and $I = p\,\mathbb{Z}_p$. Then $A_0 = \pi_0(A)/I$ is the finite field $\mathbb{F}_p$ and a map $f_0: A_0 \to B_0$ of discrete rings is relative perfect if and only if B_0 is a perfect $\mathbb{F}_p$-algebra. If this condition is satisfied, then Theorem 2.4 allows us to lift B_0 to an $\mathbb{E}_\infty$-$H\mathbb{Z}_p$-algebra which is complete with respect to the ideal $p\,\mathbb{Z}_p$ and for which the quotient $\pi_0(B)/p\,\pi_0(B)$ is isomorphic to B_0. This $\mathbb{Z}_p$-algebra is in fact the Eilenberg-Maclane spectrum of the ring of Witt vectors $W(B_0)$. See also [Ser79, Section II.5, Proposition 10] for a classical description of this universal property.

Example 2.6. (Spherical Witt vectors, [Lur18a, Example 5.2.7]) In the statement of Theorem 2.4 take $A = \mathbb{S}_p^\wedge$ and $I = (p)$. Then $A_0 = \pi_0(A)/I$ is the finite field $\mathbb{F}_p$ and a morphism $f_0: A_0 \to B_0$ is relative perfect if and only if B_0 is a perfect $\mathbb{F}_p$-algebra. If this condition is satisfied, Theorem 2.4 allows us to lift B_0 to an $\mathbb{E}_\infty$-$\mathbb{S}_p^\wedge$-algebra which is complete with respect to the ideal (p) and the tensor product $H\mathbb{F}_p \otimes_{\mathbb{S}_p^\wedge} B \simeq \pi_0(B)/p\,\pi_0(B)$ is isomorphic to B_0. This is the $\mathbb{E}_\infty$-ring $W^+(B_0)$ of "spherical" Witt vectors.

Proposition 2.7. *$\pi_0(W^+(k))$ is isomorphic to $W(k)$, the ring of Witt vectors, and $HW(k) \simeq W^+(k) \otimes_{\mathbb{S}_p^\wedge} H\mathbb{Z}_p$ for any perfect $\mathbb{F}_p$-algebra k.*

Proof. First, we have a commutative diagram

$$\begin{array}{ccc} \mathbb{S}_p^\wedge & \longrightarrow & W^+(k) \\ \downarrow & & \\ H\mathbb{Z}_p & & \downarrow \\ \downarrow & & \\ H\mathbb{F}_p & \longrightarrow & Hk \end{array}$$

Figure 2.1.

where the outer square is given by Theorem 2.4. The right vertical map $W^+(k) \to Hk$ factors through the pushout $W^+(k) \otimes_{\mathbb{S}_p^\wedge} H\mathbb{Z}_p$ in the category of $\mathbb{E}_\infty$-rings. Note that $\mathbb{S}_p^\wedge$ is a coherent ring as in Definition A.10, and $H\mathbb{Z}_p \simeq H\pi_0(\mathbb{S}_p^\wedge)$ is an almost perfect $\mathbb{S}_p^\wedge$-module by Corollary A.12, which implies that $W^+(k) \otimes_{\mathbb{S}_p^\wedge} H\mathbb{Z}_p$ is an almost perfect $W^+(k)$-module by Proposition A.8. By Definition 2.1, $W^+(k)$ is a p-complete $\mathbb{E}_\infty$-$\mathbb{S}_p^\wedge$-algebra, therefore by Proposition A.27, the spectrum $W^+(k) \otimes_{\mathbb{S}_p^\wedge} H\mathbb{Z}_p$ is p-complete. Now we take $A = \mathbb{S}_p^\wedge$, $A' = H\mathbb{Z}_p$, $A_0 = H\mathbb{F}_p$, $B = W^+(k)$, $B' = W^+(k) \otimes_{\mathbb{S}_p^\wedge} H\mathbb{Z}_p$ and $B_0 = Hk$ in Remark 2.3, we deduce that the lower square

$$\begin{array}{ccc} H\mathbb{Z}_p & \longrightarrow & W^+(k) \otimes_{\mathbb{S}_p^\wedge} H\mathbb{Z}_p \\ \downarrow & & \downarrow \\ H\mathbb{F}_p & \longrightarrow & Hk \end{array}$$

constitutes a commutative diagram of thickening as in Definition 2.1. Then it follows from Remark 2.2 and Example 2.5 that $W^+(k) \otimes_{\mathbb{S}_p^\wedge} H\mathbb{Z}_p$ is equivalent to $HW(k)$ as $\mathbb{E}_\infty$-$H\mathbb{Z}_p$-algebras, which implies that $W(k) \cong \pi_0(HW(k)) \cong \mathrm{Tor}_0^{\pi_0(\mathbb{S}_p^\wedge)}(\pi_0(W^+(k)),\, \pi_0(H\mathbb{Z}_p)) \cong \mathrm{Tor}_0^{\mathbb{Z}_p}(\pi_0(W^+(k)), \mathbb{Z}_p) \cong \pi_0(W^+(k))$. $\square$

Proposition 2.8. (Recognition of Thickenings, [Lur18a, Proposition 5.2.9]) *Let A be a connective $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 A$ be a finitely generated ideal, and set $A_0 = \pi_0(A)/I$. Suppose that A_0 is an $\mathbb{F}_p$-algebra which is almost perfect as an A-module and that the Frobenius map $\varphi_{A_0}: A_0 \to A_0$ is flat. Suppose we are given a commutative diagram of connective $\mathbb{E}_\infty$-rings σ:*

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow & \sigma & \downarrow \\ A_0 & \xrightarrow{f_0} & B_0 \end{array}$$

where f_0 is a relative perfect morphism of commutative $\mathbb{F}_p$-algebras. Then σ exhibits f as an A-thickening of f_0 if and only if the following conditions are satisfied:

i. The $\mathbb{E}_\infty$-ring B is I-complete as an A-module;

ii. The diagram σ is a pushout square.

3. Perfect rings being Thom spectra

We first admit a (superficially) slightly stronger Hopkins-Mahowald's theorem for sake of convenience. Given a perfect $\mathbb{F}_p$-algebra k and an invertible element $u \in \mathrm{GL}_1(W(k))$, as a special case of Remark 1.12, we have a map $f_{k,pu}: \Omega^2 S^3 \to \mathrm{BGL}_1(W^+(k))$.

Theorem 3.1. (Hopkins-Mahowald for k) *The Eilenberg-Maclane spectrum Hk is the $\mathbb{E}_2$-Thom spectrum associated to the map $f_{k,pu}$.*

For technical reasons, we start with the special case that $u \in \mathrm{GL}_1(\mathbb{Z}_p) \subseteq \mathrm{GL}_1(W(k))$. In this case, it is a direct consequence of that for $\mathbb{F}_p$.

Lemma 3.2. *Theorem 3.1 is true when $u \in \mathrm{GL}_1(\mathbb{Z}_p) \subseteq \mathrm{GL}_1(W(k))$.*

Proof. We note that the image of the multiplication map $m_{1-pu}: \mathbb{S}_p^\wedge \to \mathbb{S}_p^\wedge$ given by $1 - pu \in \pi_0(\mathbb{S}_p^\wedge) \cong \mathbb{Z}_p$ under the canonical (symmetric monoidal) functor $W^+(k) \otimes_{\mathbb{S}_p^\wedge} -: \mathrm{Mod}_{\mathbb{S}_p^\wedge} \to \mathrm{Mod}_{W^+(k)}$ is still a multiplication map $m_{1-pu}: W^+(k) \to W^+(k)$ given by $1 - pu \in \pi_0(W^+(k)) \cong W(k)$, and therefore the map $f_{k,pu}$ coincides with the composition map

$$\Omega^2 S^3 \xrightarrow{f_{\mathbb{F}_p,pu}} \mathrm{BGL}_1(\mathbb{S}_p^\wedge) \xrightarrow{W^+(k) \otimes_{\mathbb{S}_p^\wedge} -} \mathrm{BGL}_1(W^+(k))$$

Since $Mf_{\mathbb{F}_p,pu} \simeq H\mathbb{F}_p$ as $\mathbb{E}_2$-ring spectra by Theorem 1.11,

$$Mf_{k,pu} \simeq W^+(k) \otimes_{\mathbb{S}_p^\wedge} Mf_{\mathbb{F}_p,pu} \simeq W^+(k) \otimes_{\mathbb{S}_p^\wedge} H\mathbb{F}_p \simeq Hk$$

as $\mathbb{E}_2$-ring spectra, where the first equivalence follows from the fact that the functor $W^+(k) \otimes_{\mathbb{S}_p^\wedge} -$ is a left adjoint therefore commutes with colimits and the last equivalence is given by the last claim in Theorem 2.4. □

In order to prove Theorem 3.1, it suffices to show that $Mf_{k,pu} \simeq Mf_{k,p}$ holds for all $u \in \mathrm{GL}_1(W(k))$, therefore $Mf_{k,pu} \simeq Mf_{k,p} \simeq Hk$ by Lemma 3.2. We will base the proof on a universal property of Thom spectra which we will not use elsewhere, and the author looks forward to an alternative proof which does not depend on this universal property.

Lemma 3.3. (Proposition 4.9 in [AB19] along with the discussions after Lemma 4.6) *The* $\mathbb{E}_2$*-Thom spectrum* $Mf_{k,pu}$ *satisfies the following universal property: For all* $\mathbb{E}_2$*-*$W^+(k)$*-algebras* A*, the mapping space* $\mathrm{Map}_{\mathrm{Alg}^{\mathbb{E}_2}_{W^+(k)}}(Mf_{k,pu}, A)$ *could be naturally identified with the space of null-homotopies of the composite map* $W^+(k) \xrightarrow{m_{pu}} W^+(k) \xrightarrow{\eta} A$ *in the category of* $W^+(k)$*-modules where* $\eta: W^+(k) \to A$ *is the canonical map given by the* $\mathbb{E}_2$*-*$W^+(k)$*-algebra structure, and* $m_{pu}: W^+(k) \to W^+(k)$ *is the multiplication map given by* $p\,u \in W(k) = \pi_0(W^+(k))$*.*

Proof of Theorem 3.1. Note that the multiplication map $m_u: W^+(k) \to W^+(k)$ is an equivalence of $W^+(k)$-modules since $u \in W(k) = \pi_0(W^+(k))$ is invertible. Hence by Lemma 3.3, the map m_u induces an equivalence of spaces $\mathrm{Map}_{\mathrm{Alg}^{\mathbb{E}_2}_{W^+(k)}}(Mf_{k,p}, A) \to \mathrm{Map}_{\mathrm{Alg}^{\mathbb{E}_2}_{W^+(k)}}(Mf_{k,pu}, A)$ which is natural in A. By the Yoneda lemma, we deduce that $Mf_{k,pu} \simeq Mf_{k,p}$ as $\mathbb{E}_2$-$W^+(k)$-algebras. □

4. Recollection of perfectoid rings

In this section, we will review basic definitions and properties of perfectoid rings.

4.1. **Basic definitions and properties.**

Definition 4.1. Let A be a ring and $I \subseteq A$ be an ideal. Then the ring A is called *I-adically complete* if the canonical map from A to the (inverse) limit of the tower

$$\cdots \to A/I^n \to \cdots \to A/I^2 \to A/I$$

is an isomorphism. The ring A is called I-adically separated if the intersection $\bigcap_{n=0}^{\infty} I^n = 0$.

Warning 4.2. In the literature, sometimes authors call a ring A is I-adically complete when the canonical map $A \to \lim_{n \in (\mathbb{N},>)} (A/I^n)$ is only supposed to be surjective, and our I-adic completeness is equivalent to their I-adic completeness plus I-adic separateness.

Definition 4.3. Let A be an $\mathbb{F}_p$-algebra. The *direct limit perfection* A_{perf} of A is the direct limit of the telescope $A \xrightarrow{\varphi} A \xrightarrow{\varphi} A \xrightarrow{\varphi} \cdots$.

Definition 4.4. An $\mathbb{F}_p$-algebra A is called *semiperfect* if the Frobenius map $\varphi: A \to A$ is surjective.

Remark 4.5. For a semiperfect $\mathbb{F}_p$-algebra A, the direct limit perfection A_{perf} coincides with $A_{\mathrm{red}} = A/\sqrt{0}$, by checking that A_{red} satisfies the universal property of A_{perf}.

Remark 4.6. The canonical map $R \to R_{\mathrm{perf}}$ is initial among all $\mathbb{F}_p$-algebra morphisms $R \to S$ such that S is a perfect $\mathbb{F}_p$-algebra. This follows directly from the universal property of direct limits in the definition of direct limit perfections.

Definition 4.7. Let R be a commutative ring which is p-adically complete. The *tilt* of R, denoted by $R^\flat$, is a perfect $\mathbb{F}_p$-algebra defined by the limit of the tower

$$\cdots \xrightarrow{\varphi} R/p \xrightarrow{\varphi} R/p \xrightarrow{\varphi} R/p$$

where $\varphi: R/p \to R/p$ is the Frobenius map. In particular, if R is an $\mathbb{F}_p$-algebra, then $R^\flat$ is the inverse limit perfection of R, and if furthermore R is semiperfect, then the canonical map $R^\flat \to R$ is a surjection.

We need the following classical proposition to define the Fontaine's pro-infinitesimal thickening map. We omit the proof which is routine. One can find a proof in, say, [HN19, Section 1.3].

Proposition 4.8. *Let R be a p-adically complete commutative ring. Then there exists a multiplicative map (that is to say, a morphism of multiplicative monoids) $R^\flat \xrightarrow{(-)^\sharp} R$ that sends $a = (x_n)_{n\in\mathbb{N}} \in R^\flat$ to $a^\sharp := \lim_{n\to\infty} y_n^{p^n}$ where $(x_n)_{n\in\mathbb{N}}$ satisfies $\varphi(x_{n+1}) = x_n$ for all $n \in \mathbb{N}$, and $(y_n)_{n\in\mathbb{N}} \in R^{\mathbb{N}}$ is any sequence such that for each $n \in \mathbb{N}$, y_n is a lift of $x_n \in R/p$ in R. We note that $a^\sharp$ does not depend on choice of $(y_n)_{n\in\mathbb{N}}$.*

Definition 4.9. *Fontaine's map* $\theta: W(R^\flat) \to R$ is given by $\theta(\sum_{i=0}^{\infty} [a_i]\, p^i) = \sum_{i=0}^{\infty} a_i^\sharp\, p^i$, where $[-]: R^\flat \to W(R^\flat)$ is the Teichmüller representative.

Definition 4.10. ([BMS18, Definition 3.5]) A commutative ring R is *perfectoid* if there exists $\pi \in R$ such that $p \in \pi^p R$, such that the ring R is (π)-adically complete, such that the $\mathbb{F}_p$-algebra R/p is semiperfect, and such that the kernel of $\theta: W(R^\flat) \to R$ is a principal ideal.

Definition 4.11. Let R be a perfectoid ring. The *special fiber*, denoted by κ, is the direct limit perfection of R/p, that is to say $\kappa := (R/p)_{\text{perf}} = R/\sqrt{p\, R}$ since R/p is semiperfect.

Notation 4.12. *Let R be a perfectoid ring. We denote by ξ a generator of Fontaine's map $\theta: W(R^\flat) \to R$.*

Proposition 4.13. ([BMS18, Lemma 3.13]) *Let R be a perfectoid ring. Then the commutative diagram*

$$\begin{array}{ccc} W(R^\flat) & \xrightarrow{\theta} & R \\ \downarrow & & \downarrow \\ W(\kappa) & \xrightarrow{\bmod p} & \kappa \end{array}$$

is a Tor*-independent pushout square.*

Corollary 4.14. *Let R be a perfectoid ring. For any generator $\xi \in \ker \theta$, there exists an invertible element $u \in \mathrm{GL}_1(W(\kappa))$ such that the image of $\xi \in W(R^\flat)$ in $W(\kappa)$ is $p\, u$.*

Proof. By Proposition 4.13, the image of $u \in W(R^\flat)$ in $W(\kappa)$ is a generator of the ideal $p\, W(\kappa)$. Since $p \in W(\kappa)$ is not a zero divisor, we deduce the result that we need. □

Proposition 4.15. *Let R be a perfectoid ring. Then the kernel of the composition $R^\flat \to R/p \to \kappa$ is $\sqrt{\xi\, R^\flat}$.*

Proof. The kernel of the composition $W(R^\flat) \to R/p \to \kappa$ is $\sqrt{p\, W(R^\flat) + \xi\, W(R^\flat)}$ whose image under the canonical map $W(R^\flat) \to R^\flat$ is $\sqrt{\xi\, R^\flat}$. □

4.2. **Universal properties of Fontaine's map (and a spherical analogue).** The results of this subsection will not be used later. However, we find it better to understand that Fontaine's map $\theta: W(R^\flat) \to R$ and its spherical analogue $W^+(R^\flat) \to \tau_{\leq 0}(W^+(R^\flat)) \simeq HW(R^\flat) \xrightarrow{H\theta} HR$ satisfy a universal property, which is related to the thickening defined in Definition 2.1. Roughly speaking, they are mixed characteristic "absolute" versions of thickenings in Definition 2.1. The following proposition is essentially due to Fontaine (see [Fon94], Theorem 1.2.1), rephrased in the modern language:

Proposition 4.16. ([HN19, Proposition 1.3.4]) *Let R be a perfectoid ring. Then Fontaine's map $\theta: W(R^\flat) \to R$ is initial among surjections $\theta_D: D \to R$ of rings such that the ring D is both p-adically complete and $\ker \theta_D$-adically complete.*

We will sketch the proof in [HN19] for the universal property, that is, assume that the p-adic completeness and the ξ-adic completeness of $W(R^\flat)$ is already given, we show that Fontaine's map $\theta: W(R^\flat) \to R$ is initial as claimed.

Proof. Let $\theta_D: D \to R$ be a map of rings such that D is both p-adically complete and $\ker \theta_D$-adically complete. We need to show that θ_D factors uniquely through $\theta: W(R^\flat) \to R$. In view of Example 2.5 and Definition 2.1, we have a bijection

$$\mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{Z}_p}}(W(R^\flat), D) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p}}(R^\flat, D/p)$$

(here everything is discrete therefore classical, but in order to avoid conflicts of notations with other parts of the article, we retain the cumbersome notations $\mathrm{CAlg}^{\heartsuit}_{\mathbb{Z}_p}$ and $\mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p}$) which is given as follows: for any map $W(R^\flat) \to D$ of discrete $\mathbb{Z}_p$-algebras, we compose it with the canonical map $D \to D/p$ to get the map $W(R^\flat) \to D/p$, which factors uniquely through $W(R^\flat) \to W(R^\flat)/p \cong R^\flat$ therefore gives rise to a map $R^\flat \to D/p$. Note that $\theta_R = \mathrm{id}_R: R \to R$ serves as a special choice of θ_D since the perfectoid ring R is p-adically complete by Definition 4.10 and tautologically $\ker(\theta_R) = (0)$-adically complete. That is to say, we also have a bijection

$$\mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{Z}_p}}(W(R^\flat), R) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p}}(R^\flat, R/p)$$

The map $\theta_D: D \to R$ gives rise to a commutative diagram

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{Z}_p}}(W(R^\flat), D) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p}}(R^\flat, D/p) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{Z}_p}}(W(R^\flat), R) & \xrightarrow{\cong} & \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p}}(R^\flat, R/p) \end{array}$$

So in order to show that the map $\theta_D: D \to R$ factors through the canonical map θ, or equivalently put, the preimage of the element $\theta \in \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{Z}_p}}(W(R^\flat), R)$ under the induced map $\mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{Z}_p}}(W(R^\flat), D) \to \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{Z}_p}}(W(R^\flat), R)$ is a singleton, it suffices to show that the preimage of the element $(R^\flat \to R/p) \in \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p}}(R^\flat, R/p)$ under the map $\mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p}}(R^\flat, D/p) \to \mathrm{Hom}_{\mathrm{CAlg}^{\heartsuit}_{\mathbb{F}_p}}(R^\flat, R/p)$ is a singleton, or equivalently put, the canonical map $\sigma: R^\flat \to R/p$ lifts uniquely through the map $\sigma_D: D/p \to R/p$ induced by the map $\theta_D: D \to R$. Note that the surjectivity of θ_D implies that of σ_D. Since the ring D is $(p, \ker \theta_D)$-adically complete, the ring D/p is $\ker \sigma_D$-adically complete.

We can conclude the existence and the uniqueness of lift of the map $\sigma: R^\flat \to R/p$ along the surjection $\sigma_D: D/p \to R/p$ simply by the fact that the $\mathbb{F}_p$-algebra $R^\flat$ is perfect and thus the cotangent complex $\mathbb{L}_{R^\flat/\mathbb{F}_p}$ is contractible, which implies the existence and the uniqueness of such lift.

However, we prefer to give a direct argument: We set the $\mathbb{F}_p$-algebra $A := R^\flat$ to stress that we only depend on the fact that A is a perfect $\mathbb{F}_p$-algebra, but not on the properties of the map $\sigma: A \to R/p$. Denote by $\varphi_B: B \to B, x \mapsto x^p$ the Frobenius map on any $\mathbb{F}_p$-algebra A. Then the Frobenius map φ_A is an isomorphism by assumption.

For each $a \in A$, we choose a sequence $(b_n)_{n=0}^\infty \in (D/p)^\mathbb{N}$ such that for each $n \in \mathbb{N}$, we have $\sigma_D(b_n) = \sigma(\varphi_A^{-n}(a))$.

Note that the sequence $(\varphi_{D/p}^n(b_n))_{n=0}^\infty$ converges $\ker \sigma_D$-adically: $\sigma_D(b_n - b_{n+1}^p) = \sigma_D(b_n) - \sigma_D(b_{n+1})^p = \sigma(\varphi_A^{-n}(a)) - \sigma\big(\varphi_A^{-(n+1)}(a)\big)^p = \sigma(\varphi_A^{-n}(a)) - \sigma(\varphi_A(\varphi_A^{-n}(a))) = 0$ and therefore $\varphi_{D/p}^n(b_n) - \varphi_{D/p}^{n+1}(b_{n+1}) = \varphi_{D/p}^n(b_n - b_{n+1}^p) \in \varphi_{D/p}^n(\ker \sigma_D) \subseteq (\ker \sigma_D)^{p^n}$. Let $b := \lim_{n\to\infty} \varphi_{D/p}^n(b_n)$.

We first note that $\sigma_D(b) = \sigma(a)$, since $\sigma(\varphi_{D/p}^n(b_n)) = \sigma(b_n^{p^n}) = \sigma(b_n)^{p^n} = \sigma(\varphi_A^{-n}(a))^{p^n} = \sigma(\varphi_A^n(\varphi_A^{-n}(a))) = \sigma(a)$ for all $n \in \mathbb{N}$.

Now the value $b \in D/p$ does not depend on the choice of (b_n), since for any other choice (c_n), we have $c_n - b_n \in \ker \sigma_D$, thus $\varphi_{D/p}^n(b_n) - \varphi_{D/p}^n(c_n) = \varphi_{D/p}^n(b_n - c_n) \in \varphi_{D/p}^n(\ker \sigma_D) \subseteq (\ker \sigma_D)^{p^n}$ which implies that $\lim_{n\to\infty} \varphi_{D/p}^n(c_n) = b$.

Combining the preceding discussions, we have shown that for each $a \in A$, we can associate a $b \in D/p$ such that $\sigma_D(b) = \sigma(a)$. It is routine to check that $a \mapsto b$ defines a map $A \to D/p$ of rings which serves as a lift of $\sigma: A \to R/p$. Furthermore, the uniqueness essentially follows from the above argument that the value $b \in D/p$ does not depend on the choice of (b_n). □

Remark 4.17. In fact, we can weaken our assumption on D to be derived p-complete and that the map $D \to R$ is Adams complete (due to [Car08] while the terminology is coined in [Bha12]) by using some basic facts about Adams complete surjective maps of animated rings.

Now we give a spherical version of Fontaine's universal property:

Proposition 4.18. *Let R be a perfectoid ring. We compose Fontaine's map $\theta: W(R^\flat) \to R$ with the 0th Postnikov section $W^+(R^\flat) \to \tau_{\leq 0}(W^+(R^\flat)) = HW(R^\flat)$, obtaining the map $\eta: W^+(R^\flat) \to HR$. Then we have*

1. *The $\mathbb{E}_\infty$-$\mathbb{S}_p^\wedge$-algebra $W^+(R^\flat)$ of spherical Witt vectors is $(p, \ker\theta)$-complete. Furthermore, the discrete ring $\pi_0(W^+(R^\flat))/p$ is $\ker\theta/(p\,\pi_0(W^+(R^\flat)) + \ker\theta)$-adically separated.*
2. *The map $\eta: W^+(R^\flat) \to HR$ is initial among all maps $\eta_D: D \to HR$ surjective on π_0 where D is an $\mathbb{E}_\infty$-$\mathbb{S}_p^\wedge$-algebra such that D is $(p, \ker\eta_D)$-complete and the discrete ring $\pi_0(D)/p$ is $\ker\theta_D/(p\,\pi_0(D) + \ker\theta_D)$-adically separated, where we denote the map $\pi_0(\eta_D): \pi_0(D) \to R$ by θ_D.*

Remark 4.19. In Proposition 4.18, the technical conditions imposed on the $\mathbb{E}_\infty$-$\mathbb{S}_p^\wedge$-algebra D are somewhat complicated. However, we can restrict to the full subcategory of η_D such that $\pi_0(D)$ is $(p, \ker\eta_D)$-adically complete, where $\eta: W^+(R^\flat) \to HR$ lives (see the proof of Proposition 4.18) and hence η is still an initial object in this full subcategory.

Remark 4.20. Using Remark 4.17, we can drop the adic completeness of $\pi_0(D)/p$ in Proposition 4.18.

Now we want to establish some computational results about homotopy groups of the ring $W^+(k)$ of spherical Witt vectors of a perfect $\mathbb{F}_p$-algebra k. First, we need the following proposition, which follows from Serre's computations of homotopy groups of spheres:

Proposition 4.21. *The sphere spectrum $\mathbb{S}$ is connective, $\pi_0(\mathbb{S}) = \mathbb{Z}$, and for all $n \in \mathbb{N}_{>0}$, the nth (stable) homotopy group $\pi_n(\mathbb{S})$ is finite.*

Thus for each $n \in \mathbb{N}$, the homotopy group $\pi_n(\mathbb{S})$ has bounded p-torsion. Combined with Milnor sequence of homotopy groups, we have

Corollary 4.22. *The p-adic sphere spectrum $\mathbb{S}_p^\wedge$ is connective, $\pi_0(\mathbb{S}_p^\wedge) = \mathbb{Z}_p$ and for all $n \in \mathbb{N}_{>0}$, the nth (stable) homotopy group $\pi_n(\mathbb{S}_p^\wedge)$ is a finite direct sum of finite abelian groups of form $\mathbb{Z}/p^r \cong \mathbb{Z}_p/p^r$ for some positive integer $r \in \mathbb{N}_{>0}$.*

We need a result announced in [Lur18a, Example 5.2.7] the argument of which we learn from Matthew Morrow:

Proposition 4.23. *Let k be a perfect $\mathbb{F}_p$-algebra. Then the ring of spherical Witt vectors $W^+(k)$ is a flat $\mathbb{S}_p^\wedge$-module.*

Proof. First, by Proposition 2.7, $\pi_0(W^+(k)) \cong W(k)$ which is a torsion-free $\mathbb{Z}_p$-module. Since $\mathbb{Z}_p$ is a valuation ring, we deduce that $W(k)$ is a flat $\mathbb{Z}_p$-module (see [Sta20, Tag 0539]). Now we consider the Postnikov tower $(\tau_{\geq n} \mathbb{S}_p^\wedge)_{n \in \mathbb{N}}$ of the p-adic sphere spectrum $\mathbb{S}_p^\wedge$, which induces a tower $X_n := (\tau_{\geq n} \mathbb{S}_p^\wedge) \otimes_{\mathbb{S}_p^\wedge} W^+(k)$. Note that $X_n / X_{n-1} \cong \Sigma^n H\pi_n(\mathbb{S}_p^\wedge) \otimes_{\mathbb{S}_p^\wedge} W^+(k)$. We have shown in Corollary 4.22 that $\pi_n(\mathbb{S}_p^\wedge)$ is a direct sum of finite abelian groups of form $\mathbb{Z}_p/p^r$, which allows us to realize $H\pi_n(\mathbb{S}_p^\wedge)$ as a direct sum of spectra of form $\mathrm{cofib}\Big(H\mathbb{Z}_p \xrightarrow{p^r} H\mathbb{Z}_p\Big)$. Note that the smash product $- \otimes_{\mathbb{S}_p^\wedge} W^+(k)$ commutes with taking cofibers, we deduce that $H\pi_n(\mathbb{S}_p^\wedge) \otimes_{\mathbb{S}_p^\wedge} W^+(k) \cong H\,\mathrm{Tor}_0^{\mathbb{Z}_p}(\pi_n(\mathbb{S}_p^\wedge), W(k))$. Thus the tower $(X_n)_{n \in \mathbb{N}}$ constitutes the Postnikov tower of the spectrum $W^+(k)$, therefore $\pi_n(W^+(k)) \cong \mathrm{Tor}_0^{\pi_0(\mathbb{S}_p^\wedge)}(\pi_n(\mathbb{S}_p^\wedge), W(k))$. ☐

Corollary 4.24. *Let k be a perfect $\mathbb{F}_p$-algebra. Then the ring of spherical Witt vectors $W^+(k)$ is connective, $\pi_0(W^+(k)) = W(k)$, and for all $n \in \mathbb{N}_{>0}$, the nth (stable) homotopy group $\pi_n(W^+(k))$ is a finite direct sum of $W(k)$-modules of form $W(k)/p^r$.*

We are now ready to prove Proposition 4.18:

Proof of Proposition 4.18. We check two statements one by one:

1. Proposition 4.16 tells us that the discrete ring $\pi_0(W^+(R^\flat)) \cong W(R^\flat)$ is $(p, \ker\theta)$-adically complete, therefore by Proposition A.29, it is $(p, \ker\theta)$-complete. Furthermore, we deduce from $(p, \ker\theta)$-adic completeness that $\pi_0(W^+(R^\flat))$ is $\ker\theta/(p\,\pi_0(W^+(R^\flat)) + \ker\theta)$-adically separated. In view of Theorem A.25, it remains to show that for each $n \in \mathbb{N}_{>0}$, the homotopy group $\pi_n(W^+(R^\flat))$ is (derived) $(p, \ker\theta)$-complete as a discrete $W(R^\flat)$-module. However, by Corollary 4.24, we have realized $\pi_n(W^+(R^\flat))$ as a direct sum of cofibers of $(p, \ker\theta)$-complete modules, therefore it is $(p, \ker\theta)$-complete.

2. This part is parallel to the proof of Proposition 4.16. We start with the following commutative diagram induced by the map $\eta_D : D \to HR$:

$$\begin{array}{ccc} \mathrm{Map}_{\mathrm{CAlg}_{\mathbb{S}_p^\wedge}}(W^+(R^\flat), D) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathrm{CAlg}^\heartsuit_{\mathbb{F}_p}}(R^\flat, \pi_0(D)/p) \\ \downarrow & & \downarrow \\ \mathrm{Map}_{\mathrm{CAlg}_{\mathbb{S}_p^\wedge}}(W^+(R^\flat), HR) & \xrightarrow{\simeq} & \mathrm{Hom}_{\mathrm{CAlg}^\heartsuit_{\mathbb{F}_p}}(R^\flat, R/p) \end{array}$$

as in the proof of Proposition 4.16. It follows from Definition 2.1 and Example 2.6 that the horizontal maps are homotopy equivalences, which implies that the connected components of each space on the left are all contractible. We pick the connected component of $\mathrm{Map}_{\mathrm{CAlg}_{\mathbb{S}_p^\wedge}}(W^+(R^\flat), HR)$ corresponds to the map $\eta : W^+(R^\flat) \to HR$. In order to show that η is an initial object, it suffices to show that there exists one and only one connected component in $\mathrm{Map}_{\mathrm{CAlg}_{\mathbb{S}_p^\wedge}}(W^+(R^\flat), D)$ which maps to the connect component corresponding to η. Note that the image of η in $\mathrm{Hom}_{\mathrm{CAlg}^\heartsuit_{\mathbb{F}_p}}(R^\flat, R/p)$ along the bottom horizontal map coincides with $\sigma \in \mathrm{Hom}_{\mathrm{CAlg}^\heartsuit_{\mathbb{F}_p}}(R^\flat, R/p)$ defined in the proof of Proposition 4.16. In view of the commutative diagram, it remains to show that the preimage of $\sigma \in \mathrm{Hom}_{\mathrm{CAlg}^\heartsuit_{\mathbb{F}_p}}(R^\flat, R/p)$ in $\mathrm{Hom}_{\mathrm{CAlg}^\heartsuit_{\mathbb{F}_p}}(R^\flat, \pi_0(D)/p)$. The rest of the proof is identical to that of Proposition 4.16. □

5. Proof of the main theorem

Fix a perfectoid ring R and a generator ξ of Fontaine's map $\theta : W(R^\flat) \to R$, the goal of this section is to prove Theorem 1.13. We first need a much weaker version which says that the 0th homotopy group of the $\mathbb{E}_2$-Thom spectrum in question, as a ring, is isomorphic to R:

Lemma 5.1. *The 0th homotopy group $\pi_0(Mf_{R,\xi})$ of the Thom spectrum associated to $f_{R,\xi}$ is isomorphic to R for any perfectoid ring R.*

Proof. We mimic a segment of the proof of Theorem A.1 in [KN]:

We note that $Mf_{R,\xi}$ is connective, so we have

$$\pi_0(Mf_{R,\xi}) \cong \pi_0(W^+(R^\flat)_{h\Omega^3 S^3}) \cong \pi_0(W^+(R^\flat))_{\pi_0(\Omega^3 S^3)}$$

where the $\pi_0(\Omega^3 S^3) \cong \mathbb{Z}$-action on $\pi_0(W^+(R^\flat)) \cong W(R^\flat)$ is given by multiplication by $1 - \xi$, hence

$$\pi_0(W^+(R^\flat))_{\pi_0(\Omega^3 S^3)} \cong W(R^\flat)/(1 - (1 - \xi)) \cong R$$ □

In view of Lemma 5.1, in order to prove Theorem 1.13, it suffices to show that

Proposition 5.2. *The 0th Postnikov section $t_{R,\xi} : Mf_{R,\xi} \to \tau_{\leq 0} Mf_{R,\xi} \simeq HR$, being an $\mathbb{E}_2$-map a priori, is an equivalence of spectra.*

To begin with, we first note that the special case when R is a perfect $\mathbb{F}_p$-algebra is already covered by previous considerations:

Lemma 5.3. *The $t_{R,\xi}$ in question is an equivalence of spectra when R is a perfect $\mathbb{F}_p$-algebra.*

Proof. Theorem 3.1 tells us that there is an equivalence $Mf_{R,\xi} \to HR$. The lemma follows from the fact that HR lives in $(\mathrm{Mod}_{W^+(R)})_{\leq 0}$ and that the 0th Postnikov section is essentially unique. □

We first note that both $Mf_{R,\xi}$ and HR admit canonical $W^+(R^\flat)$-module structures. Our strategy breaks up into several steps:

1. Prove some finiteness and completeness results of $Mf_{R,\xi}$ and HR as $W^+(R^\flat)$-modules;
2. Show that $t_{R,\xi}$ is an equivalence after the base change along $W^+(R^\flat) \to W^+(\kappa)$, and hence an equivalence after a further base change along $W^+(\kappa) \to H\kappa$ to the special fiber $H\kappa$;
3. The composition $W^+(R^\flat) \to W^+(\kappa) \to H\kappa$ coincides with the composition $W^+(R^\flat) \to HR^\flat \to H\kappa$, and a Nakayama-like argument shows that $t_{R,\xi}$ is an equivalence after base change along $W^+(R^\flat) \to HR^\flat$;
4. Deduce that $t_{R,\xi}$ is an equivalence by completeness.

To proceed, by Corollary 4.14, we also fix an invertible element $u \in \mathrm{GL}_1(W(\kappa))$ associated to ξ so that the image of ξ in $W(\kappa)$ is $p\,u$.

5.1. Finiteness and completeness of $Mf_{R,\xi}$ and HR as $W^+(R^\flat)$-modules.

Lemma 5.4. *$HW(k)$ is an almost perfect $W^+(k)$-module for any perfect $\mathbb{F}_p$-algebra k.*

Proof. If $k = \mathbb{F}_p$, then $W^+(\mathbb{F}_p) \simeq \mathbb{S}_p^\wedge$ is a coherent ring as in Definition A.10, and $HW(\mathbb{F}_p) \simeq H\mathbb{Z}_p \simeq H\pi_0(W^+(\mathbb{F}_p))$ is an almost perfect $\mathbb{S}_p^\wedge$-module by Corollary A.12.

In general, by Proposition 2.7, we have $HW(k) \simeq W^+(k) \otimes_{\mathbb{S}_p^\wedge} H\mathbb{Z}_p$, hence $HW(k)$ is almost perfect by Proposition A.8. □

Corollary 5.5. *HR is an almost perfect $W^+(R^\flat)$-module.*

Proof. HR is the cofiber of the multiplication map $m_\xi : HW(R^\flat) \to HW(R^\flat)$ where the domain and the codomain are almost perfect (Lemma 5.4), hence HR is also almost perfect (Proposition A.7). □

We need a nontrivial input from algebraic topology:

Proposition 5.6. *There exists a Kan complex $X_\bullet$ which is homotopy equivalent to the double loop space $\Omega^2 S^3$ of the 3-sphere such that X_n is a finite set for each $[n] \in \Delta^{\mathrm{op}}$.*

Proof. It follows from [Wal65, Theorem A and Theorem B] and the computational results about $H^*(\tau_{\geq 2}\Omega^2 S^3, \mathbb{Z})$ that $\Omega^2 S^3$ is homotopy equivalent to a CW complex of finite type (that is, the number of cells in each dimension is finite). Now we apply the simplicial approximation theorem to conclude [details needed]. □

Lemma 5.7. *$Mf_{R,\xi}$ is an almost perfect $W^+(R^\flat)$-module.*

Proof. We first pick up a Kan complex $X_\bullet$ representing $\Omega^2 S^3$ where each X_n is a finite set as in Proposition 5.6. It follows from Bousfield-Kan formula (see, for example, Corollary 12.3 in [Sha18]) that $Mf_{R,\xi}$ could be written as the geometric realization of a simplicial object $N_\bullet$ where each N_n is a free $W^+(R^\flat)$-module of finite rank, hence almost perfect by Proposition A.7. □

Corollary 5.8. $\operatorname{cofib}(t_{R,\xi})$ *is an almost perfect* $W^+(R^\flat)$*-module.*

Proof. The subcategory of almost perfect modules are closed under taking cofibers and base changes (Proposition A.7). The statement then follows from Corollary 5.5 and Lemma 5.7. □

Lemma 5.9. *The spectrum* HR *is* p*-complete.*

Proof. By definition of perfectoid rings, R is p-adically complete, therefore HR is p-complete by Proposition A.29. □

Lemma 5.10. *The spectrum* $Mf_{R,\xi}$ *is* p*-complete.*

Proof. We note that $W^+(R^\flat)$ is p-complete by definition of spherical Witt vectors, and $Mf_{R,\xi}$ is almost perfect, therefore p-complete by Proposition A.27. □

Corollary 5.11. *The spectrum* $\operatorname{cofib}(t_{R,\xi})$ *is* p*-complete.*

Proof. It follows from Corollary 5.8 and Proposition A.27. □

5.2. $t_{R,\xi}$ is an equivalence after the base change along $W^+(R^\flat)\to W^+(\kappa)$.

The proof is similar to that of Theorem 3.1, except that we need to be more careful to identify the maps.

Lemma 5.12. *There is a canonical equivalence* $Mf_{\kappa,pu}\xrightarrow{\simeq}W^+(\kappa)\otimes_{W^+(R^\flat)}Mf_{R,\xi}$ *of* $W^+(\kappa)$*-modules.*

Proof. We first note that the image of the multiplication map $m_{1-\xi}:W^+(R^\flat)\to W^+(R^\flat)$ under the base change functor $W^+(\kappa)\otimes_{W^+(R^\flat)}-:\operatorname{Mod}_{W^+(R^\flat)}\to\operatorname{Mod}_{W^+(\kappa)}$ is the multiplication map $m_{1-pu}:W^+(\kappa)\to W^+(\kappa)$.

Therefore $f_{\kappa,pu}$ coincides with the composition

$$\Omega^2S^3\xrightarrow{f_{R,\xi}}\mathrm{BGL}_1(W^+(R^\flat))\xrightarrow{W^+(\kappa)\otimes_{W^+(R^\flat)}-}\mathrm{BGL}_1(W^+(\kappa))$$

Along with the fact that the functor $W^+(\kappa)\otimes_{W^+(R^\flat)}-:\operatorname{Mod}_{W^+(R^\flat)}\to\operatorname{Mod}_{W^+(\kappa)}$ commutes with small colimits, or to be more precise, that the natural transformation $\operatorname{colim}_i(W^+(\kappa)\otimes_{W^+(R^\flat)}M_i)\to W^+(\kappa)\otimes_{W^+(R^\flat)}(\operatorname{colim}_iM_i)$ is an equivalence for any diagram $(M_i)_i$ in $\operatorname{Mod}_{W^+(R^\flat)}$, we deduce that there is a canonical equivalence $Mf_{\kappa,pu}\xrightarrow{\simeq}W^+(\kappa)\otimes_{W^+(R^\flat)}Mf_{R,\xi}$ as $W^+(\kappa)$-modules. □

Lemma 5.13. *Given a morphism of perfect* $\mathbb{F}_p$*-algebras* $k\to K$*, the commutative diagram of* $\mathbb{E}_\infty$*-rings*

$$\begin{array}{ccc}W^+(k) & \longrightarrow & W^+(K)\\ \downarrow & & \downarrow\\ HW(k) & \longrightarrow & HW(K)\end{array}$$

is a pushout square.

Proof. Consider the commutative diagram of $\mathbb{E}_\infty$-rings

$$\begin{array}{ccccc}\mathbb{S}_p^\wedge & \longrightarrow & W^+(k) & \longrightarrow & W^+(K)\\ \downarrow & & \downarrow & & \downarrow\\ H\mathbb{Z}_p & \longrightarrow & HW(k) & \longrightarrow & HW(K)\end{array}$$

By Proposition 2.7, we know that the left square and the outer square are pushout squares, therefore so is the right square. □

Lemma 5.14. *There is a canonical equivalence $W^+(\kappa)\otimes_{W^+(R^\flat)} HR \to H\kappa$ of $W^+(\kappa)$-modules.*

Proof. Combining two pushout squares in the category of $\mathbb{E}_\infty$-rings:

$$\begin{array}{ccc} W^+(R^\flat) & \longrightarrow & W^+(\kappa) \\ \downarrow & \sigma & \downarrow \\ HW(R^\flat) & \longrightarrow & HW(\kappa) \\ \downarrow & \tau & \downarrow \\ HR & \longrightarrow & H\kappa \end{array}$$

where σ is a pushout square by Lemma 5.13 and τ is a pushout square by Proposition 4.13. □

Lemma 5.15. *The map $W^+(\kappa)\otimes_{W^+(R^\flat)} t_{R,\xi} : W^+(\kappa)\otimes_{W^+(R^\flat)} Mf_{R,\xi} \to W^+(\kappa)\otimes_{W^+(R^\flat)} HR$ is equivalent to $t_{\kappa,pu}: Mf_{\kappa,pu}\to H\kappa$.*

Proof. In view of Lemma 5.12 and Lemma 5.14, we only need to show that $t_{\kappa,pu}: Mf_{\kappa,pu}\to H\kappa$ coincides with the composition of the equivalences $Mf_{\kappa,pu}\to W^+(\kappa)\otimes_{W^+(R^\flat)} Mf_{R,\xi}$, $W^+(\kappa)\otimes_{W^+(R^\flat)} t_{R,\xi}$ and $W^+(\kappa)\otimes_{W^+(R^\flat)} HR\to H\kappa$. In other words, it suffices to show that the composition in question is the 0th Postnikov section. We only need to check that the composition induces an isomorphism on π_0 by basic properties of t-structures, since $\tau_{\leq 0} Mf_{\kappa,pu}\simeq H\kappa$. It suffices to show that $W^+(\kappa)\otimes_{W^+(R^\flat)} t_{R,\xi}$ induces an isomorphism on π_0, and this follows from the fact that all spectra in question are connective and that $t_{R,\xi}$ induces an isomorphism on π_0 by Lemma 5.1. □

Corollary 5.16. *$H\kappa\otimes_{W^+(R^\flat)} t_{R,\xi}$ is an equivalence of spectra.*

Proof. It follows from Lemma 5.15 and Lemma 5.3. □

5.3. $t_{R,\xi}$ is an equivalence after the base change along $W^+(R^\flat)\to HR^\flat$.

Lemma 5.17. *Let M be an $HR^\flat$-module which is bounded below and almost perfect. If there exists an $r\in\mathbb{N}$ such that $\xi^r\pi_n(M)=0$ for all $n\in\mathbb{Z}$, and $H\kappa\otimes_{HR^\flat} M\simeq 0$, then $M\simeq 0$.*

Proof. We show inductively on n that $\pi_n M=0$.

- Since M is bounded below, $\pi_n M=0$ for $n\ll 0$;
- Suppose that for $m<n$ we have $\pi_m M=0$. Then by unrolling Definition A.6, $\pi_n M$ is a compact object in the category of discrete $R^\flat$-modules, therefore is finitely presented and in particular finitely generated. Now we have

  $$0=\pi_n(H\kappa\otimes_{HR^\flat} M)=\mathrm{Tor}_0^{R^\flat}(\kappa,\pi_n M).$$

  By Proposition 4.15 and that $\xi^r\pi_n(M)=0$, we have

  $$\mathrm{Tor}_0^{R^\flat}(\kappa,\pi_n M)=\mathrm{Tor}_0^{R^\flat/\xi^r}(\kappa,\pi_n M)$$

We note that $\ker(R^\flat/\xi^r \to \kappa)$ lies in the (nil-radical, therefore) Jacobson radical of $R^\flat/\xi^r$, thus $\pi_n M = 0$, by Nakayama's lemma along with the fact that $\pi_n M$ is finitely generated. □

Remark 5.18. Matthew Morrow told us that in Lemma 5.17, the hypothesis $\xi^r \pi_n(M) = 0$ is redundant, since the kernel of the map $R^\flat \to \kappa$ of $\mathbb{F}_p$-algebras lies in the radical of the ideal $\xi R^\flat \subseteq \mathrm{Rad}(R^\flat)$ where $\mathrm{Rad}(R^\flat)$ is the Jacobson radical of the $\mathbb{F}_p$-algebra $R^\flat$ and the inclusion $\xi R^\flat \subseteq \mathrm{Rad}(R^\flat)$ is deduced from the $\xi R^\flat$-adically completeness of the $\mathbb{F}_p$-algebra $R^\flat$. Since the Jacobson radical is "radical", the kernel of the map $R^\flat \to \kappa$ also lies in the Jacobson radical $\mathrm{Rad}(R^\flat)$. We decide to preserve the original version to reflect our real thoughts.

Corollary 5.19. *$HR^\flat \otimes_{W^+(R^\flat)} t_{R,\xi}$ is an equivalence of spectra.*

Proof. Note that

$$\pi_0(HR^\flat \otimes_{W^+(R^\flat)} Mf_{R,\xi}) = \mathrm{Tor}_0^{W(R^\flat)}(R^\flat, \pi_0(Mf_{R,\xi})) = R^\flat/\xi R^\flat$$

and

$$\pi_0(HR^\flat \otimes_{W^+(R^\flat)} HR) = \mathrm{Tor}_0^{W(R^\flat)}(R^\flat, R) = R^\flat/\xi R^\flat$$

and that $HR^\flat \otimes_{W^+(R^\flat)} Mf_{R,\xi}$, $HR^\flat \otimes_{W^+(R^\flat)} HR$ are connective $\mathbb{E}_\infty$-rings, we conclude that the homotopy groups of these $\mathbb{E}_\infty$-rings are ξ-torsion groups, which implies that for all $n \in \mathbb{Z}$,

$$\xi^2 \pi_n(\mathrm{cofib}(HR^\flat \otimes_{W^+(R^\flat)} t_{R,\xi})) = 0$$

In addition, since the subcategory of almost perfect modules are closed under base changes (Proposition A.8), we deduce from Corollary 5.8 that $\mathrm{cofib}(HR^\flat \otimes_{W^+(R^\flat)} t_{R,\xi}) \simeq HR^\flat \otimes_{W^+(R^\flat)} \mathrm{cofib}(t_{R,\xi})$ is almost perfect. On the other hand, being the cofiber of a map of connective spectra, it is also connective. Then we invoke Lemma 5.17 along with Corollary 5.16 to conclude that $\mathrm{cofib}(HR^\flat \otimes_{W^+(R^\flat)} t_{R,\xi}) \simeq 0$. □

5.4. Conclude: $t_{R,\xi}$ is an equivalence.

We are now at the final stage to conclude a proof of Proposition 5.2, and consequently, Theorem 1.13.

Proof of Proposition 5.2. We recall that by Theorem 2.4 and Example 2.6, there is a pushout square of $\mathbb{E}_\infty$-rings:

$$\begin{array}{ccc} \mathbb{S}_p^\wedge & \longrightarrow & W^+(R^\flat) \\ \downarrow & & \downarrow \\ H\mathbb{F}_p & \longrightarrow & HR^\flat \end{array}$$

Therefore by Corollary 5.19 we have

$$0 \simeq \mathrm{cofib}(HR^\flat \otimes_{W^+(R^\flat)} t_{R,\xi}) \simeq HR^\flat \otimes_{W^+(R^\flat)} \mathrm{cofib}(t_{R,\xi}) \simeq H\mathbb{F}_p \otimes_{\mathbb{S}_p^\wedge} \mathrm{cofib}(t_{R,\xi})$$

We then invoke Corollary A.33 with Corollary 5.11 to deduce that $\mathrm{cofib}(t_{R,\xi}) \simeq 0$. □

5.5. An intermezzo: Identifying $\mathrm{THH}(-/W^+(k))$ with $\mathrm{THH}(-)$ after p-completion.

In this subsection, we will show that Proposition 1.17 follows from Proposition 1.16. It suffices to prove the following lemma:

Lemma 5.20. *Let R be an $\mathbb{E}_1$-algebra over $W^+(k)$ where k is a perfect $\mathbb{F}_p$-algebra. Then the canonical map $\mathrm{THH}(R) \to \mathrm{THH}(R/W^+(k))$ induced by $\mathbb{S} \to \mathbb{S}_p^\wedge$ is an equivalence after p-completion.*

Proof. Note that $\mathrm{THH}(R/W^+(k)) \simeq W^+(k) \otimes_{\mathrm{THH}(W^+(k))} \mathrm{THH}(R)$. We are left to show that the canonical map $\mathrm{THH}(W^+(k)) \to W^+(k)$ is an equivalence after p-completion. In view of Corollary A.33, we only need to check it after tensoring with $H\mathbb{F}_p$. We note that the base changed map $H\mathbb{F}_p \otimes \mathrm{THH}(W^+(k)) \to H\mathbb{F}_p \otimes W^+(k)$ fits into the commutative diagram

$$\begin{array}{ccc} H\mathbb{F}_p \otimes \mathrm{THH}(W^+(k)) & \longrightarrow & H\mathbb{F}_p \otimes W^+(k) \\ \downarrow & \sigma & \| \\ \mathrm{THH}(H\mathbb{F}_p \otimes W^+(k)/H\mathbb{F}_p) & \longrightarrow & H\mathbb{F}_p \otimes W^+(k) \\ \downarrow & \tau & \downarrow \\ \mathrm{THH}(Hk/H\mathbb{F}_p) & \longrightarrow & Hk \end{array}$$

where the commutativity of σ follows from the functoriality of the base change functor of THH, and the commutative of τ follows from the functoriality of the natural transformation $\mathrm{THH}(-/H\mathbb{F}_p) \to (-)$. All vertical maps are equivalences of spectra: the upper left map is the base change equivalence, and the lower right map is the equivalence by Proposition 2.7, and the lower left map is the image of this equivalence under the functor $\mathrm{THH}(-/H\mathbb{F}_p)$ and hence also an equivalence. The bottom horizontal map is an equivalence by the fact that k is a perfect $\mathbb{F}_p$-algebra. □

6. Analogues

It is worth to note that in Bhatt and Scholze's recent work [BS19], they introduced the concept of prisms (A, I) which serves as a "non-perfect" version of perfectoid rings. Especially, the category of perfect prisms (A, I) is equivalent to that of perfectoid rings A/I, and given a perfectoid ring R, the corresponding perfect prism is given by $(W(R^\flat), \ker\theta)$. It is interesting to know whether we can generalize our description for general orientable prisms (A, I), that is to say,

Question 1. Given an orientable prism $(A, I = (d))$. When can we find an $\mathbb{E}_\infty$-ring spectrum A^+ (which satisfies some hypotheses related to A. A naive guess would be that $\pi_0(A^+) = A$) and a map $\Omega^2 S^3 \to \mathrm{BGL}_1(A^+)$ to which the associated $\mathbb{E}_2$-Thom spectrum (possibly after p-completion) coincides with A/I.

We don't know the answer in this generality. However, we will discuss another special class of prism (related to Breuil-Kisin cohomology) for which an analogue holds. This result is more-or-less known by experts. In fact, it is essentially equivalent to Remark 3.4 in [KN19] of which no proof is presented. In this section, we will first recall some basic facts about complete discrete valuation rings, then we will indicate briefly how to adapt our proof above to this special class.

6.1. **Preparations.**

Definition 6.1. **([Ser79, Section I.1])** A ring A is called a *discrete valuation ring*, or a *DVR*, if it is a principal ideal domain that has a unique non-zero prime ideal $\mathfrak{m}$. In this case, since A is local, we also denote the DVR A by $(A,\mathfrak{m})$. The field $A/\mathfrak{m}$ is called the *residue field* of A. A generator of $\mathfrak{m}$, unique up to multiplication by an invertible element, is called a *uniformizer*, usually denoted by ϖ.

Definition 6.2. A DVR $(A,\mathfrak{m})$ is called *of mixed characteristics* $(0,p)$ if the field of fraction $\mathrm{Frac}(A)$ of A is of characteristics 0 while the residue field $A/\mathfrak{m}$ is of characteristics p, which implies that $0\neq p\in\mathfrak{m}$.

Definition 6.3. **([Ser79, Section I.1])** Let $(A,\mathfrak{m})$ be a DVR. The *valuation* of an element $x\in A\setminus 0$ is defined to be the maximal integer $n\in\mathbb{N}$ such that $x\in\mathfrak{m}^n$, which always exists, denoted by $v(x)\in\mathbb{N}$.

Definition 6.4. **([Ser79, Section II.5])** Let $(A,\mathfrak{m})$ be a DVR of mixed characteristics $(0,p)$. Then the integer $e=v(p)$ is called the *absolute ramification index* of A.

Definition 6.5. **([Ser79, Chapter II])** A DVR $(A,\mathfrak{m})$ is called *complete* if it is complete with respect to the $\mathfrak{m}$-adic topology, that is to say, the canonical map from A to the limit of the tower

$$\cdots\to A/\mathfrak{m}^n\to\cdots\to A/\mathfrak{m}^2\to A/\mathfrak{m}$$

is an isomorphism.

Proposition 6.6. **([Ser79, Section II.5, Theorem 4])** *Let $(A,\mathfrak{m})$ be a complete DVR of mixed characteristics $(0,p)$ with residue field k being perfect. Let e be its absolute ramification index. Let $\varpi\in\mathfrak{m}$ be a uniformizer. Then there exists an Eisenstein $W(k)$-polynomial $E(u)\in W(k)[u]$ (that is, a $W(k)$-polynomial $E(u)=u^e+\sum_{j=0}^{e-1}a_j u^j$ such that $p\mid a_j$ for $j=0,\ldots,e-1$ and $p^2\nmid a_0$, where $W(k)$ is the ring of Witt vectors as before) along with an isomorphism $W(k)[u]/(E(u))\xrightarrow{\sim}A$ which maps u to the uniformizer $\varpi\in\mathfrak{m}$.*

In the rest of this section, we will fix a complete DVR $(A,\mathfrak{m})$ of mixed characteristics $(0,p)$ with residue field k being perfect, absolute ramification index e and a uniformizer $\varpi\in\mathfrak{m}$. We also fix a choice of an Eisenstein $W(k)$-polynomial $E(u)\in W(k)[u]$ as in Proposition 6.6. We first note that

Proposition 6.7. *The element $1-E(u)\in W(k)[[u]]$ is invertible.*

Proof. Write $E(u)=u^e+\sum_{j=0}^{e-1}a_j u^j$ as in Proposition 6.6. Note that $W(k)$ is p-adically complete, therefore $1-a_0$ is invertible in $W(k)$, which implies that $1-E(u)\in W(k)[[u]]$ is invertible. □

Let $W^+(k)[u]$ be the "single variable polynomial $W^+(k)$-algebra", that is, the $\mathbb{E}_\infty$-$W^+(k)$-algebra $W^+(k)\otimes_{\mathbb{S}}\mathbb{S}[\mathbb{N}]$. Since the space $\mathbb{N}$ is endowed with discrete topology, we have

Proposition 6.8. *As a $W^+(k)$-module, $W^+(k)[u]$ is equivalent to the direct sum $\bigoplus_{j=0}^{\infty}u^jW^+(k)$, a free $W^+(k)$-module. The graded homotopy group $\pi_*(W^+(k)[u])$, as a (graded-commutative) $\pi_*(W^+(k))$-algebra, is equivalent to $\pi_*(W^+(k))[u]$, where $\deg u=0$.*

Now let $W^+(k)[[u]]$ be the (u)-completion of the $\mathbb{E}_\infty$-$W^+(k)$-algebra $W^+(k)[u]$. To study $W^+(k)[[u]]$, we need some preparations.

Proposition 6.9. *Let $n\in\mathbb{N}$ be a natural number. Let $m_{u^n}: W^+(k)[u]\to W^+(k)[u]$ be the multiplication map given by $u^n\in\pi_0(W^+(k)[u])=W(k)[u]$. Then the $W^+(k)[u]$-module* $\operatorname{cofib}(m_{u^n})$ *as a $W^+(k)$-module is a free $W^+(k)$-module $\bigoplus_{j=0}^{n-1} u^j W^+(k)$ of rank n, which admits an $\mathbb{E}_\infty$-$W^+(k)[u]$-algebra structure. In particular, we have the cofiber sequence*

$$W^+(k)[u]\xrightarrow{m_u} W^+(k)[u]\to W^+(k)$$

of $W^+(k)[u]$-modules.

Proof. For any space $X\in\mathcal{S}$, we let $X_+\in\mathcal{S}_*$ denote the pointed discrete space $\{*\}\cup X$. Especially, $\mathbb{N}_+=\{*\}\cup\mathbb{N}$ and $(\mathbb{N}_{<n})_+=\{*\}\cup\mathbb{N}_{<n}$. The addition map $\mathbb{N}\to\mathbb{N}, m\mapsto n+m$ induces a map of pointed spaces $\alpha_n:\mathbb{N}_+\to\mathbb{N}_+$. Note that in the ∞-category $\mathcal{S}$ of spaces, we have a pushout diagram

$$\begin{array}{ccc} \mathbb{N}_+ & \xrightarrow{\alpha_n} & \mathbb{N}_+ \\ \downarrow & & \downarrow \\ \{*\} & \longrightarrow & (\mathbb{N}_{<n})_+ \end{array}$$

to which we apply the functor $\Sigma^\infty:\mathcal{S}_*\to\mathrm{Sp}$, left adjoint of the functor $\Omega^\infty_*:\mathrm{Sp}\to S_*$ therefore preserving colimits, we get a cofiber sequence $\mathbb{S}[u]\xrightarrow{u^n}\mathbb{S}[u]\to\bigoplus_{j=0}^{n-1}u^j\,\mathbb{S}$. A further base change to $W^+(k)$ gives rise to the result. In addition, the multiplication structure could be seen from the fact that the addition map $\mathbb{N}\to\mathbb{N}, m\mapsto n+m$ in fact defines a monoidal action. □

Corollary 6.10. *Let $n\in\mathbb{N}$ be a natural number. Let $m_{u^n}: W^+(k)[u]\to W^+(k)[u]$ be the multiplication map. Then homotopy groups $\pi_*(\operatorname{cofib}(m_{u^n}))$ of the cofiber as $\pi_*(W^+(k))$ could be identified with $\pi_*(W^+(k))[u]/(u^n)$, and the long exact sequence of homotopy groups associated to the cofiber sequence $W^+(k)[u]\xrightarrow{m_{u^n}}W^+(k)[u]\to$* $\operatorname{cofib}(m_{u^n})$ *decomposes as short exact sequences, which assemble to a short exact sequence of graded $\pi_*(W^+(k))[u]$-modules:*

$$0\to\pi_*(W^+(k))[u]\xrightarrow{u^n}\pi_*(W^+(k))[u]\to\pi_*(W^+(k))[u]/(u^n)\to 0$$

Furthermore, this sequence is functorial in $n\in(\mathbb{N},>)$.

Proposition 6.11. *The $\mathbb{E}_\infty$-$W^+(k)$-algebra $W^+(k)[[u]]$ is connective. The zeroth homotopy group of $\pi_0(W^+(k)[[u]])$ is isomorphic to the (u)-adic completion of the polynomial $W(k)$-algebra $W(k)[u]$, that is, the formal power series $W(k)$-algebra $W(k)[[u]]$, as $W(k)$-algebras.*

Our proof is incomplete: we only identify the $W(k)$-module structures on homotopy groups. A formal identification of algebra structures would require more rudiments about the symmetric monoidal structure on the completion functor than we know.

Proof. We reinterpret Proposition A.19 as follows: since the limit functor is exact, it commutes with cofibers, therefore we can rewrite $W^+(k)[[u]]=(W^+(k)[u])^\wedge_{(u)}$ as the limit of the tower

$$\cdots\to\operatorname{cofib}\Big(W^+(k)[u]\xrightarrow{u^2}W^+(k)[u]\Big)\to\operatorname{cofib}\big(W^+(k)[u]\xrightarrow{u}W^+(k)[u]\big)$$

After passage to homotopy groups, by Corollary 6.10, we get the tower of graded $\pi_*(W^+(k))[u]$-modules

$$\cdots \to \pi_*(W^+(k))[u]/(u^n) \to \cdots \to \pi_*(W^+(k))[u]/(u^2) \to \pi_*(W^+(k))[u]/(u) \qquad (6.1)$$

which is degree-wise a tower of surjective maps. It follows from Milnor's sequence that the graded $\pi_*(W^+(k))[u]$-module $\pi_*(W^+(k)[[u]])$ is isomorphic to the (ordinary) inverse limit of the tower (6.1), that is, $\pi_*(W^+(k))[[u]]$. Take $*=0$, we get the result. □

The following lemma serves as a key tool in our proof:

Lemma 6.12. *Let M be a $W^+(k)[u]$-(or $W^+(k)[[u]]$-)module (spectrum). If the spectrum $W^+(k)\otimes_{W^+(k)[u]} M$ (or $W^+(k)\otimes_{W^+(k)[[u]]} M$ respectively) is contractible, then so is the (u)-completion of the spectrum M. In particular, if furthermore $W^+(k)[u]$-(or $W^+(k)[[u]]$-)module M is assumed to be (u)-complete, then the spectrum M is contractible.*

Proof. We first assume that the spectrum $W^+(k)\otimes_{W^+(k)[u]} M$ is contractible. In this case, we apply the exact functor $-\otimes_{W^+(k)[u]} M$ to the cofiber sequence

$$W^+(k)[u] \xrightarrow{m_u} W^+(k)[u] \to W^+(k) \qquad (6.2)$$

indicated in Proposition 6.9 obtaining that the base-changed map $M \xrightarrow{m_u \otimes_{W^+(k)[u]} M} M$ is an equivalence of spectra. Note that this map is just the multiplication map, denoted by $m_{M,u}$. Now we look at Proposition A.19: the (u)-completion of the $W^+(k)[u]$-module M is the cofiber of the canonical map $T(M)\to M$, where $T(M)$ is the limit of the tower

$$\cdots \xrightarrow{m_{M,u}} M \xrightarrow{m_{M,u}} M \xrightarrow{m_{M,u}} M$$

Since all maps in the tower are equivalences of spectra, we deduce that the canonical map $T(M)\to M$ is an equivalence of spectra, which implies that the (u)-completion of the $W^+(k)[u]$-module M is contractible. In particular, the $W^+(k)[u]$-module M is assumed to be (u)-complete, therefore the spectrum M is contractible.

If, on the other hand, $W^+(k)\otimes_{W^+(k)[[u]]} M$ is contractible, then to adopt the proof above, it suffices to establish the cofiber sequence

$$W^+(k)[[u]] \xrightarrow{m_u} W^+(k)[[u]] \to W^+(k) \qquad (6.3)$$

We apply the (u)-complete functor to the cofiber sequence (6.2), and note that the $W^+(k)[u]$-module $W^+(k)$ is (u)-nilpotent (in fact, multiplying u is the zero map on $W^+(k)$), therefore $W^+(k)$ is (u)-complete by Corollary A.17, which leads to the cofiber sequence (6.3). The rest of the proof is same as before. □

6.2. The Breuil-Kisin case.

As before, we fix a complete DVR $(A,\mathfrak{m})$ of mixed characteristics $(0,p)$ with residue field k being perfect, absolute ramification index e, a uniformizer $\varpi\in\mathfrak{m}$ and an Eisenstein $W(k)$-polynomial $E(u)\in W(k)[u]$ which induces an isomorphism $W(k)[u]/(E(u)) \xrightarrow{\sim} A, u\mapsto\varpi$ as in Proposition 6.6. As in Remark 1.9 and Remark 1.12, $1-E(u)\in W(k)[[u]]=\pi_1(\mathrm{BGL}_1(W^+(k)[[u]]))$ gives rise to a map $f_E:\Omega^2 S^3\to \mathrm{BGL}_1(W^+(k)[[u]])$. The proof of Lemma 5.1 results in the following analogue:

Lemma 6.13. *The zeroth homotopy group of the $\mathbb{E}_2$-Thom spectrum Mf_E associated to the map f_E is isomorphic to the $W(k)$-algebra $W(k)[[u]]/(E(u)) \cong W(k)[u]/(E(u)) \cong A$.*

The $W(k)[u]$-module structure on A gives rise to a $W^+(k)[u]$-module structure on HA. Since A is $\mathfrak{m} = (\varpi)$-adically complete, the $W(k)[u]$-module structure on A also gives rise to a $W(k)[[u]]$-module structure on A and consequently a $W^+(k)[[u]]$-module structure on HA. We readily [details needed] check that these structures are compatible, in the sense that the $W^+(k)[u]$-module structure on HA coincides with the image of the $W^+(k)[[u]]$-module HA under the forgetful functor $\mathrm{Mod}_{W^+(k)[[u]]} \to \mathrm{Mod}_{W^+(k)[u]}$. Matthew Morrow proposed the following analogue of the Hopkins-Mahowald theorem:

Theorem 6.14. *The truncation map $t_E: Mf_E \to H\pi_0(Mf_E) \cong HA$ of $\mathbb{E}_2$-$W^+(k)[[u]]$-algebras is an equivalence of spectrum. Thus the Eilenberg-Maclane spectrum HA is the $\mathbb{E}_2$-Thom spectrum Mf_E associated to the map $f_E: \Omega^2 S^3 \to \mathrm{BGL}_1(W^+(k)[[u]])$.*

Corollary 6.15. (see [KN19, Remark 3.4]) *The $\mathbb{E}_2$-HA-algebra $HA \otimes_{W^+(k)[[u]]} HA$ is a free $\mathbb{E}_2$-HA-algebra on a single generator in degree 1.*

Proof. The strategy is already covered in the proof of Lemma 3.2 and Lemma 5.12. Since this pattern will appear again soon, we find it beneficial to present again. Let's recall that the $\mathbb{E}_2$-Thom spectrum Mf_E is the colimit of the composite functor

$$\Omega^2 S^3 \xrightarrow{f_E} \mathrm{BGL}_1(W^+(k)[[u]]) \to \mathrm{Mod}_{W^+(k)[[u]]}$$

which by abuse of notation will be still denoted by f_E.

Since the base change functor $HA \otimes_{W^+(k)[[u]]} - : \mathrm{Mod}_{W^+(k)[[u]]} \to \mathrm{Mod}_{HA}$ is a left adjoint, it commutes with colimits, we deduce that $HA \otimes_{W^+(k)[[u]]} Mf_E \simeq M(f_E \otimes_{W^+(k)[[u]]} HA)$, where $f_E \otimes_{W^+(k)[[u]]} HA$ is the map $\Omega^2 S^3 \to \mathrm{BGL}_1(HA)$.

As in the proof of Lemma 3.2, we can identify map as follows: we pick the image of $1 - E(u) \in \mathrm{GL}_1(W(k)[[u]])$ under the map $\mathrm{GL}_1(W(k)[[u]]) \to \mathrm{GL}_1(A)$, that is, the element $1 \in \mathrm{GL}_1(A) \cong \pi_1(\mathrm{BGL}_1(HA))$, which gives rise to the constant map $S^1 \to \mathrm{BGL}_1(HA)$ and consequently the constant map $f_A: \Omega^2 S^3 \to \mathrm{BGL}_1(HA)$, as in Remark 1.9 and Remark 1.12.

In conclusion, the map $f_E \otimes_{W^+(k)} HA: \Omega^2 S^3 \to \mathrm{BGL}_1(HA)$ coincides with the constant map f_A, and the $\mathbb{E}_2$-Thom spectrum Mf_A is thus the colimit of a constant map, which evaluates to $HA \otimes \Omega^2 S^3$, the free $\mathbb{E}_2$-HA-algebra on a single generator in degree 1. □

Recall that $E(u) \in W(k)[u]$ is an Eisenstein $W(k)$-polynomial. Let a_0 denote the constant term of $E(u)$. By assumption, $p \mid a_0$ but $p^2 \nmid a_0$. Let $a_0 = p\, b_0$ where $b_0 \in W(k)$. Since p is not a zero-divisor in $W(k)$, we have $p \nmid b_0$, which implies that the image of b_0 in $W(k)/p \cong k$ is invertible since k is a field. Now since $W(k)$ is p-adically complete, we have $b_0 \in \mathrm{GL}_1(W(k))$.

The strategy to prove Theorem 6.14 is similar to the approach to attack Theorem 1.13. We first show that the base change of the truncation map t_E along the map $W^+(k)[[u]] \to W^+(k)$ coincides with the truncation map t_{k,a_0}, then it follows from Lemma 5.3 that the base changed map $W^+(k) \otimes_{W^+(k)[[u]]} t_E \simeq t_{k,a_0}$ is an equivalence of spectra, and by completeness, we deduce that the map t_E is also an equivalence of spectra by Lemma 6.12.

Lemma 6.16. *There is a canonical equivalence $Mf_{k,a_0} \xrightarrow{\simeq} W^+(k) \otimes_{W^+(k)[[u]]} Mf_E$ of $W^+(k)$-modules.*

Proof. We will duplicate the proof of Lemma 5.12. The image of the multiplication map $m_{1-E(u)}: W^+(k)[[u]] \to W^+(k)[[u]]$ under the base change functor $W^+(k) \otimes_{W^+(k)[[u]]} - : \mathrm{Mod}_{W^+(k)[[u]]} \to \mathrm{Mod}_{W^+(k)}$ is the multiplication map $m_{1-a_0}: W^+(k) \to W^+(k)$. Note also that the base change functor is symmetric monoidal. Now we conclude that the map f_{k,a_0} coincides with the composite map

$$\Omega^2 S^3 \xrightarrow{f_E} \mathrm{BGL}_1(W^+(k)[[u]]) \xrightarrow{W^+(k)\otimes_{W^+(k)[[u]]}-} \mathrm{BGL}_1(W^+(k))$$

Thus by commuting the colimit and the base-change, we obtain

$$\begin{aligned} Mf_{k,a_0} &= \mathrm{colim}(W^+(k) \otimes_{W^+(k)[[u]]} f_E) \\ &\xrightarrow{\simeq} W^+(k) \otimes_{W^+(k)[[u]]} \mathrm{colim}\, f_E \\ &= W^+(k) \otimes_{W^+(k)[[u]]} Mf_E \end{aligned}$$

where by abuse of notation, the colimit of the maps f_{k,a_0} (or f_E respectively) are understood as the colimit of the maps f_{k,a_0} (or f_E respectively) composed with the functor $\mathrm{BGL}_1(W^+(k)) \to \mathrm{Mod}_{W^+(k)}$ (or $\mathrm{BGL}_1(W^+(k)[[u]]) \to \mathrm{Mod}_{W^+(k)[[u]]}$ respectively) as in the definition of Thom spectra. □

Lemma 6.17. *There is a canonical equivalence $W^+(k) \otimes_{W^+(k)[[u]]} HA \xrightarrow{\simeq} Hk$ of $W^+(k)$-modules.*

Proof. As in the proof of Lemma 6.12, we identify $W^+(k)$ with the cofiber of the multiplication map $m_u: W^+(k)[[u]] \to W^+(k)[[u]]$ which gives us an equivalence

$$W^+(k) \otimes_{W^+(k)[[u]]} HA \simeq \mathrm{cofib}\big(HA \xrightarrow{m_{HA,u}} HA\big)$$

Now by the definition of the $W^+(k)[[u]]$-module structure on HA and that u is not a zero-divisor in A, we have the equivalence $\mathrm{cofib}\big(HA \xrightarrow{m_{HA,u}} HA\big) \simeq H\big(\mathrm{coker}\big(A \xrightarrow{\varpi} A\big)\big) \simeq Hk$. Thus we obtain an equivalence $W^+(k) \otimes_{W^+(k)[[u]]} HA \simeq Hk$. We can readily check [details needed] that this equivalence could be described as follows: consider the commutative diagram in the ∞-category of $\mathbb{E}_\infty$-rings

$$\begin{array}{ccc} W^+(k)[[u]] & \longrightarrow & W^+(k) \\ \downarrow & & \downarrow \\ HA & \longrightarrow & Hk \end{array}$$

where the left vertical map is the composite map $W^+(k)[[u]] \to H(\pi_0(W^+(k)[[u]])) \simeq H(W(k)[[u]]) \xrightarrow{u \mapsto \varpi} HA$ (where the first map is the Postnikov section). The commutative diagram induces a map $W^+(k) \otimes_{W^+(k)[[u]]} HA \to Hk$ (note that the left hand side is a pushout of $\mathbb{E}_\infty$-rings), which coincides with the equivalence obtained above. □

Lemma 6.18. *The equivalences in Lemma 6.16 and Lemma 6.17 assembles into a commutative diagram:*

$$\begin{array}{ccc} Mf_{k,a_0} & \xrightarrow{t_{k,a_0}} & Hk \\ \downarrow\simeq & & \uparrow\simeq \\ W^+(k) \otimes_{W^+(k)[[u]]} Mf_E & \longrightarrow & W^+(k) \otimes_{W^+(k)[[u]]} HA \end{array}$$

where the top horizontal map is the 0th Postnikov section t_{k,a_0} defined in Proposition 5.2 and the bottom horizontal map is the base-changed 0th Postnikov section $W^+(k) \otimes_{W^+(k)[[u]]} t_E$.

Proof. As in the proof of Lemma 5.15, it suffices to show that the composite map on the 0th homotopy group $\pi_0(Mf_{k,a_0}) \to \pi_0(W^+(k) \otimes_{W^+(k)[[u]]} Mf_E) \to \pi_0(W^+(k) \otimes_{W^+(k)[[u]]} HA) \to \pi_0(Hk) \cong k$ is an isomorphism, which follows from an explicit element chasing. □

Combined with Lemma 5.3, we obtain that

Corollary 6.19. *The base-changed map $W^+(k) \otimes_{W^+(k)[[u]]} t_E$: $W^+(k) \otimes_{W^+(k)[[u]]} Mf_E \to W^+(k) \otimes_{W^+(k)[[u]]} HA$ is an equivalence of $W^+(k)$-modules.*

Apply Lemma 6.12 to the cofiber $\operatorname{cofib}(t_E)$, we deduce that

Corollary 6.20. *The map $t_E : Mf_E \to HA$ is an equivalence of spectra after (u)-completion.*

As in Lemma 5.9, we deduce from Theorem A.25 that

Lemma 6.21. *The $W^+(k)[[u]]$-module HA is (u)-complete.*

Now, given the nontrivial topological input Proposition 5.6, as in Lemma 5.10 and Corollary 5.11, we deduce that

Lemma 6.22. *The $W^+(k)[[u]]$-module Mf_E is (u)-complete.*

Corollary 6.23. *The cofiber $\operatorname{cofib}(t_E)$ is a (u)-complete $W^+(k)[[u]]$-module, and thus the map t_E is an equivalence of spectra by Corollary 6.20.*

This completes the proof of Theorem 6.14.

6.3. **Complete regular local rings.**

Inspired by [KN19, Section 9], we will provide a Hopkins-Mahowald theorem for complete regular local rings of mixed characteristic. We will show how to modify our proof of Theorem 6.14 to deduce this. Note that this is also a special case of Question 1, by [BS19, Remark 3.11].

We need some preparations in higher algebra:

Let $W^+(k)[u_1, \ldots, u_n]$ be the "n-variate polynomial $W^+(k)$-algebra", that is, the $\mathbb{E}_\infty$-$W^+(k)$-algebra $W^+(k) \otimes_{\mathbb{S}} \mathbb{S}[\mathbb{N}^n]$. Since the space $\mathbb{N}^n$ is endowed with discrete topology, parallel to Proposition 6.8, we have

Proposition 6.24. *As a $W^+(k)$-module, $W^+(k)[u_1, \ldots, u_n]$ is equivalent to the direct sum $\bigoplus_{\alpha \in \mathbb{N}^n} u^\alpha W^+(k)$, a free $W^+(k)$-module. The graded homotopy group $\pi_*(W^+(k)[u_1, \ldots, u_n])$, as a (graded-commutative) $\pi_*(W^+(k))$-algebra, is equivalent to $\pi_*(W^+(k))[u_1, \ldots, u_n]$, where $\deg u_1 = \cdots = \deg u_n = 0$.*

Now let $W^+(k)[[u_1, \ldots, u_n]]$ be the $(u_1, \ldots, u_n)$-completion of the $\mathbb{E}_\infty$-$W^+(k)$-algebra $W^+(k)[u_1, \ldots, u_n]$. By induction on $n \in \mathbb{N}_{>0}$ and argue as in Proposition 6.11 [details needed], we obtain:

Proposition 6.25. *The $\mathbb{E}_\infty$-$W^+(k)$-algebra $W^+(k)[[u_1, \ldots, u_n]]$ is connective. The zeroth homotopy group of $\pi_0(W^+(k)[[u_1, \ldots, u_n]])$ is isomorphic to the $(u_1, \ldots, u_n)$-adic completion of the polynomial $W(k)$-algebra $W(k)[u_1, \ldots, u_n]$, that is, the formal power series $W(k)$-algebra $W(k)[[u_1, \ldots, u_n]]$, as $W(k)$-algebras.*

Similarly, argue inductively on $n \in \mathbb{N}_{>0}$ as in Lemma 6.12, we obtain:

Lemma 6.26. *Let M be a $W^+(k)[u_1, \dots, u_n]$-(or $W^+(k)[[u_1, \dots, u_n]]$-)module (spectrum). If the spectrum $W^+(k) \otimes_{W^+(k)[u_1,\dots,u_n]} M$ (or $W^+(k) \otimes_{W^+(k)[[u_1,\dots,u_n]]} M$ respectively) is contractible, then so is the $(u_1, \dots, u_n)$-completion of the spectrum M. In particular, if furthermore $W^+(k)[u_1, \dots, u_n]$-(or $W^+(k)[[u_1, \dots, u_n]]$-)module M is assumed to be $(u_1, \dots, u_n)$-complete, then the spectrum M is contractible.*

We note that in these inductive arguments, we heavily depend on Proposition A.23.

Now we are ready to formulate the Hopkins-Mahowald theorem for complete regular local rings. We fix a positive integer $n \in \mathbb{N}_{>0}$, a perfectoid ring R. As in Section 5, let $\theta : W(R^\flat) \to R$ be Fontaine's pro-infinitesimal thickening. Let $\phi \in W(R^\flat)[[u_1, \dots, u_n]]$ be formal power series such that $\phi(0, \dots, 0) \in W(R^\flat)$ is a generator of $\ker \theta$. We recall that $\ker \theta$ is principal by definition. We note that the element $1 - \phi(u_1, \dots, u_n) \in W(R^\flat)[[u_1, \dots, u_n]]$ is invertible, since $1 - \phi(0, \dots, 0) \in W(R^\flat)$ is invertible as the ring $W(R^\flat)$ is $\ker \theta$-adically complete. As in Remark 1.9 and Remark 1.12, the element $1 - \phi(u_1, \dots, u_n) \in \mathrm{GL}_1(W(R^\flat)[[u_1, \dots, u_n]])$ gives rise to an $\mathbb{E}_2$-map $f : \Omega^2 S^3 \to \mathrm{BGL}_1(W(R^\flat)[[u_1, \dots, u_n]])$. The proof of Lemma 5.1 results in the following analogue:

Lemma 6.27. *The zeroth homotopy group of the $\mathbb{E}_2$-Thom spectrum Mf associated to the map f is isomorphic to the $W(R^\flat)$-algebra $W(R^\flat)[[u_1, \dots, u_n]]/(\phi(u_1, \dots, u_n))$.*

We now phrase the following variant of the Hopkins-Mahowald theorem:

Theorem 6.28. *The truncation map $t : Mf \to H\pi_0(Mf) \cong HW(R^\flat)[[u_1, \dots, u_n]]/(\phi(u_1, \dots, u_n))$ of $\mathbb{E}_2$-$W^+(R^\flat)[[u_1, \dots, u_n]]$-algebras is an equivalence of spectrum. Thus the Eilenberg-Maclane spectrum $HW(R^\flat)[[u_1, \dots, u_n]]/(\phi(u_1, \dots, u_n))$ is the $\mathbb{E}_2$-Thom spectrum Mf associated to the map $f : \Omega^2 S^3 \to \mathrm{BGL}_1(W^+(R^\flat)[[u_1, \dots, u_n]])$.*

The proof is parallel to that of Theorem 6.14, which we will omit. Now let $(A, \mathfrak{m})$ be a complete regular local ring with residue field $k = A/\mathfrak{m}$ being perfect of characteristic p. We also assume that $p \neq 0$ in A. Let $(a_1, \dots, a_n) \subseteq \mathfrak{m}$ be a regular sequence which generates the maximal ideal $\mathfrak{m}$. We need the following lemma:

Lemma 6.29. ([KN19, Lemma 9.2]) *There exists a map $W(k)[[u_1, \dots, u_n]] \to A$ of rings given by $u_i \mapsto a_i$ for $i = 1, \dots, n$, which is surjective with kernel being principal, generated by a formal power series $\phi \in W(k)[[u_1, \dots, u_n]]$ with $\phi(0, \dots, 0) = p$.*

Proof. First, the isomorphism $k \to A/\mathfrak{m}$ lifts to a map $W(k) \to A$ since A is $\mathfrak{m}$-adically complete, see Example 2.5 or [Ser79, Section II.5, Proposition 10]. The map $W(k)[[u_1, \dots, u_n]] \to A$ is then well-defined since A is $\mathfrak{m}$-adic complete. Let C, K be the cokernel and the kernel of the map $W(k)[[u_1, \dots, u_n]] \to A$ of $W(k)[[u_1, \dots, u_n]]$-modules. By right-exactness of classical tensor products, we have

$$\mathrm{Tor}_0^{W(k)[[u_1,\dots,u_n]]}(C, W(k)) \cong \mathrm{coker}(W(k) \to k) \cong 0$$

Now by inspecting the exact sequence $0\to\mathfrak{m}\to A\to k\to 0$, we deduce that A is a finitely generated $W(k)[[u_1,\ldots,u_n]]$-module, therefore so is C. We deduce from Nakayama's lemma that $C\cong 0$, therefore the map $W(k)[[u_1,\ldots,u_n]]$ is surjective. Now we obtain a short exact sequence of $W(k)[[u_1,\ldots,u_n]]$-modules

$$0\to K\to W(k)[[u_1,\ldots,u_n]]\to A\to 0$$

which gives rise to an exact sequence of $W(k)$-modules

$$\mathrm{Tor}_1^{W(k)[[u_1,\ldots,u_n]]}(A,W(k))\to\mathrm{Tor}_0^{W(k)[[u_1,\ldots,u_n]]}(K,W(k))\to W(k)\to k\to 0$$

Since $(a_1,\ldots,a_n)$ is a regular sequence, it is also Koszul regular [Sta20, Tag 062F], hence $\mathrm{Tor}_1^{W(k)[[u_1,\ldots,u_n]]}(A,W(k))\cong 0$. Thus

$$\mathrm{Tor}_0^{W(k)[[u_1,\ldots,u_n]]}(K,W(k))\cong\ker(W(k)\to k)\cong p\,W(k)$$

We pick a lift $\phi\in K$ of $p\in p\,W(k)$. By Nakayama's lemma, the $W(k)[[u_1,\ldots,u_n]]$ module (and hence the ideal) K is generated by the element $\phi\in K$. Furthermore, by multiplying an invertible element in $W(k)$, we can assume that the lift ϕ is so chosen that $\phi(0,\ldots,0)=p$. □

Remark 6.30. Our proof of Lemma 6.29 leads to a more general result: Let A be a commutative ring with an ideal $I\subseteq A$ generated by a (Koszul) regular sequence $(a_1,\ldots,a_n)\subseteq I$. If A is both p-adically complete and I-adically complete, and $R:=A/I$ is a perfectoid ring, then by Proposition 4.16, there exists a unique map $W(R^\flat)\to A$ such that the composite map $W(R^\flat)\to A\to R$ coincides with Fontaine's map, which allows us to view A as a $W(R^\flat)$-algebra. Now we consider the map $\varphi: W(R^\flat)[[u_1,\ldots,u_n]]\to A$ of $W(R^\flat)$-algebras given by $u_i\mapsto a_i$ for $i=1,\ldots,n$. Our proof of Lemma 6.29 implies that the map φ is surjective with kernel being principal, generated by a formal power series $\phi\in W(R^\flat)[[u_1,\ldots,u_n]]$ such that $\phi(0,\ldots,0)$ generates the kernel $\ker(\theta)$ of Fontaine's map $\theta: W(R^\flat)\to R$.

Corollary 6.31. *Let $\phi\in W(k)[[u_1,\ldots,u_n]]$ be a power series as described in Lemma 6.29. Let $f:\Omega^2 S^3\to\mathrm{BGL}_1(W^+(k)[[u_1,\ldots,u_n]])$ be the map given by the element $1-\phi(u_1,\ldots,u_n)\in\mathrm{GL}_1(W(k)[[u_1,\ldots,u_n]])$. Then the $\mathbb{E}_2$-Thom spectrum Mf associated to the map f is as an $\mathbb{E}_2$-$W^+(k)[[u_1,\ldots,u_n]]$-algebra equivalent to the Eilenberg-Maclane spectrum HA of the complete regular local ring A (of mixed characteristic).*

Proof. It follows from Theorem 6.28 by taking $R=k$ and Lemma 6.29. □

7. Characterizing Thom spectra as quotients of free $\mathbb{E}_2$-algebras

In this section, we will discuss an alternative characterization of Thom spectra which we learn from [AB19]. This characterization will enable us to peel off some redundant restraints in the definition of Thom spectra. We will rephrase Question 1 more broadly, and give a toy example related to the Breuil-Kisin case. We note that in fact, we have already used this characterization in Lemma 3.3.

We first present a theorem which we learn from Antolín-Camarena and Barthel's paper:

Remark 7.1. Let R be an $\mathbb{E}_\infty$-ring. Let $R[\Omega^2 S^2]$ be the free $\mathbb{E}_2$-R-algebra on a single generator in degree 0. Then for all $\mathbb{E}_2$-R-algebra S and elements $x \in \pi_0(S)$, the universal property of free $\mathbb{E}_2$-R-algebras gives rise to a map $R[\Omega^2 S^2] \to S$ which maps the generator (in fact, a connected component) to x. We will call this map the evaluation map of $R[\Omega^2 S^2]$ at x.

Theorem 7.2. **([AB19, Theorem 4.10])** *Let R be an $\mathbb{E}_\infty$-ring and $\alpha \in \pi_1(\mathrm{BGL}_1(R)) \cong \mathrm{GL}_1(\pi_0 R)$. Let $q: S^1 \to \mathrm{BGL}_1(R)$ a loop representing $\alpha \in \pi_1(\mathrm{BGL}_1(R))$. Let $f: \Omega^2 S^3 \to \mathrm{BGL}_1(R)$ be the double loop map associated to q (see Remark 1.2). Then the $\mathbb{E}_2$-Thom spectrum Mf associated to the $\mathbb{E}_2$-map f fits into a pushout diagram of $\mathbb{E}_2$-R-algebras:*

$$\begin{array}{ccc} R[\Omega^2 S^2] & \longrightarrow & R \\ \downarrow & & \downarrow \\ R & \longrightarrow & Mf \end{array}$$

where $R[\Omega^2 S^2] \cong R \otimes_{\mathbb{S}} \Sigma^\infty_+ S^2$ is the free $\mathbb{E}_2$-R-algebra on a single generator in degree 0, and two maps $R[\Omega^2 S^2] \to R$ are evaluation maps of $R[\Omega^2 S^3]$ at $0 \in \pi_0 R$ and $1 - \alpha \in \pi_0 R$ respectively.

Remark 7.3. Theorem 7.2 shows that the Thom spectrum description is equivalent to the pushout-diagram description. However, we note that the pushout-diagram description is more general in the sense that even if $\alpha \in \pi_0 R$ is not invertible, the pushout-diagram description is still valid while we can no longer, at least superficially, give a Thom spectrum description. We find it easier to write down proofs for Thom spectrum description so we adapted the Thom spectrum description for perfectoid rings.

We can now rephrase Question 1 as follows:

Question 2. Given an orientable prism $(A, I = (d))$. When can we find an $\mathbb{E}_\infty$-ring spectrum A^+ (which satisfies some hypotheses related to A. A naive guess would be that $\pi_0(A^+) = A$) so that the Eilenberg-Maclane spectrum $H(A/I)$ as an $\mathbb{E}_2$-A^+-algebra fits into a pushout diagram

$$\begin{array}{ccc} A^+[\Omega^2 S^2] & \longrightarrow & A^+ \\ \downarrow & & \downarrow \\ A^+ & \longrightarrow & H(A/I) \end{array}$$

such that two maps $A^+[\Omega^2 S^2] \to A^+$ are evaluation maps of the free $\mathbb{E}_2$-A^+-algebra $A^+[\Omega^2 S^2]$ at $0 \in \pi_0(A^+)$ and $d \in \pi_0(A^+)$ respectively.

Remark 7.4. Theorem 7.2 shows that Theorem 1.13 answers this question affirmatively when (A, I) is a perfect prism $(W(R^\flat), \ker\theta)$, with $A^+ := W^+(R^\flat)$.

Remark 7.5. Similarly, Theorem 6.14 answers this question affirmatively when (A, I) is a prism $(W(k)[[u]], (E(u)))$ associated to Breuil-Kisin cohomology where k is a perfect $\mathbb{F}_p$-algebra and $E(u) \in W(k)[u]$ is an Eisenstein polynomial.

We now announce a toy example of a variant of Theorem 6.14. As there, we fix a complete DVR $(A, \mathfrak{m})$ of mixed characteristics $(0, p)$ with residue field k being perfect, absolute ramification index e, a uniformizer $\varpi \in \mathfrak{m}$ and an Eisenstein $W(k)$-polynomial $E(u) \in W(k)[u]$ which induces an isomorphism $W(k)[u]/(E(u)) \xrightarrow{\sim} A$, $u \mapsto \varpi$ as in Proposition 6.6.

Theorem 7.6. *The (u)-completion of the total cofiber of the commutative diagram of $\mathbb{E}_2$-$W^+(k)[u]$-algebras*

$$\begin{array}{ccc} W^+(k)[u]\otimes_{\mathbb{S}}\mathbb{S}[\Omega^2 S^2] & \longrightarrow & W^+(k)[u] \\ \downarrow & & \downarrow \\ W^+(k)[u] & \longrightarrow & HA \end{array}$$

is contractible, where two maps $W^+(k)[u]\otimes_{\mathbb{S}}\mathbb{S}[\Omega^2 S^3]\to W^+(k)[u]$ are given by evaluation maps at $0\in\pi_0(W^+(k)[u])$ and $E(u)\in\pi_0(W^+(k)[u])$ respectively. Equivalently put, the commutative diagram above induces an equivalence of $W^+(k)[u]$-modules from the $\mathbb{E}_2$-pushout of the diagram $W^+(k)[u]\leftarrow W^+(k)[u]\otimes_{\mathbb{S}}\mathbb{S}[\Omega^2 S^2]\to W^+(k)[u]$ to the Eilenberg-Maclane spectrum HA after (u)-completion.

Corollary 7.7. ([KN19, Remark 3.4]) *The $\mathbb{E}_2$-HA-algebra $HA\otimes_{W^+(k)[u]}HA$ is the (p)-completion of the free $\mathbb{E}_2$-HA-algebra on a single generator in degree 1.*

Proof. Note that $E(u)$ vanishes after tensoring HA, and that (u)-completion coincides with (p)-completion for HA since ϖ^e/p is an invertible element, the result follows. ☐

We sketch a proof of Theorem 7.6, which is totally parallel to that of Theorem 6.14.

A sketch of a proof of Theorem 7.6. Let X be the pushout of the diagram $W^+(k)[u]\leftarrow W^+(k)[u]\otimes_{\mathbb{S}}\mathbb{S}[\Omega^2 S^2]\to W^+(k)[u]$ in question. We first check that the induced map $X\to HA$ is the 0th Postnikov section. Then we perform a base change $W^+(k)\otimes_{W^+(k)[u]}-$. We show that after such a base change, the induced map $X\to HA$ becomes an equivalence, given by Theorem 7.2 and Lemma 5.3. We then conclude the result by Corollary 6.20. ☐

Appendix A. Recollection of Higher Algebra

This appendix is devoted to a recollection of basic facts in Higher Algebra needed in the main text. Our main reference is [Lur17], [Lur18b] and [Lur18a].

A.1. Finiteness properties of rings and modules.

We will include some definitions and properties from [Lur17, Section 7.2.4].

Definition A.1. ([Lur17, Notation 7.1.1.10, Proposition 7.1.1.13]) Given a connective $\mathbb{E}_1$-ring R, there is a canonical accessible t-structure on LMod_R determined by subcategories $(\mathrm{LMod}_R)_{\geq 0}$ and $(\mathrm{LMod}_R)_{\leq 0}$, where $(\mathrm{LMod}_R)_{\geq 0}$ is the full subcategory of LMod_R spanned by those left R-modules M for which $\pi_n M\cong 0$ for $n<0$, and $(\mathrm{LMod}_R)_{\leq 0}$ is the full subcategory of LMod_R spanned by those left R-modules M for which $\pi_n M\cong 0$ for $n>0$.

Proposition A.2. ([Lur17, Proposition 7.1.1.13]) *Let R be a connective $\mathbb{E}_1$-ring, then the subcategories $(\mathrm{LMod}_R)_{\geq 0}$, $(\mathrm{LMod}_R)_{\leq 0}\subseteq\mathrm{LMod}_R$ are stable under small products and small filtered colimits.*

Definition A.3. ([Lur17, Proposition 7.2.2.10]) Let M be a left module over an $\mathbb{E}_1$-ring R. We will say that M is *flat* if the following conditions are satisfied:

1. The homotopy group $\pi_0 M$ is flat as a left module over $\pi_0 R$ in the usual sense.
2. For each $n\in\mathbb{Z}$, the natural map $\mathrm{Tor}_0^{\pi_0 R}(\pi_n R,\pi_0 M)\to\pi_n M$ is an isomorphism of abelian groups.

Definition A.4. **([Lur17, Definition 7.2.4.1])** Let R be an $\mathbb{E}_1$-ring. We let $\mathrm{LMod}_R^{\mathrm{perf}}$ denote the smallest stable subcategory of LMod_R which contains R (regarded as a left module over itself) and is closed under retracts. We will say that a left R-module M is perfect if it belongs to $\mathrm{LMod}_R^{\mathrm{perf}}$.

Definition A.5. **([Lur17, Definition 7.2.4.8])** Let $\mathcal{C}$ be a compactly generated ∞-category. We will say that an object $C \in \mathcal{C}$ is *almost compact* if $\tau_{\leq n} C$ is a compact object of $\tau_{\leq n}\mathcal{C}$ for all $n \geq 0$.

Definition A.6. **([Lur17, Definition 7.2.4.10])** Let R be a connective $\mathbb{E}_1$-ring. We will say that a left R-module M is *almost perfect* if there exists an integer k such that $M \in (\mathrm{LMod}_R)_{\geq k}$ and is almost compact as an object of $(\mathrm{LMod}_R)_{\geq k}$. We let $\mathrm{LMod}_R^{\mathrm{aperf}}$ denote the full subcategory of LMod_R spanned by the almost perfect left R-modules.

Proposition A.7. **([Lur17, Proposition 7.2.4.11])** *Let R be a connective $\mathbb{E}_1$-ring. Then:*

1. *The full subcategory $\mathrm{LMod}_R^{\mathrm{aperf}} \subseteq \mathrm{LMod}_R$ is closed under translation and finite colimits, and is therefore a stable subcategory of LMod_R;*
2. *The full subcategory $\mathrm{LMod}_R^{\mathrm{aperf}} \subseteq \mathrm{LMod}_R$ is closed under retracts;*
3. *Every perfect left R-module is almost perfect;*
4. *The full subcategory $(\mathrm{LMod}_R^{\mathrm{aperf}})_{\geq 0} \subseteq \mathrm{LMod}_R$ is closed under geometric realizations of simplicial objects;*
5. *Let M be a left R-module which is connective and almost perfect. Then M can be obtained as the geometric realization of a simplicial left R-module $P_\bullet$ such that each P_n is a free R-module of finite rank.*

Proposition A.8. *Let $f: A \to A'$ be a map of connective $\mathbb{E}_1$-rings. Let M be a connective left A-module and set $M' = A' \otimes_A M$. If M is an almost perfect left A-module, then M' is an almost perfect left A'-module.*

Proof. Since M is connective and almost perfect, by Proposition A.7, there exists a simplicial object $P_\bullet$ in LMod_A such that each P_n is a free A-module of finite rank and M is equivalent to the geometric realization of $P_\bullet$. Therefore M' is equivalent to the geometric realization of $A' \otimes_A P_\bullet$, by the fact the tensor products commute with small colimits. On the other hand, each $A' \otimes_A P_n$ is a free A'-module of finite rank, hence perfect, thus almost perfect. Now M' is equivalent to the geometric realization of almost perfect modules, therefore M' is almost perfect by Proposition A.7. □

Definition A.9. **([Lur17, Definition 7.2.4.13])** A discrete associative ring R is *left coherent* if every finitely generated left ideal of R is finitely presented as a left R-module.

Definition A.10. **([Lur17, Definition 7.2.4.16])** Let R be an $\mathbb{E}_1$-ring. We will say that R is *left coherent* if the following conditions are satisfied:

1. The $\mathbb{E}_1$-ring R is connective;
2. The discrete associative ring $\pi_0 R$ is left coherent;
3. For each $n \geq 0$, the homotopy group $\pi_n R$ is finitely presented as a left module over $\pi_0 R$.

Proposition A.11. **([Lur17, Proposition 7.2.4.17])** *Let R be an $\mathbb{E}_1$-ring and M a left R-module. Suppose that R is left coherent. Then M is almost perfect if and only if the following conditions are satisfied:*

i. For $m \ll 0$, $\pi_m M = 0$;

ii. For every integer m, $\pi_m M$ is finitely presented as a left $\pi_0 R$-module.

Corollary A.12. *Let R be a left coherent $\mathbb{E}_1$-ring, then $H\pi_0(R)$ as a left R-module is almost perfect.*

A.2. **Nilpotent, local and complete modules.**

We will include several definitions and propositions from [Lur18b], Chapter 7.

Definition A.13. **([Lur18b, Definition 7.1.1.1, Example 7.1.1.2])** Let R be a connective $\mathbb{E}_\infty$-ring and let $x \in \pi_0 R$. An R-module M is *x-nilpotent* if the localization $M[1/x]$ vanishes. Equivalently, M is x-nilpotent if and only if the action of x on $\pi_* M$ is locally nilpotent, that is, if and only if for each $y \in \pi_j M$, there exists an integer $n \gg 0$ such that $x^n y = 0$ in $\pi_j M$ for all $j \in \mathbb{Z}$.

Definition A.14. **([Lur18b, Definition 7.1.1.6])** Let R be a connective $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0 R$ be an ideal. We say that an R-module M is *I-nilpotent* if it is x-nilpotent for each $x \in I$.

Definition A.15. **([Lur18b, Definition 7.2.4.1])** Let R be a connective $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0 R$ be an ideal. We say that an R-module M is *I-local* if for every I-nilpotent R-module N, the mapping space $\mathrm{Map}_{\mathrm{Mod}_R}(N, M)$ is contractible.

Definition A.16. **([Lur18b, Definition 7.3.1.1])** Let R be a connective $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0 R$ be an ideal. We will say that an R-module M is *I-complete* if for every I-local R-module N, the mapping space $\mathrm{Map}_{\mathrm{Mod}_R}(N, M)$ is contractible.

Corollary A.17. *Let R be a connective $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0 R$ be an ideal. If M is an I-nilpotent R-module, then it is also an I-complete R-module.*

Proposition A.18. **([Lur18b, Proposition 7.3.1.4 and Notation 7.3.1.5])** *Let R be a connective $\mathbb{E}_\infty$-ring and let $I \subseteq \pi_0 R$ be a finitely generated ideal. Then every left R-module M fits into an (essentially unique) fiber sequence $M' \to M \to M''$, where M' is I-local and M'' is I-complete. Moreover, there is a functor, called the I-completion functor, $\mathrm{Mod}_R \to \mathrm{Mod}_R$, which maps M to M''. We denote by $M_I^\wedge$ the image of M under the I-completion functor.*

We can compute the I-completion functor when I is principal:

Proposition A.19. **([Lur18b, Proposition 7.3.2.1])** *Let R be a connective $\mathbb{E}_\infty$-ring and let $x \in \pi_0 R$ be an element. For any R-module $M \in \mathrm{Mod}_R$, let $T(M)$ denote the limit of the tower*

$$\cdots \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} M$$

Then $T(M)$ is (x)-local and the (x)-completion of M can be identified with the cofiber of the canonical map $\theta: T(M) \to M$.

Corollary A.20. (**[Lur18b, Corollary 7.3.2.2]**) *Let R be a connective $\mathbb{E}_\infty$-ring and let $x \in \pi_0 R$ be an element. The following conditions on an R-module $M \in \mathrm{Mod}_R$ are equivalent:*

1. *The module M is (x)-complete.*
2. *The limit of the tower*
$$\cdots \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} M$$
vanishes.

Corollary A.21. (**[Lur18b, Corollary 7.3.2.3]**) *Let R be a connective $\mathbb{E}_\infty$-ring, $I \subseteq \pi_0 R$ an ideal and $x \in \pi_0 R$ an element. Then the (x)-completion functor $\mathrm{Mod}_R \to \mathrm{Mod}_R, M \mapsto M^\wedge_{(x)}$ carries I-complete modules to I-complete modules.*

Corollary A.22. (**[Lur18b, Corollary 7.3.2.4]**) *Let R be a connective $\mathbb{E}_\infty$-ring, $x \in \pi_0 R$ and let M be an R-module.*

1. *If the R-module M is connective, then the (x)-completion $M^\wedge_{(x)}$ is connective.*
2. *If $M \in (\mathrm{Mod}_R)_{\leq 0}$, then $M^\wedge_{(x)} \in (\mathrm{Mod}_R)_{\leq 1}$.*

Proof. Let $T(M)$ be the limit of the tower $\left(\cdots \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} M\right)$. Then by Proposition A.19, we have the cofiber sequence $T(M) \to M \to M^\wedge_{(x)}$ which gives rise to a long exact sequence

$$\cdots \to \pi_n(T(M)) \to \pi_n(M) \to \pi_n(M^\wedge_{(x)}) \to \pi_{n-1}(T(M)) \to \pi_{n-1}(M) \to \pi_{n-1}(M^\wedge_{(x)}) \to \cdots$$

Furthermore, let $T_n(M)_*$ be the tower

$$\cdots \xrightarrow{x} \pi_n(M) \xrightarrow{x} \pi_n(M) \xrightarrow{x} \pi_n(M)$$

Then there is a Milnor sequence

$$0 \to \lim{}^1 T_{n+1}(M)_* \to \pi_n(T(M)) \to \lim T_n(M)_* \to 0$$

Especially, if M is assumed to be connective, then $T_n(M)_*$ is a tower of 0 for $n < 0$, which implies that $\pi_{n-1}(T(M))$ vanishes when $n < 0$. We deduce from the long exact sequence that $\pi_n(M^\wedge_{(x)})$ vanishes when $n < 0$. Similarly, if $M \in (\mathrm{Mod}_R)_{\leq 0}$, then $T_n(M)_*$ is a tower of 0 for $n > 0$, thus $\pi_n(T(M))$ vanishes when $n \geq 0$. We deduce from the long exact sequence that $\pi_n(M^\wedge_{(x)})$ vanishes when $n > 0$. □

Proposition A.23. (**[Lur18b, Corollary 7.3.3.3]**) *Let R be a connective $\mathbb{E}_\infty$-ring and $I \subseteq \pi_0 R$ be a finitely generated ideal. Let M be an R-module. Then the following conditions on M are equivalent:*

1. *M is I-complete;*
2. *For each $x \in I$, M is (x)-complete;*
3. *There exists a set of generators $x_1, \ldots, x_n$ for the ideal I such that M is (x_i)-complete for $i = 1, \ldots, n$.*

Remark A.24. (**[Lur18b, Corollary 7.3.3.6]**) Let $\phi: R \to R'$ be a morphism of connective $\mathbb{E}_\infty$-rings, $I \subseteq \pi_0 R$ a finitely generated ideal and $I' = \phi(I)\,\pi_0(R')$ the ideal generated by the image of I. Then

1. An R'-module M is I'-complete if and only if it is I-complete as an R-module;

2. For every R'-module M, the canonical map $M \to M_I^\wedge$ exhibits $M_I^\wedge$ as an I-completion of M, regarded as a morphism of R-modules.

Theorem A.25. **([Lur18b, Theorem 7.3.4.1])** *Let R be an $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal and let M be an R-module. The following conditions are equivalent:*

a) The R-module M is I-complete;

b) For every integer k, the homotopy group $\pi_k M$ satisfies the condition that for each $x \in I$, we have $\mathrm{Ext}^0_A(A[1/x], \pi_k M) = 0 = \mathrm{Ext}^1_A(A[1/x], \pi_k M)$ where $A = \pi_0 R$.

Proposition A.26. **([Lur18b, Proposition 7.3.4.8])** *Let R be a connective $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and let $x \in \pi_0 R$ be an element whose image in $(\pi_0 R)/I$ is invertible. If M is an I-complete left R-module, then multiplication by x induces an equivalence from M to itself.*

Proposition A.27. **([Lur18b, Proposition 7.3.5.7])** *Let R be a connective $\mathbb{E}_\infty$-ring, let $I \subseteq \pi_0 R$ be a finitely generated ideal, and let M be an almost perfect R-module. If R is I-complete, then so is M.*

Definition A.28. **([Lur18b, Section 7.3.6])** Let R be a discrete commutative ring and M be a (discrete) R-module. Let $I \subseteq R$ be a finitely generated ideal. The *I-adic completion* of M, denoted by $\mathrm{Cpl}(M; I)$, is defined to be the limit of tower $\lim M/I^n M$. There is a canonical map $M \to \mathrm{Cpl}(M; I)$. An R-module M is called *I-adically complete* if the canonical map $M \to \mathrm{Cpl}(M; I)$ is an isomorphism, and M is called I-adically separated if the canonical map $M \to \mathrm{Cpl}(M; I)$ is injective.

Proposition A.29. **([Lur18b, Corollary 7.3.6.3])** *Let R be a discrete commutative ring, let $I \subseteq R$ be a finitely generated ideal, and let M be a discrete R-module. The following conditions are equivalent:*

a) The module M is I-adically complete;

b) The module HM is I-complete and M is I-adically separated.

Warning A.30. By Proposition A.29, the concept of I-adic completeness does not coincide with the concept of I-completeness for discrete modules over discrete commutative rings. Rather, the former is stronger than the latter.

Definition A.31. A spectrum X is called *p-complete* if it is (p)-complete as an $\mathbb{S}$-module. For any spectrum X, the p-completion of X, denoted by $X_p^\wedge$, is the (p)-completion of X as an $\mathbb{S}$-module.

Remark A.32. When M is an R-module for a connective $\mathbb{E}_\infty$-ring R, (p) is also an ideal of $\pi_0 R$. In this case, it follows from Remark A.24 that M is (p)-complete as an $\mathbb{S}$-module if and only if it is (p)-complete as an R-module, so there is completely no ambiguity to talk about p-completeness. Similarly, Remark A.24 implies the p-completion of an R-module M is the underlying spectrum of the (p)-completion of M as an R-module.

Corollary A.33. *Let X be a bounded below spectrum. If X is p-complete and $H\mathbb{F}_p \otimes X \simeq 0$, then $X \simeq 0$.*

Proof. We will show inductively on $n \in \mathbb{Z}$ that $\pi_n X = 0$.

1. Since X is bounded below, $\pi_n X = 0$ for $n \ll 0$.
2. Suppose now that for every $m < n$, we have $\pi_m X = 0$. We will show that $\pi_n X = 0$. In this case, we have $0 = \pi_n(H\mathbb{F}_p \otimes X) \cong \mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{F}_p, \pi_n X)$. Thus for each $x \in \pi_n X$, there exists (by axiom of choice) a sequence $(x_j)_{j \in \mathbb{N}} \in (\pi_n X)^{\mathbb{N}}$ such that $x_0 = x$ and $x_j = p\, x_{j+1}$ for all $j \in \mathbb{N}$, which gives rise to a map $\varphi_x : \mathbb{Z}[1/p] \to \pi_n X$ of abelian groups given by $\varphi(1/p^j) = x_j$. Theorem A.25 tells us that $\varphi_x = 0$, and especially, $x = 0$. In conclusion, we have proved that $x = 0$ for each $x \in \pi_n X$, thus $\pi_n X = 0$. □

Bibliography

[AB19] Omar Antolín-Camarena and Tobias Barthel. A simple universal property of Thom ring spectra. *Journal of Topology*, 12(1):56–78, 2019.

[Bha12] B. Bhatt. Completions and derived de Rham cohomology. *ArXiv e-prints*, jul 2012.

[BMS18] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. Integral p-adic Hodge theory. *Publications Mathématiques. Institut de Hautes Études Scientifiques*, 128:219–397, 2018.

[BMS19] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. Topological Hochschild homology and integral p-adic Hodge theory. *Publications Mathématiques. Institut de Hautes Études Scientifiques*, 129:199–310, 2019.

[Bot82] Raoul Bott. *Differential Forms in Algebraic Topology*. Springer New York, New York, NY, 1982.

[BS19] Bhargav Bhatt and Peter Scholze. Prisms and Prismatic Cohomology. *ArXiv e-prints*, May 2019.

[Car08] Gunnar Carlsson. Derived completions in stable homotopy theory. *Journal of Pure and Applied Algebra*, 212(3):550–577, 2008.

[Fon94] Jean-Marc Fontaine. Le corps des périodes p-adiques. Number 223, pages 59–111. 1994. With an appendix by Pierre Colmez, Périodes p-adiques (Bures-sur-Yvette, 1988).

[HN19] Lars Hesselholt and Thomas Nikolaus. Topological cyclic homology. *ArXiv e-prints*, May 2019.

[KN] Achim Krause and Thomas Nikolaus. Lectures on topological Hochschild homology and cyclotomic spectra. `https://www.uni-muenster.de/IVV5WS/WebHop/user/nikolaus/Papers/Lectures.pdf`.

[KN19] Achim Krause and Thomas Nikolaus. Bökstedt periodicity and quotients of DVRs. *ArXiv e-prints*, Jul 2019.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009.

[Lur17] Jacob Lurie. Higher Algebra. `https://www.math.ias.edu/~lurie/papers/HA.pdf`, sep 2017.

[Lur18a] Jacob Lurie. Elliptic Cohomology II: Orientations. `https://www.math.ias.edu/~lurie/papers/Elliptic-II.pdf`, apr 2018.

[Lur18b] Jacob Lurie. Spectral Algebraic Geometry. `https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf`, feb 2018.

[Ser79] Jean-Pierre Serre. *Local fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York-Berlin, 1979. Translated from the French by Marvin Jay Greenberg.

[Sha18] Jay Shah. Parametrized higher category theory and higher algebra: Exposé II - Indexed homotopy limits and colimits. *ArXiv e-prints*, Sep 2018.

[Sta20] StacksProjectAuthors. The Stacks project. `https://stacks.math.columbia.edu`, 2020.

[Wal65] C. T. C. Wall. Finiteness conditions for CW-complexes. *Annals of Mathematics. Second Series*, 81:56–69, 1965.