GENERIC RANK OF BETTI MAP AND UNLIKELY INTERSECTIONS

ZIYANG GAO

Abstract. Let $A \to S$ be an abelian scheme over an irreducible variety over $\mathbb{C}$ of relative dimension $g$. For any simply-connected subset $\Delta$ of $S^{an}$ one can define the Betti map from $A_\Delta$ to $\mathbb{T}^{2g}$, the real torus of dimension $2g$, by identifying each closed fiber of $A_\Delta \to \Delta$ with $\mathbb{T}^{2g}$ via the Betti homology. Computing the generic rank of the Betti map restricted to a subvariety $X$ of $A$ is useful to study Diophantine problems, e.g. proving the Geometric Bogomolov Conjecture over characteristic 0 and studying the relative Manin-Mumford conjecture. In this paper we give a geometric criterion to detect this rank. As applications we answer a question of André-Corvaja-Zannier, and prove that this generic rank is maximal after taking a large enough fibered power (if $X$ satisfies some mild conditions). As a byproduct we reduce the relative Manin-Mumford conjecture to a simpler conjecture. The tools are functional transcendence and unlikely intersections techniques for mixed Shimura varieties (the universal abelian variety).

Contents

1. Introduction 1
2. Convention and Notation 8
3. Universal abelian variety and Betti map 9
4. Betti map on arbitrary abelian schemes 11
5. Bi-algebraic system associated with $A_g$ 12
6. From Betti map to the $t$-th degeneracy locus 17
7. Zariski closeness of the $t$-th degeneracy locus 19
8. Criterion of degenerate subvarieties 21
9. Application of the criterion to fibered powers 25
10. Generic rank of the Betti map 29
11. Link with relative Manin-Mumford 33
Appendix A. Discussion when the base takes some simple form 35
References 37

1. Introduction

Let $S$ be an irreducible variety over $\mathbb{C}$, and let $\pi_S : A \to S$ be an abelian scheme of relative dimension $g$. Through the whole introduction, let $X$ be a closed irreducible subvariety of $A$ such that $\pi_S(X) = S$.

For any $s \in S(\mathbb{C})$, there exists an open neighborhood $\Delta \subseteq S^{an}$ of $s$ with a real-analytic map, which we call the Betti map,

$$b_\Delta : A_\Delta = \pi_S^{-1}(\Delta) \to \mathbb{T}^{2g},$$

where $\mathbb{T}^{2g}$ is the real torus of dimension $2g$; see (4.3) for definition. An alternative way to see the Betti map is to define it as a canonical real-analytic map $b_\tilde{S} : A_{\tilde{S}} \to \mathbb{T}^{2g}$ where $\tilde{S} \to S^{an}$ is the universal cover and $A_{\tilde{S}}$ be the pullback of $A \to S$ under the universal cover. The relation

2000 Mathematics Subject Classification. 11G10, 11G50, 14G25, 14K15.
between these two points of view will be explained at the end of §4. For most applications one can use either $b_\Delta$ or $b_\Delta^\perp$. We prefer to state the final results in terms of $b_\Delta$ in this paper.

The goal of our paper is to fully study the generic rank of the Betti map when restricted a subvariety of $\mathcal{A}$. As a byproduct we reduce the relative Manin-Mumford conjecture to a simpler conjecture on unlikely intersections. Let us briefly explain the question.

Let $\mathcal{A}_g$ be the moduli space of principally polarized abelian varieties of dimension $g$ (with some high level structure) and let $\mathfrak{A}_g \to \mathcal{A}_g$ be the universal abelian variety. The abelian scheme $\pi_S: \mathcal{A} \to S$ induces a modular map $\varphi$ in the following way. There exists a principally polarizable abelian scheme $\mathcal{A}^! / S$ equipped with an isogeny $\mathcal{A} \to \mathcal{A}^!$. Then up to taking a finite covering of $S$ and the associated base change of $\mathcal{A} \to S$, we have the following diagram

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\varphi} & \mathcal{A}^! \\
\pi_S & \downarrow & \pi \\
S & \xrightarrow{\varphi_S} & \mathfrak{A}_g
\end{array}
$$

In some number theory and algebro-geometric applications, we wish to characterize when

$$
(1.2) \quad \max_{x \in X^{\text{sm}}(\mathbb{C}) \cap \mathcal{A}_\Delta} (\text{rank}_R (db_\Delta|_{X \cap \mathcal{A}_\Delta})_x) = 2l
$$

for $l \in \{1, \ldots, \min(\dim \varphi(X), g)\}$.[1] In other words to compute the generic rank of $b_\Delta|_{X \cap \mathcal{A}_\Delta}$.

Betti map was first studied and used in [Zan12]. Then it was used to study the relative Manin-Mumford conjecture by Bertrand, Corvaja, Masser, Pillay and Zannier in a series of works [MZ12, MZ14, BMPZ16, CMZ18, MZ18], to prove the geometric Bogomolov conjecture over char 0 by Gao-Habegger [GH18] and Cantat-Gao-Habegger-Xie [CGHX18] with (1.2) for $l = \dim \varphi(X) - \dim \varphi_S(S)$, and to prove the denseness of torsion points on sections of Lagrangian fibrations by Voisin [Voi18] using André-Corvaja-Zannier’s result [ACZ18] on (1.2) for $l = g$.

In [ACZ18] the question of determining (1.2) was explicitly asked and systematically studied for the first time. The submersivity problem, namely $l = g$, was solved for all $A / S$ with $\text{End}(A / S) = \mathbb{Z}$ and $\dim \varphi_S(S) \geq g$. A clear connection to functional transcendence (André’s independence of abelian logarithms [And92] and pure Ax-Schanuel [MPT17]) was presented.

In our paper we completely solve the generic rank problem by giving a criterion (equivalent condition) to (1.2) for each $l$ in simple terms of the geometry of $X$: see Theorem 1.4 and above, and its simplification (1.4) for some particular $A / S$. Our approach is independent of [ACZ18]. It uses functional transcendence (in a different vein, mixed Ax-Schanuel [Gao18]), combined with a finiteness result [Gao18, Theorem 1.4]. We refer to the end of §1.5 for more details.

### 1.1. Two results on the generic rank of the Betti map.

Before explaining the criterion, let us see two applications in this subsection.

The following question was asked by André-Corvaja-Zannier [ACZ18, Conjecture 2.1.2].

**ACZ Question.** Assume that $A / S$ has no fixed part over any finite covering of $S$ and that $\mathbb{Z}X = \bigcup_{N \in \mathbb{Z}} \{[N]x : x \in X(\mathbb{C})\}$ is Zariski dense in $A$. Does (1.2) hold for $l = \min(\dim \varphi(X), g)$?

The original question was on sections of $A \to S$. It is equivalent to our formulation as for general $X$ we may take the pullback of $A \to S$ under $\pi_S|_X: X \to S$, thus making $X$ into a section. We prefer the current statement for a reason which we will see below. Many cases were solved when $\dim \varphi_S(S) \geq g$ in [ACZ18], e.g., any $A / S$ with $\text{End}(A / S) = \mathbb{Z}$; see §1.5 for more details.

Our **first application** is to answer the ACZ Question: it holds in many cases, but may fail in general.

---

[1] Note that the left hand side is clearly $\leq 2 \min(\dim \varphi(X), g)$ since $b_\Delta$ factors through $\varphi$; see (4.3).
Theorem 1.1. We have:

1. The ACZ Question has a positive answer if:
   (i) Either $A \rightarrow S$ is geometrically simple;
   (ii) Or each Hodge generic curve $C \subseteq \varphi_S(S)$ satisfies the following property: $\varphi(A)|_C := \pi^{-1}(C) \rightarrow C$ has no fixed part over any finite covering of $C$.

2. There exist a closed irreducible subvariety $S \subseteq \mathbb{A}_4$ of dimension 4 and a section $\zeta$ of $\mathbb{A}_g|_S \rightarrow S$ such that $\mathbb{A}_g|_S \rightarrow S$ has no fixed part over any finite covering of $S$, $\mathbb{Z}_\xi$ is Zariski dense in $\mathbb{A}_g|_S$, and $\text{rank}(\mathbb{H}|_{\xi(S) \cap A_\Delta}) \leq 8$ for all $x \in \xi(S)$.

Part (1) is Theorem 10.4; see Remark 10.5 for more explanation on part (1.ii). Part (2) is a simple construction presented in Example 10.6. Note that this counterexample is the simplest one: In this example $A/S$ has maximal variation and $X$ is the image of a section, and by [ACZ18, Theorem 2.3.1] no such examples exist for $g \leq 3$.

Now let us turn to the second application. In some number theory and algebro-geometric applications, it is particularly important to understand when (1.2) holds for $l = \dim X$, namely

\[ \max_{x \in X^{\text{cm}}(C) \cap A_\Delta} (\text{rank}_X(\mathbb{H}|_{X \cap A_\Delta})_x) = 2 \dim X. \]

Moreover we cannot assume that $A/S$ has no fixed part in general. For example for $C \rightarrow S$ a family of curve of genus $g \geq 2$ (with a section $\sigma$), its Jacobian $\text{Jac}(C) \rightarrow S$ may have fixed part.

In order for (1.3) to hold, $\varphi|_X$ must be generically finite because the left hand side of (1.3) is at most $2 \dim \varphi(X)$. Let us assume that $\varphi|_X$ is generically finite. Even under this assumption it is clearly true that achieving (1.3) requires more. For a toy case, if $\dim X = \dim \varphi(X) > g$, then (1.3) cannot hold since its left hand side is at most $2g$.

Because of this problem, our main result towards (1.3) is in the following philosophy. Instead of proving (1.3) for $X$, we raise $X$ to a large enough fibered power so that (1.3) holds for this fibered power. Note that part (2) of Theorem 1.1 says that one cannot expect (1.3) to hold in general, so we still need to put some extra assumptions on $X$.

We illustrate this philosophy with the following example. Let $M_g$ be the moduli space of curves of genus $g$. Up to taking a finite covering, we have the universal curve $C_g \rightarrow M_g$ and a section of it. Embed $C_g \hookrightarrow \text{Jac}(C_g)$ via this section, then we obtain a diagram

\[
\begin{array}{ccc}
C_g & \hookrightarrow & \text{Jac}(C_g) \\
\downarrow & & \downarrow \pi \\
M_g & \rightarrow & \mathbb{A}_g
\end{array}
\]

Let $T_g \subseteq \mathbb{A}_g$ be the image of the bottom arrow and by abuse of notation denote by $C_g$ the image of the top arrows. Then $C_g$ contains the zero section of $\mathbb{A}_g|_{T_g} := \pi^{-1}(T_g) \rightarrow T_g$.

Theorem 1.2. Let $S \subseteq T_g$ be a closed irreducible subvariety, and let $A = \pi^{-1}(S)$, $C_S = C_g \times T_g$. For any $m \geq 1$, denote by $A^{[m]} = A \times_S \ldots \times_S A$ (m-copies), $C_S^{[m]} = C_S \times_S \ldots \times_S C_S$ (m-copies), and $b^{[m]}_A = (b_A, \ldots, b_A) : A^{[m]}_\Delta \rightarrow \mathbb{T}^{2mg}$. Assume $g \geq 2$. Then

1. $\max_{x \in (C_S^{[m]}|_{\text{cm}}(C) \cap A^{[m]}} (\text{rank}_{C_S}(b^{[m]}_A|_{C_S^{[m]}|_{\text{cm}}(C) \cap A^{[m]}})_x) = 2 \dim(C_S^{[m]})$ for all $m \geq \dim S$

2. $\max_{x \in (C_S^{[m]} - C_S^{[m]})|_{\text{cm}}(C) \cap A^{[m]}} (\text{rank}_{C_S}(b^{[m]}_A|_{(C_S^{[m]} - C_S^{[m]})|_{\text{cm}}(C) \cap A^{[m]}})_x) = 2 \dim(C_S^{[m]} - C_S^{[m]})$ for all $m \geq \dim S$ if moreover $g \geq 3$.

Theorem 1.2 is a particular case of part (1) of the following Theorem 1.3 (applied to $\varphi = \text{id}$, $X = C_S$ and $X = C_S - C_S = (C_S - C_S) \times T_g S$), which is the general result towards (1.3).
For any integer $m \geq 1$, denote by $\mathcal{A}^{[m]} = A \times_S \ldots \times_S A$ ($m$-copies), $X^{[m]} = X \times_S \ldots \times_S X$ ($m$-copies) and by $b^{[m]}_\Delta = (b_\Delta, \ldots, b_\Delta): A^{[m]}_\Delta \to \mathbb{T}^{2mg}$.

**Theorem 1.3.** Assume that $\varphi|_X$ is generically finite. Assume furthermore that $X$ satisfies:

(i) We have $\dim X > \dim S$.

(ii) There exists $s \in S$ such that no proper subgroup of $A_s$ contains $X_s$.

(iii) We have $X + A' \not\subseteq X$ for any non-isotrivial abelian subscheme $A'$ of $A \to S$.

Then we have

1. $\max_{x \in (X^{[m]})^{sm}(C) \cap A^{[m]}_\Delta} \rank R(\db^{[m]}_\Delta|_{X^{[m]} \cap A^{[m]}_\Delta})_x = 2 \dim X^{[m]}$ for all $m \geq \dim S$.

2. If furthermore $X$ contains the image of $\mathcal{S}$ under a constant section $\sigma$ of $A \to S$, then $\max_{x \in (D^m_\Delta(X^{[m]}) \cap A^{[m]}_\Delta)} \rank R(\db^{[m]}_\Delta|_{D^m_\Delta(X^{[m]}) \cap A^{[m]}_\Delta})_x = 2 \dim D^m_\Delta(X^{[m]})$ for all $m \geq \dim X$. Here $D^m_\Delta: A^{[m]+1} \to A^{[m]}$ is the $m$-th Faltings-Zhang map fiberwise defined by $(P_0, P_1, \ldots, P_m) \mapsto (P_1 - P_0, \ldots, P_m - P_0)$.

This theorem follows directly from (the slightly more general) Theorem 10.3. In practice, the bound for $m$ can often be improved; see Remark A.4 for some improvement of Theorem 1.2 and Theorem 1.2'. Here hypothesis (i) is crucial: if $X$ is the image of a multi-section of $A \to S$, then $X^{[m]}$ is contained in the diagonal of $A \to A^{[m]}$, so essentially no new objects are constructed with the operation. This is why we prefer to state the ACZ question in the current form.

We close this subsection with the following result, which is a direct corollary of part (2) of Theorem 1.3 applied to $\varphi = \text{id}$ and $X = \mathcal{C}_S$.

**Theorem 1.2'.** Under the notation of Theorem 1.2. Let $D^m_\Delta$ be the $m$-th Faltings-Zhang map for $\pi: \mathfrak{A}_g \to \mathfrak{H}_g$, namely

$$D^m_\Delta: \mathfrak{A}_g \times_{\mathfrak{A}_g} \mathfrak{A}_g \times_{\mathfrak{A}_g} \ldots \times_{\mathfrak{A}_g} \mathfrak{A}_g \to \mathfrak{A}_g \times_{\mathfrak{A}_g} \ldots \times_{\mathfrak{A}_g} \mathfrak{A}_g$$

fiberwise defined by $(P_0, P_1, \ldots, P_m) \mapsto (P_1 - P_0, \ldots, P_m - P_0)$. Assume $g \geq 2$. Then we have $\max_{x \in (D^m_\Delta(\mathcal{C}_S^{[m+1]}) \cap A^{[m]}_\Delta\mathcal{A}_{\Delta}^{[m+1]})} \rank R(\db^{[m]}_\Delta|_{D^m_\Delta(\mathcal{C}_S^{[m+1]}) \cap A^{[m]}})_x = 2 \dim D^m_\Delta(\mathcal{C}_S^{[m+1]})$ for all $m \geq \dim(\mathcal{C}_S) = 1 + \dim S$.

1.2. **Criterion for generic Betti rank.** Denote for simplicity by

$$\rank R(\db^{[m]}_\Delta|_X) = \max_{x \in X^{[m]}(C) \cap A^{[m]}_\Delta} \rank R(\db^{[m]}|_{X^{[m]} \cap A^{[m]}_\Delta})_x.$$

In this subsection we state the criterion for (1.2) for each $l$. An equivalent (but simpler formulated) way to do this is to give an equivalent condition for $\rank R(\db^{[m]}_\Delta|_X) < 2l$ for each $l$.

Let $\mathcal{A}_X$ be the translate of an abelian subscheme of $A \to S$ by a torsion section which contains $X$, minimal for this property. Then $\mathcal{A}_X \to S$ itself is an abelian scheme (up to taking a finite covering of $S$), whose relative dimension we denote by $g_\mathcal{B}$ and see §2 for our convention of abelian subschemes and torsion sections.

**Theorem 1.4.** Let $l \in \{1, \ldots, \min(\dim(\varphi(X), g))\}$ or $l = \dim X$. Then $\rank R(\db^{[m]}_\Delta|_X) < 2l$ if and only if the following condition holds: there exists an abelian subscheme $B$ of $\mathcal{A}_X \to S$ (whose relative dimension we denote by $g_B$) such that for the quotient abelian scheme $p_B: \mathcal{A}_X \to \mathcal{A}_X/B$ and the modular map $\iota_B: \mathcal{A}_X/B \to \mathfrak{A}_{g_X - g_B}$ (as below), we have $\dim(\iota_B \circ p_B)(X) < l - g_B$.

$$\begin{array}{ccc}
\mathcal{A}_X/B & \xrightarrow{\iota_B} & \mathfrak{A}_{g_X - g_B} \\
\downarrow & & \\
S & \xrightarrow{\iota_B/S} & \mathfrak{H}_{g_X - g_B}
\end{array}$$
This criterion is proven as Theorem 10.2. In fact we can prove more: we are able to write the locus of $X$ where $\text{rank}_S(\text{db}_{\Delta}(X))$ is smaller than $2l$ for the integers $l$ listed in Theorem 1.4. We will explain this in the next subsection.

We finish this subsection by making the following remark: when $S$ takes some simple form, this geometric criterion can be much simplified. More precisely for each $t$ of Definition 1.2. See Appendix A for some discussion.

Let $\mathcal{A}$ be the modular map $\mathcal{A} \equiv \mathcal{A}(d)$. We have

(1.4) \[
\text{rank}_S(\text{db}_{\Delta}(X)) < 2l \iff \dim(\langle \varphi(X) \rangle_{\text{gen-sp}} - \dim \varphi_S(S) < l
\]

if $\dim \varphi_S(S) = 1$ or $\varphi_S(S)$ has simple connected algebraic monodromy group.

See Corollary A.3 for the statement and Definition A.1 for the definition of $\langle \varphi(X) \rangle_{\text{gen-sp}}$.

1.3. The $t$-th degeneracy locus. Our method to study the generic rank of the Betti map is to translate the problem into studying the $t$-th degeneracy locus defined below. Let us explain it in this subsection.

Definition 1.5. A closed irreducible subvariety $Z$ of $\mathcal{A}$ is called a generically special subvariety of sg type of $\mathcal{A}$ if there exists a finite covering $S' \rightarrow S$, inducing a morphism $\rho: \mathcal{A}' = \mathcal{A} \times_S S' \rightarrow \mathcal{A}$, such that $Z = \rho(\sigma' + \sigma_0 + \beta')$, where $\beta'$ is an abelian subscheme of $\mathcal{A}'/S'$, $\sigma'$ is a torsion section of $\mathcal{A}'/S'$, and $\sigma_0$ is a constant section of $\mathcal{A}'/S'$.

We briefly explain the meaning of constant section here. Let $C' \times S'$ be the largest constant abelian subscheme of $\mathcal{A}'/S'$. We say that a section $\sigma_0': S' \rightarrow \mathcal{A}'$ is a constant section if there exists $c' \in C'(\mathbb{C})$ such that $\sigma_0$ is the composite of $S' \rightarrow C' \times S'$, $s' \mapsto (c', s')$, and the inclusion $C' \times S' \subseteq \mathcal{A}'$.

Definition 1.5 is closely related to the generically special subvarieties defined in [GH18, Definition 1.2]. See Appendix A for some discussion.

For any locally closed irreducible subvariety $Y$ of $\mathcal{A}$, denote by $\langle Y \rangle_{\text{sg}}$ the smallest generically special subvariety of sg type of $\mathcal{A}|_{\pi_S(Y)} = \pi_S^{-1}(\pi_S(Y))$ which contains $Y$.

Definition 1.6. Let $X$ be a closed irreducible subvariety of $\mathcal{A}$. For any $t \in \mathbb{Z}$, define the $t$-th degeneracy locus of $X$, denoted by $X^{\text{deg}}(t)$, to be the union of positive dimensional closed irreducible subvarieties $Y \subseteq X$ such that $\dim(Y)_{\text{sg}} - \dim \pi_S(Y) < \dim Y + t$. When $t = 0$, we abbreviate $X^{\text{deg}}(0)$ as $X^{\text{deg}}$. We say that $X$ is degenerate if $X^{\text{deg}}$ is Zariski dense in $X$.

The $t$-th degeneracy locus $X^{\text{deg}}(t)$ thus defined is a priori a complicated subset of $X$. However we prove that it is Zariski closed in $X$ in most interesting cases (as Theorem 7.1).

Theorem 1.7. Assume $S$ is a locally closed irreducible subvariety of $\mathcal{A}_g$ and $\mathcal{A} = \mathfrak{A}_g|_{\mathfrak{A}_g}$. Then $X^{\text{deg}}(t)$ is Zariski closed in $X$ for each $t \in \mathbb{Z}$.

Now we are ready to describe the locus on which $\text{rank}_S(\text{db}_{\Delta}(X))$ is smaller than expected. Recall the modular map $\varphi: \mathcal{A} \rightarrow \mathfrak{A}_g$ (1.1). We have (Theorem 10.1):

Theorem 1.8. Let $x \in X^{\text{sm}}(\mathbb{C}) \cap \mathcal{A}_{\Delta}$. Then

1. We have

$$\text{rank}_S(\text{db}_{\Delta}(X)) = 2 \dim X \iff x \notin X^{\text{deg}} \iff \varphi|_{X^{\text{sm}}} \text{ is injective around } x \text{ and } \varphi(x) \notin \varphi(X)^{\text{deg}}.$$ 

In particular if $x \notin \varphi^{-1}(Z \cup \varphi(X)^{\text{deg}}) = X^{\text{deg}}$ where $Z$ is the Zariski closed subset of $\varphi(X)$ on which $\varphi|_X$ has positive dimensional fibers, then $\text{rank}_S(\text{db}_{\Delta}(X)) = 2 \dim X$.

2. Denote by $d = \dim \varphi(X)$. For each integer $l \in \{1, \ldots, \min(d, g)\}$, we have

$$\text{rank}_S(\text{db}_{\Delta}(X)) = 2l \iff \varphi(x) \notin \varphi(X)^{\text{deg}}(l - d).$$

In fact our proof of Theorem 1.4 is via Theorem 1.8: We first prove Theorem 1.8, so that the question of the generic rank of the Betti map becomes characterizing for which $X$ we have $\varphi(X)^{\text{deg}}(t) = \varphi(X)$ ($t \leq 0$). Then we prove the desired characterization in Theorem 8.1 and thus obtain Theorem 1.4.
1.4. Relation with the relative Manin-Mumford conjecture. Another application of studying the $t$-th degeneracy locus is to prove the relative Manin-Mumford conjecture. In this application we need to take $t = 1$. Let us state the result.

Denote by $A_{\text{tor}}$ the set of points $x \in A(\mathbb{C})$ such that $[N]x$ lies in the zero section of $A \to S$ for some integer $N$. Zannier [Zan12] proposed the following relative Manin-Mumford conjecture.

Relative Manin-Mumford Conjecture. Assume that $Z_X := \bigcup_{N \in \mathbb{Z}} \{[N]x : x \in X(\mathbb{C})\}$ is Zariski dense in $A$. If $(X \cap A_{\text{tor}})^{\text{Zar}} = X$, then $\text{codim}_A(X) \leq \dim S$.

In this paper we will reduce this conjecture to another simpler conjecture.

Conjecture 1.9. Assume $S$ is a locally closed irreducible subvariety of $A_g$ defined over $\mathbb{Q}$ and $A = \mathfrak{A}_g \times_{\mathfrak{A}_g} S$. Assume $X$ is defined over $\mathbb{Q}$. If $(X \cap A_{\text{tor}})^{\text{Zar}} = X$, then $X^{\text{deg}}(1) = X$.

Proposition 1.10. Conjecture 1.9 implies the relative Manin-Mumford conjecture.

A more precise version of this reduction is Proposition 11.2.

Proposition 1.10 suggests that there is a strong link between the Betti map and the relative Manin-Mumford conjecture. The existence of such a link already appeared in previous works on relative Manin-Mumford: the Betti coordinate played a key role in the proofs of many particular cases of the conjecture by Masser-Zannier [MZ12, MZ14, MZ15] and Corvaja-Masser-Zannier [CMZ18] (pencils of abelian surfaces, first over $\overline{\mathbb{Q}}$ then over $\mathbb{C}$; passing from $\overline{\mathbb{Q}}$ to $\mathbb{C}$ is highly non-trivial as it enlarges the base), Bertrand-Masser-Pillay-Zannier [BMPZ16] (semi-abelian surfaces), and Masser-Zannier [MZ18] (any abelian scheme over a curve). See also [Zan14].

Here let me say a few more words about how reducing to Conjecture 1.9 helps prove the relative Manin-Mumford conjecture.

The Pila-Zannier method to solve “special point” kind of problem is as follows. One first associates each special point with a complexity and shows that the size of the Galois orbit of each special point grows at least polynomially in its complexity (with the constants independent of the special point); then one uses the Pila-Wilkie counting theorem (and its variants) and some functional transcendence result to produce positive dimensional “weakly special/optimal subvarieties” and shows that their union is Zariski dense; the last step is non-standard and is different case by case.

In view of the relative Manin-Mumford conjecture, “special points” are precisely those points in $A_{\text{tor}}$, and the complexity is the order of the torsion point. Conjecture 1.9 can serve as the last step of the Pila-Zannier method above.

1.5. Historical notes on the Betti map. As we said, [ACZ18] was the first paper which explicitly asked the question of determining the generic rank of the Betti map and systematically studied it, obtaining some rather general results for the case $l = g$ (submersivity). However the idea of using Betti map (and Betti coordinates) to solve number theory and algebro-geometric problems dates back long. Now let us step back and put these into a historic perspective.

In extending Deligne’s theory of Shimura varieties, Pink [Pin89] studied the uniformizing space of the universal abelian variety (and the universal relatively ample line bundle on it), where one can see clearly the blueprint of the Betti map. In fact we will use his idea to define the Betti map; see §3.3. An evidence that the Betti map plays an important role in the geometry of $\mathfrak{A}_g$ is briefly discussed in §3.4 (after Mok [Mok91]).

On the other hand, the idea of Betti map already goes back to Manin [Man63]. In modern language he characterized in loc.cit. the cases when the Betti map restricted to a section of an
abelian scheme is locally constant.\footnote{In particular he proved that in the case of abelian schemes with no constant part, the Betti map restricted to a section is locally constant if and only if the section is torsion. This celebrated paper linked the Betti map with differential operators, leading to the Gauss-Manin connection (after Grothendieck).} Some explicit Betti maps, as those related to hyperelliptic curves, were also studied; see [Ser18].

This aspect was re-observed independently by Zannier to study unlikely intersection problems in [Zan12], where Betti map (and Betti coordinates) was first studied and used to solve Diophantine problems. However the name Betti map/coordinates was introduced only later by Bertrand (see [ACZ18, footnote 6]) and appeared for the first time in [BMPZ16]. The relative Manin-Mumford conjecture was proposed based on some works of Masser, and Betti coordinates were used to prove certain cases of the conjecture by Masser-Zannier [MZ12, MZ14, MZ15] and Corvaja-Masser-Zannier [CMZ18] (pencils of abelian surfaces, first over \( \mathbb{Q} \) then over \( \mathbb{C} \); passing from \( \mathbb{Q} \) to \( \mathbb{C} \) is highly non-trivial as it enlarges the base), Bertrand-Masser-Pillay-Zannier [BMPZ16] (semi-abelian surfaces), and Masser-Zannier [MZ18] (any abelian scheme over a curve). In these works one sees a blueprint of the ACZ Question. See also [Zan14].

Meanwhile the Betti map has also been used by Gao-Habegger [GH18] and Cantat-Gao-Habegger-Xie [CGHX18] to prove the geometric Bogomolov conjecture over characteristic 0. In [GH18] one also sees a height inequality which goes beyond GBC and is new in spirit.

The ACZ Question was explicitly proposed and systematically studied by André-Corvaja-Zannier in [ACZ18], by relating the Betti map to the Kodaira- Spencer map [And16]. A theorem on the independence of abelian logarithms by André [And92] also played an important role; see also [And17]. They got a sufficient condition (which we call Condition ACZ) in terms of the derivations on the base [ACZ18, Corollary 2.2.2]. Then they proved Condition ACZ in loc.cit. when \( \dim \varphi_S(S) \geq g \) and \( g \leq 3 \), and a link to pure Ax-Schanuel was observed. Inspired by several discussion with André-Corvaja-Zannier, the author proved in Appendix II of loc.cit. Condition ACZ when \( \dim \varphi_S(S) \geq g \) and \( \varphi_S(S) \) is Hodge generic in \( \mathbb{A}_g \), by using the pure Ax-Schanuel theorem [MPT17]. Thus a result of André-Corvaja-Zannier [ACZ18, Theorem 8.1.1] gives an affirmative answer to the ACZ Question when \( \dim \varphi_S(S) \geq g \) and \( \text{End}(A/S) = \mathbb{Z} \); see [ACZ18, Theorem 2.3.2].

Thus when \( \dim \varphi_S(S) \geq g \), many cases are solved in [ACZ18]. The proof is very beautiful as Condition ACZ is purely on the base, and hence it suffices to study the moduli space instead of the universal abelian variety in these cases. This of course much simplifies the problem. In particular [ACZ18] only uses pure Ax-Schanuel, avoiding the more complicated mixed Ax-Schanuel on which everything builds in the current paper. Before moving on, let us point out that some results in [ACZ18] were also used by Voisin [Voi18] to prove the denseness of torsion points on sections of Lagrangian fibrations.

However it is hardly possible to fully answer the ACZ Question by applying Condition ACZ or only the pure Ax-Schanuel theorem. Firstly \( \varphi(X) \) usually has positive dimensional fibers under \( \pi: \mathbb{A}_g \to \mathbb{A}_g \), and hence it is hardly possible to completely turn the question into one on the base. For the level of difficulty there are essentially 3 cases to study for the ACZ Question: \( \dim \varphi_S(S) \geq g \), \( \dim \varphi_S(S) < g \) but \( \dim \varphi(X) \geq g \), and \( \dim \varphi(X) < g \). Condition ACZ does not apply to the last two cases. Secondly even in the first case, there are examples for which the ACZ Question is true but Condition ACZ is violated: we can construct, by using Shimura varieties of PEL type, such examples when \( \varphi_S \) is generically finite, \( X \) is a section, \( \dim S \geq g \), and \( A/S \) is a geometrically simple abelian scheme.

Our proof of Theorem 1.1 is independent of [ACZ18]. It is a generalization of the method presented in [Gao18, Theorem 1.3], which proves the ACZ Question when \( \varphi_S \) is generically finite, \( X \) is a section, \( \dim S \geq g \), and \( A/S \) is a geometrically simple abelian scheme. The proof of [Gao18, Theorem 1.3] is a simple application of the mixed Ax-Schanuel theorem for...
the universal abelian varieties [Gao18, Theorem 1.2]. In this paper we shall moreover apply a finiteness result à la Bogomolov [Gao18, Theorem 1.4] and use extensively Deligne-Pink’s language of mixed Shimura varieties. We point out that the Kodaira-Spencer map is not used and our method does not need or make any new contribution to Condition ACZ.

1.6. Outline of the paper. In §2 we set up some convention of the paper. In §3 we recall the universal abelian variety and define the Betti map for this case. In §4 we define the Betti map for a general abelian scheme. These are the basic setting up of the paper.

In §5 we explain in details the main tools which we use to study the Betti map. There are two parts. The first part §5.1-5.2 is to introduce the functional transcendence theorem (called weak Ax-Schanuel), and the second part §5.3-5.4 is Deligne-Pink’s language of mixed Shimura varieties. Then §6 uses weak Ax-Schanuel to transfer the problem of the generic rank of the Betti map into studying the t-th degeneracy locus for some particular t’s.

We prove the Zariski closeness of the t-th degeneracy locus in §7. The proof in this section will also be used in §8, where the criterion to compute the generic rank is proven. Then we apply this criterion to prove that the t-th degeneracy locus is proper after taking a large fibered power of a given subvariety (under some mild conditions) in §9. In §10 we summarize the previous results to state our results concerning the generic rank of the Betti map.

In §11 we reduce the relative Manin-Mumford conjecture to a simpler conjecture about the 1-st degeneracy locus.

In the Appendix we further simplify the geometric criterion of computing the Betti map (or the properness of the t-th degeneracy locus) when the base takes some simple form.

Acknowledgements. The author would like to thank Umberto Zannier for relevant discussions, especially on the historical notes on the Betti map. The author would like to thank Fabrizio Barroero, Philipp Habegger, and Umberto Zannier on relevant discussions on relative Manin-Mumford. The author would like to thank Ngaiming Mok for relevant discussions on §3.4. The author would like to thank Yves André, Daniel Bertrand, Emmanuel Ullmo, Xinyi Yuan and Shouwu Zhang for their comments on the manuscript. The author would like to thank the Institute for Advanced Studies (NJ, USA) and the Morningside Center of Mathematics (Beijing, China) for their hospitality during the preparation of this work.

2. Convention and Notation

Let $g \geq 0$ be an integer. When $g = 0$, let $\mathfrak{A}_0$ and $\mathfrak{A}_0$ be a point. When $g \geq 1$. For any integer $N \geq 3$, let $\mathfrak{A}_g(N)$ denote the moduli space of principally polarized abelian varieties of dimension $g$ with level-$N$-structure. Then $\mathfrak{A}_g(N)$ is a fine moduli space, and hence admits a universal family $\pi: \mathfrak{A}_g(N) \to \mathfrak{A}_g$. Since level structure is not important for our purpose, we shall omit “(N)” in the rest of the paper.

Let $S$ be an irreducible variety over $\mathbb{C}$ and let $\mathcal{A}/S$ be an abelian scheme of relative dimension $g$. Since level structures are not important in this paper, we will use the following abuse of notation through the whole paper.

(1) We say that a subvariety $\mathcal{B}$ of $\mathcal{A}$ is an abelian subscheme of $\mathcal{A} \to S$ if there exists a finite covering $S' \to S$, inducing a morphism $\rho: \mathcal{A}' = \mathcal{A} \times_S S' \to \mathcal{A}$, such that $\mathcal{B} = \rho(\mathcal{B}')$ where $\mathcal{B}'$ is an abelian subscheme of $\mathcal{A}'/S'$ in the usual sense.\(^{[3]}\)

\(^{[3]}\)Namely $\mathcal{B}'$ is an irreducible subgroup scheme of $\mathcal{A}' \to S'$ which is proper, flat and dominant to $S'$. In particular each fiber of $\mathcal{B}' \to S'$ is an abelian subvariety of the corresponding fiber of $\mathcal{A}' \to S'$. 

We say that $\sigma$ is a section of $\mathcal{A} \to S$ if there exists a finite covering $S' \to S$, inducing a morphism $\rho: \mathcal{A}' = \mathcal{A} \times_S S' \to \mathcal{A}$, such that $\sigma = \rho \circ \sigma'$ where $\sigma': S' \to \mathcal{A}'$ is a section of $\mathcal{A}' \to S'$ in the usual sense.\[^4\] Denote by $\sigma(S) := (\rho \circ \sigma')(S')$.

(3) In (2), call $\sigma$ a torsion section if $\sigma'(s')$ is a torsion point on $\mathcal{A}'$, for each $s' \in S'(\mathbb{C})$; call $\sigma$ a constant section if $\sigma'$ is the composite of $S' \to C' \times S', s' \mapsto (c', s')$, and the inclusion $C' \times S' \subseteq \mathcal{A}'$, where $C' \times S'$ is a constant abelian subscheme of $\mathcal{A}' \to S'$.

The following definition is convenient to study constant sections.

**Definition 2.1.** An abelian scheme $\mathcal{C} \to S$ (of relative dimension $g$) is said to be isotrivial if one of the following condition holds:

1. The fibers $\mathcal{C}_s$ are isomorphic to each other for all $s \in S(\mathbb{C})$,
2. There exists a finite covering $S' \to S$ such that $\mathcal{C} \times_S S'$ is a constant abelian scheme, namely $\mathcal{C} \times_S S' = C \times S'$ for some abelian variety $C$ over $\mathbb{C}$.
3. The image of the modular map $S \to \mathbb{A}_g$ induced by $C \to S$ is a point.

In particular the zero section of any abelian scheme $\mathcal{A} \to S$ is an isotrivial abelian subscheme. If $\dim S = 0$, then $\mathcal{A} \to S$ is isotrivial.

Since the sum of two isotrivial abelian subschemes of $\mathcal{A} \to S$ is isotrivial, we can define the largest isotrivial abelian subscheme of $\mathcal{A} \to S$ which we denote by $\mathcal{C}$. Then any constant section of $\mathcal{A} \to S$ (defined above) has image in $\mathcal{C}$.

We will use the following fact several times in the paper: Assume $S \subseteq \mathbb{A}_g$, then $\mathbb{A}_g|_S$ is isotrivial if and only if $S$ is a point. The “if” part is clearly true. The “only if” part is proven as follows: the homomorphism of fundamental groups $\pi_1(S) \to \pi_1(\mathbb{A}_g)$ has finite image by Deligne’s Theorem of the Fixed Part [Del71, Corollaire 4.1.2], so $S$ is a point because it is an irreducible subvariety of $\mathbb{A}_g$.

3. Universal abelian variety and Betti map

We recall some facts on the universal abelian variety in this section.

3.1. Uniformizing space of $\mathbb{A}_g$. Let $\mathfrak{H}_g^+$ be the Siegel upper half space

$$\{Z = X + \sqrt{-1}Y \in M_{g \times g}(\mathbb{C}) : Z = Z^\top, \ Y > 0\}.$$ 

It is well-known that the uniformization of $\mathbb{A}_g$ in the category of complex varieties is given by

$$\mathfrak{u}_G: \mathfrak{H}_g^+ \to \mathbb{A}_g.$$ 

Let us look closer at this uniformization.

Let $\text{Sp}_{2g}$ be the $\mathbb{Q}$-group

$$\left\{h \in \text{GL}_{2g} : h \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix} h^\top = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}\right\},$$

and let $\text{GSp}_g$ be the image of $\mathbb{G}_m \times \text{Sp}_{2g}$ under the central isogeny $\mathbb{G}_m \times \text{SL}_{2g} \to \text{GL}_{2g}$. Then $\text{GSp}_{2g}(\mathbb{R})^+$ acts on $\mathfrak{H}_g^+$ by the formula

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} Z = (AZ + B)(CZ + D)^{-1}, \ \forall \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{GSp}_{2g}(\mathbb{R})^+ \text{ and } Z \in \mathfrak{H}_g^+.$$ 

It is known that the action of $\text{GSp}_{2g}^{\text{der}}(\mathbb{R}) = \text{Sp}_{2g}(\mathbb{R})$ on $\mathfrak{H}_g^+$ thus defined is transitive, and the uniformization (3.1) is obtained by identifying $(\mathbb{A}_g)^{\text{an}}$ with the quotient space $\Gamma_{\text{GSp}_{2g}}\mathfrak{H}_g^+$ for a suitable congruence group $\Gamma_{\text{Sp}_{2g}}$ of $\text{Sp}_{2g}(\mathbb{Z})$.

\[^4\]In other words $\sigma$ is a multi-section in the usual sense.
3.2. Uniformizing space of $\mathfrak{A}_g$. To obtain the uniformization of $\mathfrak{A}_g$, let us construct the following complex space $X^{+}_{2g,a}$.

(i) As a semi-algebraic space, $X^{+}_{2g,a} = \mathbb{R}^{2g} \times \mathfrak{H}^{+}_g$.

(ii) The complex structure of $X^{+}_{2g,a}$ is the one given by

$$X^{+}_{2g,a} = \mathbb{R}^{g} \times \mathbb{R}^{g} \times \mathfrak{H}^{+}_g, \quad (a, b, Z) \mapsto (a + Zb, Z).$$

The uniformization of $\mathfrak{A}_g$ in the category of complex varieties is then given by

$$u : X^{+}_{2g,a} \to \mathfrak{A}_g.$$

Similar to the discussion on $u_G$, there exists a $\mathbb{Q}$-group which we call $P_{2g,a}$ such that $P_{2g,a}^{\text{der}}(\mathbb{R})$ acts transitively on $X^{+}_{2g,a}$ and $u$ is obtained by identifying $(\mathfrak{A}_g)^{\text{an}}$ with the quotient space $\Gamma \backslash X^{+}_{2g,a}$ for a suitable congruence subgroup $\Gamma = \mathbb{Z}^{2g} \rtimes \Gamma_{\text{Sp}_{2g}}$ of $P_{2g,a}^{\text{der}}(\mathbb{Z})$. Let us briefly explain this.

Use $V_{2g}$ to denote the $\mathbb{Q}$-vector group of dimension $2g$. Then the natural action of $\text{GSp}_{2g}$ on $V_{2g}$ defines a $\mathbb{Q}$-group

$$P_{2g,a} = V_{2g} \rtimes \text{GSp}_{2g}.$$

The action of $P_{2g,a}(\mathbb{R})^+$ on $X^{+}_{2g,a}$ is defined as follows: for any $(v, h) \in P_{2g,a}(\mathbb{R})^+ = V_{2g}(\mathbb{R}) \times \text{GSp}_{2g}(\mathbb{R})^+$ and any $(v', x) \in X^{+}_{2g,a}$, we have

$$(v, h) \cdot (v', x) = (v + hv', hx)$$

where $\text{GSp}_{2g}(\mathbb{R})^+$ acts on $\mathbb{R}^{2g}$ as above (3.2).

The natural projection of complex spaces $\tilde{\pi} : X^{+}_{2g,a} \to \mathfrak{H}^{+}_g$ is equivariant with respect to the natural projection of groups $P_{2g,a} \to \text{GSp}_{2g}$. Hence by abuse of notation we also denote by $\tilde{\pi} : P_{2g,a} \to \text{GSp}_{2g}$.

To summarize notation, we have the following commutative diagram

$$\begin{align*}
\begin{array}{ccc}
X^{+}_{2g,a} & \longrightarrow & \mathfrak{H}^{+}_g \\
\downarrow u & & \downarrow u_G \\
\mathfrak{A}_g & \longrightarrow & \mathfrak{A}_g
\end{array}
\end{align*}$$

where $u$ is from (3.3) and $u_G$ is from (3.1).

3.3. Betti map. We define the Betti map in this section. We will start by defining the universal uniformized Betti map on $X^{+}_{2g,a}$, and then descend it to $\mathfrak{A}_g^{\ast}$, the pullback of $\mathfrak{A}_g/\mathfrak{A}_g$ under $u_G : \mathfrak{H}^{+}_g \to \mathfrak{A}_g$. Note that $\mathfrak{A}_g^{\ast}$ is a family of abelian varieties over $\mathfrak{H}^{+}_g$.

Recall that $X^{+}_{2g,a}$ is defined to be $\mathbb{R}^{2g} \times \mathfrak{H}^{+}_g$ with the complex structure determined by (3.2). The universal uniformized Betti map $\tilde{b}$ is defined to be the natural projection

$$\tilde{b} : X^{+}_{2g,a} \to \mathbb{R}^{2g}.$$

Then $\tilde{b}$ is semi-algebraic. For the complex structure on $X^{+}_{2g,a}$ given by (3.2), it is clear that $\tilde{b}^{-1}(r)$ is complex analytic for each $r \in \mathbb{R}^{2g}$.

Recall that $(\mathfrak{A}_g)^{\text{an}} \simeq \Gamma \backslash X^{+}_{2g,a}$ as complex spaces for a suitable congruence subgroup $\Gamma = \mathbb{Z}^{2g} \rtimes \Gamma_{\text{Sp}_{2g}}$ of $P_{2g,a}^{\text{der}}(\mathbb{Z})$. The family of abelian varieties $\mathfrak{A}_g^{\ast}$ defined as at the beginning of this subsection can be identified with the quotient space $(\mathbb{Z}^{2g} \rtimes \{1\}) \backslash X^{+}_{2g,a}$. Now taking quotient by $\mathbb{Z}^{2g}$ on both sides of (3.6), we obtain the universal Betti map

$$b : \mathfrak{A}_g^{\ast} \to \mathbb{T}^{2g}$$
where $\mathbb{T}^{2g}$ denotes the real torus of dimension $2g$. By the discussion below (3.6), we have the following properties for $\tilde{b}$ and $b$.

(i) Both $\tilde{b}$ and $b$ are real-analytic.

(ii) For each $r \in \mathbb{R}^{2g}$, resp. each $t \in \mathbb{T}^{2g}$, we have that $\tilde{b}^{-1}(r)$, resp. $b^{-1}(t)$, is complex analytic.

(iii) For each $\tau \in \mathcal{H}^+_g$, the restriction $b|_{(\mathcal{A}_{\mathbb{H}^+_g})_\tau}$ is a group isomorphism. (We omit the similar statement for $\tilde{b}_S$).

3.4. Betti map in complex geometry. Before moving on, let us point out the following evidence that the Betti map is useful to study the geometry of $\mathcal{A}_g$. This subsection is independent and does not interact with the rest of the paper. Our main reference is Mok [Mok91]; see also [CGHX18, §2.3].

On $\mathcal{X}_{2g,a} = \mathbb{R}^g \times \mathbb{R}^g \times \mathcal{H}^+_g$ one has the following 2-form $da \wedge db$ under the coordinates given by the left hand side of (3.2), which is clearly $V_{2g}(\mathbb{R}) \times \text{Sp}_{2g}(\mathbb{R})$-invariant. Under the coordinate $(w, Z)$ on the right hand side of (3.2), one can compute by change of coordinates that $da \wedge db$ becomes

$$
\mu := \sqrt{-1} \partial \overline{\partial} \left( (\text{Im} w)^\top (\text{Im} Z)^{-1} (\text{Im} w) \right).
$$

This is exactly the canonical Kähler $(1,1)$-form $\mu$ defined by Mok [Mok91, pp. 374]. It can be checked that $\mu$ is semi-positive (by direct computation), and at each point of $x \in \mathcal{X}_{2g,a}^+$ the kernel of the anti-symmetric pairing

$$
T_x(\mathcal{X}_{2g,a}^+) \times T_x(\mathcal{X}_{2g,a}^+) \to \mathbb{R}
$$

associated with $\mu$ is precisely $T_x(\tilde{b}^{-1}(\tilde{b}(x)))$ (by looking at the real-coordinates).

Recall that every point in the moduli space $\mathfrak{A}_g(\mathbb{C})$ parametrizes an abelian variety with a polarization. Hence on the universal abelian variety $\mathfrak{A}_g$ there exists a canonical relatively ample line bundle $\mathcal{L}_g$ which is trivial along the zero section. Since $\mu$ is $V_{2g}(\mathbb{R}) \times \text{Sp}_{2g}(\mathbb{R})$-invariant, it descends to a Kähler $(1,1)$-form $\omega$ on $\mathfrak{A}_g$. It can be shown that $\omega$ is a representative of $c_1(\mathcal{L}_g)$. By the discussion in the previous paragraph, the “horizontal” directions cut out by $c_1(\mathcal{L}_g)$ are precisely those defined by the Betti map (or the foliation associated with the Betti map if we work on $\mathfrak{A}_g$; note that this foliation is holomorphic by property (ii) of the Betti map). Note that $\mathcal{L}_g$ is a canonical line bundle on $\mathfrak{A}_g$. This indicates that the Betti map is useful to study the geometry of $\mathfrak{A}_g$.

4. BETTI MAP ON ARBITRARY ABELIAN SCHEMES

The goal of this section is to extend the definition of Betti map to an arbitrary abelian scheme. Moreover we choose to work on the original abelian scheme instead of on the pullback to the universal covering.

Let $F$ be an algebraically closed subfield of $\mathbb{C}$. Let $S$ be an irreducible quasi-projective variety over $F$, and let $\pi_S: \mathcal{A} \to S$ be an abelian scheme of relative dimension $g$. There exists a principally polarizable abelian scheme $\mathcal{A}^f/S$ defined over $F$ equipped with an isogeny $\mathcal{A} \to \mathcal{A}^f$, say of degree $D_0$. Then we have the following diagram with each morphism defined over $F$

\[
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\pi} & \mathfrak{A}_g \\
\pi_S \downarrow & & \downarrow \\
S & \xrightarrow{\varphi_S} & \mathfrak{A}_g
\end{array}
\]
By abuse of notation, we will still write this diagram as a “pullback” as taking the isogeny $A \to A'$ is harmless to our discussion.

(4.2)

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi} & \mathfrak{A}_g \\
\downarrow{\pi_S} & & \downarrow{\pi} \\
S & \xrightarrow{\varphi_S} & \mathfrak{A}_g
\end{array}
\]

Let $\Delta_0$ be a simply-connected open subset in $\varphi_S(S^{an})$. Fix a component of $\tilde{\Delta}_0$ of $\mathfrak{u}_G^{-1}(\Delta_0)$ under the uniformization $\mathfrak{u}_G : \tilde{\mathfrak{N}}^+_g \to \mathfrak{A}_g$. The fact that $\Delta_0$ is simply-connected implies that $\mathfrak{u}_G|_{\tilde{\Delta}_0}$ is an isomorphism in the category of complex spaces. Thus the universal Betti map (3.7) induces a map $b_{\Delta_0} : \varphi(A)|_{\Delta_0} \to T^{2g}$ by identifying $\varphi(A)|_{\Delta_0} = \pi^{-1}(\Delta_0)$ with $A_{\mathfrak{p}^+}|_{\Delta_0}$.

Let $\Delta$ be a non-empty connected open subset in $S^{an}$ such that $\varphi_S(\Delta) \subseteq \Delta_0$. Let $\mathcal{A}_\Delta = \pi_S^{-1}(\Delta)$. Define

(4.3)

$$b_\Delta : \mathcal{A}_\Delta \to T^{2g}$$

This will be our setup for the following sections.

Before moving on, let us see another way to define the Betti map. Let $\mathfrak{u}_S : \tilde{S} \to S^{an}$ be the universal covering, and let $\mathcal{A}_{\tilde{S}}$ be the pullback of $A \to S$ under $\mathfrak{u}_S$. Then the modular map $\varphi : A \to \mathfrak{A}_g$ induces a natural morphism $\tilde{\varphi} : \mathcal{A}_{\tilde{S}} \to \mathcal{A}_{\mathfrak{p}^+}$; see (3.7) for notation. Then one can define $b_{\tilde{S}} : \mathcal{A}_{\tilde{S}} \to T^{2g}$ to be $\tilde{\varphi}$ composed with the universal Betti map (3.7). Note that $b_{\tilde{S}}$ is uniquely determined, contrary to $b_\Delta$. Now $b_{\tilde{S}}$ (4.3) can be obtained as follows: Identify $\mathcal{A}_\Delta$ and $\mathcal{A}_{\tilde{S}}$ by identifying $\Delta$ with a component $\tilde{\Delta}$ of $\mathfrak{u}_S^{-1}(\Delta)$, then $b_\Delta$ is $b_{\tilde{S}}$ restricted to $\mathcal{A}_\Delta$.

5. Bi-algebraic system associated with $\mathfrak{A}_g$

The goal of this section is to give some further background knowledge on the universal abelian varieties, which will serve as our main tools to study the Betti map. There are two parts. The first part §5.1-5.2 is to introduce the functional transcendence theorem (called weak Ax-Schanuel), and the second part §5.3-5.4 is Deligne-Pink’s language of mixed Shimura varieties.\footnote{For readers not familiar with the language of Shimura varieties but only want to see how to study the generic rank of the Betti map or the relative Manin-Mumford conjecture via $X^{\text{dlog}}(t)$, it is probably a better idea to skip §5.3-5.4 as these two subsections are complicated and will only be used in §7 and §8 (whose proofs we also suggest to skip at first).}
5.1. Generically special subvarieties of sg type and bi-algebraic subvarieties. The goal of this subsection is to explain the relation between generically special subvarieties of sg type (see Definition 1.5) and bi-algebraic subvarieties of $\mathfrak{A}_g$.

Let us start with defining bi-algebraic subvarieties of $\mathfrak{A}_g$. By [Gao17b, §4] $X_{2g,a}^+$ can be embedded as an open (in the usual topology) semi-algebraic subset of a complex algebraic variety $X_{2g,a}^\vee$.

**Definition 5.1.** (i) A subset $Y$ of $X_{2g,a}^+$ is said to be **irreducible algebraic** if it is a complex analytic irreducible component of $X_{2g,a}^+ \cap W$ for some algebraic subvariety $W$ of $X_{2g,a}^+$.

(ii) An irreducible subvariety $Y$ of $\mathfrak{A}_g$ is said to be **bi-algebraic** if one (and hence any) complex analytic irreducible component $Y$ of $u^{-1}(Y)$ is algebraic.

It is not hard to show that the intersection of two bi-algebraic subvarieties of $\mathfrak{A}_g$ is a finite union of irreducible bi-algebraic subvarieties of $\mathfrak{A}_g$. Hence for any subset $Z$ of $\mathfrak{A}_g$, there exists a smallest bi-algebraic subvariety $\mathfrak{A}_g$ which contains $Z$. We use $Z^{\text{biZar}}$ to denote it. Note that $Z^{\text{biZar}} \supseteq Z^{\text{Zar}}$.

**Remark 5.2.** There is a canonical way to endow $\tilde{X}_g^+$ with an algebraic structure which is compatible with $\tilde{\pi} : X_{2g,a}^+ \rightarrow \tilde{X}_g^+$ and the algebraic structure on $X_{2g,a}^+$ defined above; see [Gao17b, §4]. Then it is clear that for any $F$ bi-algebraic in $\mathfrak{A}_g$, we have that $\pi(F)$ is bi-algebraic in $\mathfrak{A}_g$.

Bi-algebraic subvarieties of $\mathfrak{A}_g$ are closely related to generically special subvarieties of sg type defined in Definition 1.5.

**Proposition 5.3** ([Gao17a, Proposition 3.3]). Let $B$ be an irreducible subvariety of $\mathfrak{A}_g$. Denote by $\mathfrak{A}_g|_B = \pi^{-1}(B)$. Then we have

$$\{\text{generically special subvarieties of sg type of } \mathfrak{A}_g|_B\} = \{\langle \mathfrak{A}_g|_B \cap F : F \text{ irreducible bi-algebraic in } \mathfrak{A}_g \text{ with } B \subseteq \pi(F)\}.$$ 

This proposition has the following immediate corollary.

**Corollary 5.4.** Let $Y$ be an irreducible subvariety of $\mathfrak{A}_g$. Then $\langle Y \rangle_{\text{sg}} = (\mathfrak{A}_g|_{\pi(Y)}) \cap Y^{\text{biZar}}$. In particular $\dim(\langle Y \rangle_{\text{sg}}) = \dim(\langle Y \rangle_{\text{biZar}})$. For simplicity. Proposition 5.3 implies that $\langle \mathfrak{A}_g|_B \rangle \cap Y^{\text{biZar}}$ is a generically special subvariety of sg type of $\mathfrak{A}_g|_B$. Hence $\langle Y \rangle_{\text{sg}} \subseteq \langle \mathfrak{A}_g|_B \rangle \cap Y^{\text{biZar}}$.

For the other inclusion, since $\langle Y \rangle_{\text{sg}}$ is a generically special subvariety of sg type of $\mathfrak{A}_g|_B$, we have by Proposition 5.3 that

$$\langle Y \rangle_{\text{sg}} = (\mathfrak{A}_g|_B) \cap F$$

for some irreducible bi-algebraic subvariety $F$ of $\mathfrak{A}_g$. Then $Y \subseteq \langle Y \rangle_{\text{sg}} \subseteq F$. Hence $Y^{\text{biZar}} \subseteq F$. Thus $\langle \mathfrak{A}_g|_B \rangle \cap Y^{\text{biZar}} \subseteq \langle \mathfrak{A}_g|_B \rangle \cap F = \langle Y \rangle_{\text{sg}}$. \hfill $\square$

5.2. **Weak Ax-Schanuel for $\mathfrak{A}_g$.** One of the most important tools we shall use to study the Betti map is the following weak Ax-Schanuel theorem for $\mathfrak{A}_g$ [Gao18, Theorem 1.1 or Theorem 3.5].

**Theorem 5.5** (Weak Ax-Schanuel for $\mathfrak{A}_g$). Let $\tilde{Z}$ be a complex analytic irreducible subset of $X_{2g,a}^\vee$. Then

$$\dim \tilde{Z}^{\text{Zar}} + \dim(u(\tilde{Z}))^{\text{Zar}} \geq \dim \tilde{Z} + \dim(u(\tilde{Z}))^{\text{biZar}}.$$
where \( \tilde{Z}^{\text{Zar}} \) means the smallest irreducible algebraic subset of \( \mathcal{X}^{+}_{2g,a} \) which contains \( \tilde{Z} \). Moreover the inequality becomes an equality if \( \tilde{Z} \) is a complex analytic irreducible component of \( \tilde{Z}^{\text{Zar}} \cap u^{-1}(u(\tilde{Z}))^{\text{Zar}} \).

In the application we will also use the following pure weak Ax-Schanuel theorem proven by Mok-Pila-Tsimerman [MPT17, Theorem 1.1]. As a statement itself, it is implied by Theorem 5.5, but we state the result separately so that later on the application will be more clear.

**Theorem 5.6 (Weak Ax-Schanuel for \( \mathbb{A}_{g} \)).** Let \( \tilde{Z}_{G} \) be a complex analytic irreducible subset of \( \mathfrak{H}^{+}_{g} \). Then

\[
\dim \tilde{Z}^{\text{Zar}}_{G} + \dim(u_{G}(\tilde{Z}_{G}))^{\text{Zar}} \geq \dim \tilde{Z}_{G} + \dim(u_{G}(\tilde{Z}_{G}))^{\text{bir Zar}},
\]

Moreover the inequality becomes an equality if \( \tilde{Z}_{G} \) is a complex analytic irreducible component of \( \tilde{Z}^{\text{Zar}}_{G} \cap u_{G}^{-1}(u_{G}(\tilde{Z}_{G}))^{\text{Zar}} \).

### 5.3. A quick introduction to Shimura varieties

We gather some notation and facts on Shimura varieties. We will need the knowledge (only) for the proofs in §7 and §8.

Recall that in §3.1, we have associated a reductive \( \mathbb{Q} \)-group \( \text{GSp}_{2g} \) and a complex space \( \mathfrak{H}^{+}_{g} \) to the moduli space \( \mathbb{A}_{g} \). In other words we have associated a pair \( (\text{GSp}_{2g}, \mathfrak{H}^{+}_{g}) \) to \( \mathbb{A}_{g} \), where

- \( \text{GSp}_{2g} \) is a reductive \( \mathbb{Q} \)-group;
- \( \mathfrak{H}^{+}_{g} \) is a complex space on which \( \text{GSp}_{2g}(\mathbb{R})^{+} \) acts transitively;
- As a complex space, \( \mathbb{A}_{g} \) is the quotient of \( \mathfrak{H}^{+}_{g} \) by a congruence subgroup of \( \text{GSp}_{2g}^{\text{der}}(\mathbb{Z}) \).

This pair \( (\text{GSp}_{2g}, \mathfrak{H}^{+}_{g}) \) is a special case of pure Shimura datum, and the third bullet point makes \( \mathbb{A}_{g} \) a pure Shimura variety. In general, a pure Shimura datum is a pair \( (G, \mathcal{X}^{+}_{G}) \) such that \( G \) is a reductive \( \mathbb{Q} \)-group and \( \mathcal{X}^{+}_{G} \) is a complex space on which \( G^{\text{der}}(\mathbb{R})^{+} \) acts transitively (along with some other properties). A pure Shimura variety is a quotient space \( \Gamma_{G} \backslash \mathcal{X}^{+}_{G} \) for some congruence subgroup \( \Gamma_{G} \) of \( G^{\text{der}}(\mathbb{Z}) \).

Given two pure Shimura data \( (H, \mathcal{Y}^{+}_{G}) \) and \( (G, \mathcal{X}^{+}_{G}) \), a map \( f: (H, \mathcal{Y}^{+}_{G}) \to (G, \mathcal{X}^{+}_{G}) \) is called a Shimura morphism if \( f \) is a group homomorphism on the underlying groups and is a holomorphic morphism on the underlying complex spaces, and that \( f(h \cdot \tilde{y}_{G}) = f(h) \cdot f(\tilde{y}_{G}) \) for any \( h \in H^{\text{der}}(\mathbb{R})^{+} \) and any \( \tilde{y}_{G} \in \mathcal{Y}^{+}_{G} \). For a Shimura morphism \( f \) of this form, we say that \( (f(H), f(\mathcal{Y}^{+}_{G})) \) is a Shimura subdatum of \( (G, \mathcal{X}^{+}_{G}) \). Apply the discussion to \( (G, \mathcal{X}^{+}_{G}) = (\text{GSp}_{2g}, \mathfrak{H}^{+}_{g}) \), we get the definition of Shimura subdata of \( (\text{GSp}_{2g}, \mathfrak{H}^{+}_{g}) \). Define the special subvarieties of \( \mathbb{A}_{g} \) to be the subvarieties of the form \( u_{G}(\mathcal{Y}^{+}_{G}) \) where \( \mathcal{Y}^{+}_{G} \) is the underlying space of some Shimura subdaum \( (H, \mathcal{Y}^{+}_{G}) \) of \( (\text{GSp}_{2g}, \mathfrak{H}^{+}_{g}) \).

Next we turn to the universal abelian variety \( \mathbb{A}_{g} \). In §3.2 we have associated with it a \( \mathbb{Q} \)-group \( P_{2g,a} \) and a complex space \( \mathcal{X}^{+}_{2g,a} \). Note that \( P_{2g,a} \) is not a reductive group. Nevertheless we have the following properties for the pair \( (P_{2g,a}, \mathcal{X}^{+}_{2g,a}) \):

- \( P_{2g,a} \) is a \( \mathbb{Q} \)-group, whose unipotent radical is a vector group;
- \( \mathcal{X}^{+}_{2g,a} \) is a complex space on which \( P_{2g,a}(\mathbb{R})^{+} \) acts transitively;
- As a complex space, \( \mathbb{A}_{g} \) is the quotient of \( \mathcal{X}^{+}_{2g,a} \) by a congruence subgroup of \( P_{2g,a}^{\text{der}}(\mathbb{Z}) \).

This makes the pair \( (P_{2g,a}, \mathcal{X}^{+}_{2g,a}) \) a mixed Shimura datum of Kuga type, and the third bullet point makes \( \mathbb{A}_{g} \) a mixed Shimura variety of Kuga type. In general a mixed Shimura datum of Kuga type is a pair \( (P, \mathcal{X}^{+}) \) such that \( P \) is a \( \mathbb{Q} \)-group whose unipotent radical is a vector group, and \( \mathcal{X}^{+} \) is a complex space on which \( P(\mathbb{R})^{+} \) acts transitively (along with some other properties). A mixed Shimura variety of Kuga type is a quotient space \( \Gamma \backslash \mathcal{X}^{+} \) for some congruence subgroup \( \Gamma \) of \( P^{\text{der}}(\mathbb{Z}) \).
It is worth pointing out that any pure Shimura datum is a mixed Shimura datum of Kuga type (such that the unipotent radical of the underlying group being trivial). Given two mixed Shimura data of Kuga type \((Q, \mathcal{Y}^+)\) and \((P, \mathcal{X}^+)\), a map \(f: (Q, \mathcal{Y}^+) \to (P, \mathcal{X}^+)\) is called a Shimura morphism if \(f\) is a group homomorphism on the underlying groups and is a holomorphic morphism on the underlying complex spaces, and that \(f(q \cdot y) = f(q) \cdot f(y)\) for any \(q \in Q(\mathbb{R})^+\) and any \(y \in \mathcal{Y}^+\). For a Shimura morphism \(f\) of this form, we say that \((f(Q), f(\mathcal{Y}^+))\) is a mixed Shimura subdatum of Kuga type of \((P, \mathcal{X}^+)\). Applying the discussion to \((P, \mathcal{X}^+) = (P_{2g,a}, \mathcal{X}_{2g,a}^+)\), we get the definition of mixed Shimura subdatum of Kuga type of \((P_{2g,a}, \mathcal{X}_{2g,a}^+)\). It is known that for each mixed Shimura subdatum of Kuga type \((Q, \mathcal{Y}^+)\) of \((P_{2g,a}, \mathcal{X}_{2g,a}^+)\), the unipotent radical of \(Q\) is \(V_{2g} \cap Q\) by weight reasons; see [Gao17b, Proposition 2.9].

Define the special subvarieties of \(\mathfrak{A}_g\) to be the subvarieties of the form \(\mathfrak{u}(\mathcal{Y}^+)\) where \(\mathcal{Y}^+\) is the underlying space of some mixed Shimura subdatum of Kuga type \((Q, \mathcal{Y}^+)\) of \((P_{2g,a}, \mathcal{X}_{2g,a}^+)\).

Let us end this subsection by the following proposition on the geometric meaning of special subvarieties of \(\mathfrak{A}_g\) and beyond. Recall the notation in and above (3.5) \(\pi: P_{2g,a} \to \text{GSp}_{2g}\) and

\[
\begin{array}{ccc}
\mathcal{X}_{2g,a}^+ & \xrightarrow{\pi} & \mathfrak{h}_g^+ \\
\mathfrak{u} & \dashv & \mathfrak{u}_G \\
\mathfrak{A}_g & \xrightarrow{\pi} & \mathfrak{h}_g
\end{array}
\]

**Proposition 5.7.** Let \(M\) be a special subvariety of \(\mathfrak{A}_g\). Then \(M_G := \pi(M)\) is a special subvariety of \(\mathfrak{h}_g\), and \(M\) is the translate of an abelian subscheme of \(\mathfrak{A}_g|_{M_G} \to M_G\) by a torsion section. Conversely, all special subvarieties of \(\mathfrak{A}_g\) are obtained in this way.

More precisely, if \(M\) is associated with the mixed Shimura subdatum of Kuga type \((Q, \mathcal{Y}^+)\) of \((P_{2g,a}, \mathcal{X}_{2g,a}^+)\) (namely \(M = \mathfrak{u}(\mathcal{Y}^+)\)), then the relative dimension of \(M \to M_G\) is \(g_Q := \frac{1}{2} \dim(V_{2g} \cap Q)\).

### 5.4. Quotient by a normal group.

The discussion in this subsection will (only) be used in the proofs in §7 and §8.

The setting is as follows. Let \((Q, \mathcal{Y}^+)\) be a mixed Shimura subdatum of \((P_{2g,a}, \mathcal{X}_{2g,a}^+)\), and let \(M = \mathfrak{u}(\mathcal{Y}^+)\) be the associated special subvariety of \(\mathfrak{A}_g\). Take a normal subgroup \(N\) of \(Q\). Pink [Pin89, 2.9] constructed the quotient mixed Shimura datum \((Q, \mathcal{Y}^+)/N\) whose underlying group is \(Q/N\). We explain the geometric meaning of taking the quotient in this subsection.

By Proposition 5.7, \(M_G := \pi(M)\) is a special subvariety of \(\mathfrak{h}_g\). Let \((G_Q, \mathcal{Y}_{G_Q}^+)\) be the pure Shimura subdatum of \((\text{GSp}_{2g}, \mathfrak{h}_g^+)\) in Proposition 5.7, then \(M_G = \mathfrak{u}_G(\mathcal{Y}_{G_Q}^+)\).

Denote by \(V_Q = V_{2g} \cap Q\) and \(V_N = V_{2g} \cap N\). We explained above (5.1) that \(V_Q\) is the unipotent radical of \(Q\). Since \(N \triangleleft Q\), it is clear by group theory that \(V_N\) is the unipotent radical of \(N\). Denote by \(G_N = N/V_N\), then \(G_N\) is a normal subgroup of \(G_Q\).

Proposition 5.7 says that \(\pi|_M: M \to M_G\) itself is an abelian scheme of relative dimension \(g_Q := \frac{1}{2} \dim(V_{2g} \cap Q)\). Let \(\varepsilon: M_G \to M\) be the zero section.\(^6\) It induces a Levi decomposition \(Q = V_Q \rtimes G_Q\) and a semi-algebraic isomorphism

\[
\mathcal{Y}^+ \simeq V_Q(\mathbb{R}) \times \mathcal{Y}_{G_Q}^+
\]

such that \(\mathfrak{u}(\{0\} \times \mathcal{Y}_{G_Q}^+) = \varepsilon(M_G)\). In the rest of this subsection, we shall use this identification of \(\mathcal{Y}^+\) with \(V_Q(\mathbb{R}) \times \mathcal{Y}_{G_Q}^+\).

\(^6\)It is a torsion section of \(\mathfrak{A}_g|_{M_G} \to M_G\).
Deligne proved that $\mathcal{Y}^+ \to \mathcal{Y}^+_{G_Q}$ is a variation of Hodge structure of type $(-1,0)+(0,-1)$; see [Del71, Rappel 4.4.3].

We have $V_N \subset Q$ since $N \subset Q$ and $V_N$ is the unipotent radical of $N$. The reductive group $G_Q$, as a subgroup of $GSp_{2g}$, acts on $V_{2g}$. Now $V_N \subset Q$ implies that $V_N$ is a $G_Q$-submodule of $V_{2g}$, so $V_N$ is a sub-Hodge structure of $V_Q$. Thus $V_N(\mathbb{R}) \times \mathcal{Y}^+_{G_Q} \to \mathcal{Y}^+_{G_Q}$ is a sub-variation of Hodge structure of $\mathcal{Y}^+ \to \mathcal{Y}^+_{G_Q}$. Thus $u(V_N(\mathbb{R}) \times \mathcal{Y}^+_{G_Q})$ is an abelian subscheme of $M \to M_G$ by [Del71, Rappel 4.4.3].

The quotient $\tilde{p}_N: (Q, \mathcal{Y}^+)/(Q, \mathcal{Y}^+)/N$ can be constructed in two steps. First we take the quotient $(Q_0, \mathcal{Y}^+_0):=(Q, \mathcal{Y}^+)/V_N$, and then we take $(Q', \mathcal{Y}^+'):=(Q_0, \mathcal{Y}^+_0)/G_N$.

The geometric meaning of the quotient $\tilde{p}_0: (Q, \mathcal{Y}^+) \to (Q_0, \mathcal{Y}^+_0)=(Q, \mathcal{Y}^+)/V_N$ is as follows. We have the following commutative diagram (where $\Gamma_0 = \tilde{p}_0(\Gamma \cap Q(\mathbb{R}))$

$$
\begin{array}{ccc}
\mathcal{Y}^+ & \xrightarrow{\tilde{p}_0} & \mathcal{Y}^+_{0} \\
\downarrow{u|_{\mathcal{Y}^+}} & & \downarrow \\
M & \xrightarrow{p_0} & M_0 := \Gamma_0 \backslash \mathcal{Y}^+_{0} \\
\downarrow{\pi|_{M_G}} & & \downarrow{\pi_0} \\
M_G & \xrightarrow{id_{M_G}} & M_G
\end{array}
$$

such that $M_0 \to M_G$ is the abelian scheme obtained by taking the quotient of $M \to M_G$ by $u(V_N(\mathbb{R}) \times \mathcal{Y}^+_{G_Q})$. Denote by $g_N = \frac{1}{2}\dim V_N$, then it is clear that the relative dimension of $M_0 \to M_G$ is $g_Q - g_N$.

To explain the quotient $\tilde{p}': (Q_0, \mathcal{Y}^+_0) \to (Q', \mathcal{Y}^+'),=(Q_0, \mathcal{Y}^+_0)/G_N$, we need the following preliminary. We know that $G_N$ is a normal subgroup of $G_Q$. We take for granted that we can do the operation $(G_Q, \mathcal{Y}^+_{G_Q})/G_N$ for pure Shimura datum, and this quotient gives a morphism $p_{G_N}: M_G \to M_G'$.[7]

For each $y'_G \in M'_G$, the inverse image $(p_{G_N})^{-1}(y'_G)$ equals $u_G(G_N(\mathbb{R})^+ \tilde{y}_G)$ for some $\tilde{y}_G \in \mathcal{Y}^+_{G_Q}$.

So the connected algebraic monodromy group of $(p_{G_N})^{-1}(y'_G)$ is $G_N^{\text{der}}$. Since $N \subset Q$, we have $G_N = N/V_N \subset Q/V_N$. Hence $G_N$ acts trivially on $V_Q/V_N$. So $M_0|_{(p_{G_N})^{-1}(y'_G)} \to (p_{G_N})^{-1}(y'_G)$ is an isotrivial abelian scheme by Deligne’s Theorem of the Fixed Part [Del71, Corollaire 4.1.2].

We have the following commutative diagram

$$
\begin{array}{ccc}
\mathcal{Y}^+_0 & \xrightarrow{\tilde{p}'} & \mathcal{Y}^+_0' \\
\downarrow{\pi|_{M_G}} & & \downarrow{\pi'} \\
M_0 = \Gamma_0 \backslash \mathcal{Y}^+_0 & \xrightarrow{p_{G_N}} & M'_0
\end{array}
$$

where $\pi'$ is an abelian scheme (of relative dimension $g_Q - g_N$) such that each closed fiber of $M_0|_{(p_{G_N})^{-1}(y'_G)} \to (p_{G_N})^{-1}(y'_G)$ is the abelian variety $(\pi')^{-1}(y'_G)$. In other words, the lower box

[7]See [Pin89, 2.9] or [UY11, Definition 2.1]. In this paper we mostly only need the notion, so we choose not go into more details on this in the preliminary part.
is an intermediate step of the modular map

\[
\begin{array}{c}
M_0 \rightarrow \mathfrak{A}_{gQ-gN} \\
\pi_0 \downarrow \downarrow \downarrow \\
M_G \rightarrow \mathfrak{A}_{gQ-gN}
\end{array}
\]

We end this subsection by summarizing the quotient \(\tilde{p}_N: (Q, \mathcal{Y}^+) \rightarrow (Q, \mathcal{Y}^+)/N\) (after taking the uniformizations) in the following commutative diagram:

\[
(5.4)
\begin{array}{c}
M \xrightarrow{\pi|_M} M_0 := \Gamma_0 \backslash \mathcal{Y}_0^+ \xrightarrow{p'} M' \xrightarrow{\pi|_{M'}} \mathfrak{A}_{gQ-gN} \\
\pi|_M \downarrow \downarrow \downarrow \downarrow \\
M_G \xrightarrow{id_{M_G}} M_G \xrightarrow{p_G \pi} M'_G \rightarrow \mathfrak{A}_{gQ-gN}
\end{array}
\]

where each vertical arrow is an abelian scheme, with the left one of relative dimension \(g_Q\) and the other three of relative dimension \(g_Q - g_N\). Moreover \(M_0|_{p_G^{-1}(b')}\) is isotrivial for any \(b' \in M'\) by the discussion above (5.3). And for any \(x \in M'\), we have that \(p_N^{-1}(x)\) is the translate of an abelian subscheme of \(M|_{\pi(p_N^{-1}(x))} \rightarrow \pi(p_N^{-1}(x))\) of relative dimension \(g_Q - (g_Q - g_N) = g_N\) by a constant section.

6. FROM BETTI MAP TO THE \(l\)-TH DEGENERACY LOCUS

In this section, we work with the following setup. Recall the diagram (3.5)

\[
(6.1)
\begin{array}{c}
X_{2g,a}^+ \xrightarrow{\tilde{\pi}} \mathcal{Y}_g^+ \\
\uparrow \uparrow \uparrow \uparrow \uparrow \\
\mathfrak{A}_g \xrightarrow{\pi} \mathfrak{a}_g
\end{array}
\]

and the uniformized universal Betti map \(\tilde{b}: X_{2g,a}^+ \rightarrow \mathbb{R}^{2g}\) defined in (3.6).

Let \(B\) be a locally closed irreducible subvariety of \(\mathfrak{a}_g\), and let \(X\) be a closed irreducible subvariety of \(\mathfrak{A}_g\) with \(\pi(X) = B\). Fix a complex analytic irreducible component \(\tilde{X}\) of \(u^{-1}(X)\). The goal of this section is to prove the following theorem, which transfers the study of the generic rank of the Betti map to the \(t\)-th degenerate locus of \(X\) for some particular \(t\)'s.

**Proposition 6.1.** Denote for simplicity by \(d = \dim X\). For any integer \(l \in \{1, \ldots, d\}\), let

\[
\tilde{X}_{<2l} = \{ \tilde{x} \in \tilde{X} : \text{rank}_\mathbb{R}(\tilde{b}(\tilde{x})) \neq 2l, \ u(\tilde{x}) \in X^\text{sm}(\mathbb{C}) \}.
\]

Then \(X^\text{deg}(l-d) \cap X^\text{sm}(\mathbb{C}) \subseteq u(\tilde{X}_{<2l}) \subseteq X^\text{deg}(l-d)\), where \(X^\text{deg}(l-d)\) is defined in Definition 1.6.

In particular if \(\text{rank}_\mathbb{R}(\tilde{b}(\tilde{x})) \neq 2l\) for all \(\tilde{x} \in \tilde{X}\) with \(u(\tilde{x}) \in X^\text{sm}(\mathbb{C})\), then \(X^\text{deg}(l-d)\) is Zariski dense in \(X\).

**Proof.** Note that the “In particular” parts of the theorems follow from the main statements easily by letting \(x\) run over all points in \(X^\text{sm}(\mathbb{C})\).

Let us prove \(X^\text{deg}(l-d) \cap X^\text{sm}(\mathbb{C}) \subseteq u(\tilde{X}_{<2l})\). Suppose not, then there exists some \(\tilde{x} \in \tilde{X}\) such that \(u(\tilde{x}) \in X^\text{deg} \cap X^\text{sm}(\mathbb{C})\) and \(\text{rank}_\mathbb{R}(\tilde{b}(\tilde{x})) = 2l\). By definition of \(X^\text{deg}(l-d)\), the point \(u(\tilde{x})\) lies in a subvariety \(Y\) of \(X\) such that \(\dim(Y)_{sg} - \dim \pi(Y) < \dim Y + (l-d)\), where \((Y)_{sg}\) is defined above Definition 1.6.
Let $\tilde{Y}$ be a complex analytic irreducible component of $u^{-1}(Y)$ with $\tilde{x} \in \tilde{Y} \subseteq \tilde{X}$, and let $(\tilde{Y})_{sg}$ be a complex analytic irreducible component of $u^{-1}((Y)_{sg})$ which contains $\tilde{Y}$. Then \[\text{rank}_{g}(b|_{\tilde{X}}) \leq \text{rank}_{R}(b|_{\tilde{Y}}) + 2(\dim X - \dim Y).\] Since \[\text{rank}_{R}(b|_{\tilde{Y}}) = 2l\] by choice of $\tilde{x}$, we have \[\text{rank}_{g}(b|_{\tilde{X}}) \geq 2(l - d + \dim Y).\] However by definition of $(Y)_{sg}$ and $b$, we have \[\text{rank}_{R}(b|_{(Y)_{sg}}) = 2(\dim Y_{sg} - \dim \pi(Y))\] everywhere on $(Y)_{sg}$. Since $\tilde{Y} \subseteq (Y)_{sg}$, we then have \[2(l - d + \dim Y) \leq 2(\dim Y_{sg} - \dim \pi(Y)).\] But then \[\dim Y_{sg} - \dim \pi(Y) \geq \dim Y + (l - d),\] which contradicts our choice of $Y$. Thus we get a contradiction.

Next we prove the other inclusion $u(X_{<2l}) \subseteq X_{\deg(t)}$. Take any $\tilde{x} \in \tilde{X}_{<2l}$. Denote by $r = b(\tilde{x}) \in \mathbb{R}^{2g}$. By assumption of the proposition, we have \[\text{dim}_{g}(\tilde{b}^{-1}(r) \cap \tilde{X}) > 2(d - l).\]

In the rest of the proof, we identify $X_{2g,a}^{+}$ as the semi-algebraic space $\mathbb{R}^{2g} \times \tilde{\mathcal{S}}_{g}^{+}$ with the complex structure defined by (3.2). In particular $\tilde{b}^{-1}(r) = \{r\} \times \tilde{\mathcal{S}}_{g}^{+}$. Property (ii) of the Betti map (see below (3.7)) implies that $\tilde{b}^{-1}(r) \cap \tilde{X}$ is complex analytic. Then by (6.2), there exists a complex analytic irreducible subset $\tilde{W}$ in $\tilde{\mathcal{S}}_{g}^{+}$ of dimension $\geq d - l + 1$ such that $\tilde{x} \in \{r\} \times \tilde{W} \subseteq \tilde{X}$.

Hence it suffices to prove the following assertion: \[Y = \left(u(\{r\} \times \tilde{W})\right)_{Zar}\] satisfies \[\dim(Y)_{sg} - \dim \pi(Y) < \dim Y + (l - d).\]

Apply weak Ax-Schanuel for $\mathfrak{A}_{g}$, namely Theorem 5.5, to $\{r\} \times \tilde{W}$. Then we get \[\dim(\{r\} \times \tilde{W})_{Zar} + \dim Y \geq \dim(\{r\} \times \tilde{W}) + \dim Y_{\text{biZar}}.\]

Apply weak Ax-Schanuel for $\mathfrak{A}_{g}$, namely Theorem 5.6, to $\tilde{C}$. Then we get \[\dim \tilde{W}_{Zar} + \dim(\mathfrak{u}_{G}(\tilde{W}))_{Zar} = \dim \tilde{W}' + \dim(\mathfrak{u}_{G}(\tilde{W}))_{\text{biZar}}\]

where $\tilde{W}'$ is the complex analytic irreducible component of $\tilde{W}_{Zar} \cap \mathfrak{u}_{G}(\tilde{W})_{Zar}$ which contains $\tilde{W}$. Since $\tilde{\pi}(\{r\} \times \tilde{W}) = \tilde{W}$, we have $\mathfrak{u}_{G}(\tilde{W}) = \pi(u(\{r\} \times \tilde{W}))$. Hence $(\mathfrak{u}_{G}(\tilde{W}))_{Zar} = \pi(Y)$. Moreover the bi-algebraic systems associated with $\mathfrak{A}_{g}$ and $\mathfrak{A}_{g}$ are compatible with (6.1) (see Remark 5.2), so $(\mathfrak{u}_{G}(\tilde{W}))_{\text{biZar}} = \pi(Y)_{\text{biZar}}$. Hence (6.5) becomes \[\dim \tilde{W}_{Zar} + \dim \pi(Y) = \dim \tilde{W}' + \dim \pi(Y)_{\text{biZar}}.\]

Let us temporarily assume $(\{r\} \times \tilde{W})_{Zar} = \{r\} \times \tilde{W}_{Zar}$ and finish the proof. Then by taking the difference (6.4) and (6.6), we obtain \[\dim Y - \dim \pi(Y) \geq (\dim Y_{\text{biZar}} - \dim \pi(Y)_{\text{biZar}}) + \dim(\{r\} \times \tilde{W}) - \dim \tilde{W}' > (\dim Y_{\text{biZar}} - \dim \pi(Y)_{\text{biZar}}) + (d - l) - \dim \pi(Y) \]
\[= \left((\dim Y_{\text{biZar}} - \dim \pi(Y)_{\text{biZar}}) + \dim \pi(Y)\right) + (d - l) - 2\dim \pi(Y) = \dim(Y)_{sg} + (d - l) - 2\dim \pi(Y)\]
by Corollary 5.4.

Here the second inequality holds because $\dim \tilde{W} \geq (d - l) + 1$ and $\mathfrak{u}_{G}(\tilde{W}') \subseteq (\mathfrak{u}_{G}(\tilde{W}))_{Zar} = \pi(Y)$. From this we obtain (6.3).

It remains to prove $(\{r\} \times \tilde{W})_{Zar} = \{r\} \times \tilde{W}_{Zar}$. First note that $\{r\} \times \tilde{W}_{Zar}$ is semi-algebraic and complex analytic. Then it is a general fact that $\{r\} \times \tilde{W}_{Zar}$ is algebraic in $X_{2g,a}^{+}$; see [KUY16, (the proof of) Lemma B.1], which moreover is easily seen to be irreducible. Thus
Lemma 7.3. are compatible under \( \tilde{\pi} \). Remark 5.2 says that the algebraic structures on \( \mathcal{X}_{2g,a}^+ \) and \( \Omega_g^+ \) are compatible under \( \tilde{\pi} \), so \( \tilde{\pi}|_{\{r\} \times \tilde{\mathcal{W}}^{\text{Zar}}} : (\{r\} \times \tilde{\mathcal{W}}^{\text{Zar}} \to \tilde{\mathcal{W}}^{\text{Zar}} \) is dominant. Now we can conclude. \( \Box \)

7. Zariski closeness of the \( t \)-th degeneracy locus

Let \( B \) be a locally closed irreducible subvariety of \( \mathbb{A}_g \), and let \( X \) be a closed irreducible subvariety of \( \mathbb{A}_g|_B = \pi^{-1}(B) \) with \( \pi(X) = B \).

Through the whole section we fix a \( t \in \mathbb{Z} \). Let \( X^{\text{deg}}(t) \) be the \( t \)-th degeneracy locus of \( X \) defined in Definition 1.6. The goal of this section is to prove the following theorems.

**Theorem 7.1.** The subset \( X^{\text{deg}}(t) \) is Zariski closed in \( X \).

Our proof is similar to Daw-Ren’s work [DR18, §7] on the anomalous subvarieties in a pure Shimura variety.

7.1. Weakly optimal subvariety. To prove Theorem 7.1, we need the following definition of weakly optimal subvarieties. The notion was first introduced by Habegger-Pila [HP16] to study the Zilber-Pink conjecture for abelian varieties and product of modular curves.

**Definition 7.2.**

(i) For any irreducible subvariety \( Z \) of \( \mathbb{A}_g \), define the weakly defect to be \( \delta_{\text{ws}}(Z) = \dim Z^{\text{biZar}} - \dim Z \).

(ii) A closed irreducible subvariety \( Z \) of \( X \) is said to be weakly optimal if the following condition holds: \( Z \subsetneq Z' \subseteq X \) with \( Z' \) irreducible closed in \( X \Rightarrow \delta_{\text{ws}}(Z') > \delta_{\text{ws}}(Z) \).

Weakly optimal subvarieties of \( X \) are closely related to \( X^{\text{deg}}(t) \) by the following lemmas.

**Lemma 7.3.** \( \dim(Z)_{\text{sg}} - \dim \pi(Z) < \dim Z + t \Leftrightarrow \delta_{\text{ws}}(Z) < \dim \pi(Z)^{\text{biZar}} + t \).

**Proof.** The condition \( \dim(Z)_{\text{sg}} - \dim \pi(Z) < \dim Z + t \) can be rewritten to be \( ((\dim(Z)_{\text{sg}} - \dim \pi(Z)) + \dim \pi(Z)^{\text{biZar}}) - \dim Z < \dim \pi(Z)^{\text{biZar}} + t \).

By Corollary 5.4, we have \( \dim Z^{\text{biZar}} = (\dim(Z)_{\text{sg}} - \dim \pi(Z)) + \dim \pi(Z)^{\text{biZar}} \). Hence the inequality above becomes \( \delta_{\text{ws}}(Z) < \dim \pi(Z)^{\text{biZar}} + t \). \( \Box \)

**Lemma 7.4.** Assume a closed irreducible subvariety \( Z \subseteq X \) satisfies \( \text{codim}(Z)_{\text{sg}}(Z) < \dim \pi(Z) + t \) and is maximal for this property. Then \( Z \) is weakly optimal.

**Proof.** For any \( Z \subseteq Z' \subseteq X \) with \( Z' \) irreducible closed in \( X \), if \( \delta_{\text{ws}}(Z') \leq \delta_{\text{ws}}(Z) \), then we have \( \delta_{\text{ws}}(Z') \leq \delta_{\text{ws}}(Z) < \dim \pi(Z)^{\text{biZar}} + t \leq \dim \pi(Z')^{\text{biZar}} + t \) where the second inequality follows from Lemma 7.3 (applied to \( Z \)). Applying Lemma 7.3 to \( Z' \), we get \( \text{codim}(Z)_{\text{sg}}(Z') < \dim \pi(Z') + t \). The maximality of \( Z \) then implies \( Z = Z' \). So \( Z \) is weakly optimal. \( \Box \)

We end this subsection by stating the following finiteness theorem concerning weakly optimal subvarieties [Gao18, Theorem 1.4]. See §5.3 for notation.

**Theorem 7.5.** There exists a finite set \( \Sigma \) consisting of elements of the form \( ((Q, \mathcal{Y}^+), N) \), where \( (Q, \mathcal{Y}^+) \) is a mixed Shimura subdatum of Kuga type of \( (P_{2g,a}, \mathcal{X}_{2g,a}^+) \) and \( N \) is a connected normal subgroup of \( Q \) whose reductive part is semi-simple, such that the following property holds. If a closed irreducible subvariety \( Z \) of \( X \) is weakly optimal, then there exists \( ((Q, \mathcal{Y}^+), N) \in \Sigma \) such that \( Z^{\text{biZar}} = u(N(\mathbb{R})^+ \hat{y}) \) for some \( \hat{y} \in \mathcal{Y}^+ \).
7.2. An auxiliary proposition. Let $X$ be as in Theorem 7.1. Let $\Sigma$ be the finite set in Theorem 7.5. Take a positive dimensional closed irreducible subvariety $Z$ of $X$ such that $\text{codim}_Z(Z)_{sg}(Z) < \dim \pi(Z) + t$ and is maximal for this property. Then $Z$ is weakly optimal by Lemma 7.4, and hence Theorem 7.5 gives $((Q, \mathcal{Y}^+), N) \in \Sigma$ such that $\mathcal{Z}^{\text{biZar}} = \mathbf{u}(N(\mathbb{R})^+\tilde{y})$ for some $\tilde{y} \in \mathcal{Y}^+$. Recall that $N$ is a connected normal subgroup of $Q$ whose reductive part is semi-simple. Note that $N \neq 1$ since $Z$ has positive dimension.

Let us consider the operation of taking quotient mixed Shimura datum $\tilde{p}_N : (Q, \mathcal{Y}^+) \to (Q, \mathcal{Y}^+)/N =: (Q', \mathcal{Y}'^+)$ discussed in §5.4 and the induced morphism on the corresponding mixed Shimura varieties of Kuga type

\begin{equation}
\mathcal{Y}^+ \xrightarrow{\tilde{p}_N} \mathcal{Y}'^+ \xrightarrow{u|_{\mathcal{Y}^+}} M \xrightarrow{p_N} M'
\end{equation}

By construction we know that $\mathcal{Z}^{\text{biZar}}$ is a fiber of $p_N$. We refer to (5.4) for the geometric meaning of $p_N : M \to M'$ and notation.

Let $X'_M$ be an irreducible component of $X \cap M$. For any integer $h$, define $E_h = \{ x \in X : (p_N|_{X_M})^{-1}(p_N(x)) \text{ has an irreducible component of dimension} \geq h \text{ that contains} x \}$. Then $E_h$ is Zariski closed in $X$, and is proper if $h > \dim X_M - \dim p_N(X_M)$.

**Proposition 7.6.** Let $h = g_N + 1 - t$, where $g_N = \frac{1}{2} \dim(V_{2g} \cap N)$ as in (5.4). Then

(i) We have $Z \subseteq E_h$.

(ii) We have $E_h \subseteq X^{\text{deg}(t)}$.

**Proof.**

(i) Recall that $Z$ satisfies $\dim(Z)_{sg} - \dim \pi(Z) < \dim Z + t$. Therefore $\delta_{\text{ws}}(Z) < \dim \pi(Z)^{\text{biZar}} + t$ by Lemma 7.3. Hence

$$\dim Z > \dim Z^{\text{biZar}} - \dim \pi(Z)^{\text{biZar}} - t.$$  

By construction we know that $Z^{\text{biZar}}$ is a fiber of $p_N$. Hence as a morphism, $Z^{\text{biZar}} \to \pi(Z)^{\text{biZar}}$ is an abelian scheme of relative dimension $g_N$ by the discussion below (5.4). So the inequality above becomes $\dim Z > g_N - t = h - 1$. Thus $\dim Z \geq h$. Hence $Z \subseteq E_h$ by definition of $E_h$.

(ii) For any $x \in E_h$ and any irreducible component $Y$ of $(p_N|_{X_M})^{-1}(p_N(x))$ of dimension $\geq h$ containing $x$, we have by definition of $E_h$

$$\dim Y \geq h.$$  

Since $(p_N|_{X_M})(Y) = x$ and $Y \subseteq X_M \subseteq M$, we know that $Y \subseteq p_N^{-1}(x)$. Note that $p_N^{-1}(x) \to \pi(p_N^{-1}(x))$ is an abelian scheme of relative dimension $g_N = h - 1 + t$ as a morphism by (5.4) and the discussion below.

Proposition 5.3 says that as a morphism, $Y^{\text{biZar}} \to \pi(Y)^{\text{biZar}}$ is an abelian scheme. Since $Y \subseteq p_N^{-1}(x)$ and $p_N^{-1}(x)$ is bi-algebraic, we have $Y^{\text{biZar}} \subseteq p_N^{-1}(x)$. Thus the relative dimension of the abelian scheme $Y^{\text{biZar}} \to \pi(Y)^{\text{biZar}}$ is at most $h - 1 + t$ (which is the relative dimension of $p_N^{-1}(x) \to \pi(p_N^{-1}(x)))$. Since $\dim Y \geq h$, we have

$$\dim Y \geq (h - 1 + t) + 1 - t \geq \dim Y^{\text{biZar}} - \dim \pi(Y)^{\text{biZar}} + 1 - t.$$  

Hence $\delta_{\text{ws}}(Y) < \dim \pi(Y)^{\text{biZar}} + t$. Thus $\dim(Y)^{\text{reg}} - \dim \pi(Y) < \dim Y + t$ by Lemma 7.3, and hence $Y \subseteq X^{\text{deg}(t)}$ by definition. By varying $x \in E_h$, we get the conclusion. ☐
7.3. **Proof of Theorem 7.1.** Now we are ready to prove Theorem 7.1. Let $X$ be as in Theorem 7.1. Let $\Sigma$ be the finite set in Theorem 7.5.

For any positive dimensional closed irreducible subvariety $Z$ of $X$ such that $\text{codim}_{X}(Z) \leq \dim \pi(Z) + t$ and is maximal for this property, we obtain some $((Q, Y^{+}), N) \in \Sigma$ from which we can construct a Zariski closed subset $E_h$ of $X$ such that $Z \subseteq E_h$ by (i) of Proposition 7.6. Since $\Sigma$ is a finite set, we have finitely many such $E_h$‘s. By definition of $X_{\text{deg}}(t)$, we then have that $X_{\text{deg}}(t)$ is contained in the union of these $E_h$‘s, which is a Zariski closed subset of $X$.

Conversely by (ii) of Proposition 7.6, each such $E_h$ is contained in $X_{\text{deg}}(t)$.

Hence $X_{\text{deg}}(t)$ is the union of these $E_h$‘s. This is a finite union with each member being a closed subset of $X$. Hence $X_{\text{deg}}(t)$ is Zariski closed in $X$.

8. **Criterion of degenerate subvarieties**

Let $B$ be a locally closed irreducible subvariety of $A_g$, and let $X$ be a closed irreducible subvariety of $\mathfrak{A}_g|_B = \pi^{-1}(B)$ with $\pi(X) = B$.

Through the whole section we fix a $t \in \mathbb{Z}$. Let $X_{\text{deg}}(t)$ be the $t$-th degeneracy locus of $X$ defined in Definition 1.6. The goal of this section is to prove the following criterion for $X = X_{\text{deg}}(t)$.

For notation let $A_X$ be the translate of an abelian subscheme of $\mathfrak{A}_g|_B \to B$ by a torsion section which contains $X$, minimal for this property. Then $A_X \to B$ itself is an abelian scheme (up to taking a finite covering of $B$), whose relative dimension we denote by $g_B$.

For any abelian subscheme $\mathcal{B}$ of $A_X \to B$ whose relative dimension we denote by $g_{\mathcal{B}}$, we obtain the following diagram

$$
\begin{array}{c}
A_X \xrightarrow{p_B} A_X/B \xrightarrow{i/B} \mathfrak{A}_{g_X-g_B}, \\
\pi|_{A_X} \downarrow \quad \quad \quad \pi|_B \downarrow \quad \quad \quad \downarrow J \\
B \xrightarrow{id_B} B \xrightarrow{i/B,G} \mathfrak{A}_{g_X-g_B}
\end{array}
$$

where $p_B$ is taking the quotient abelian scheme, and the right box is the modular map.

**Theorem 8.1.** Assume either $t \leq 0$, or $t = 1$ and $A_X = \mathfrak{A}_g|_B$. Then $X = X_{\text{deg}}(t)$ if and only if the following condition holds: There exists an abelian subscheme $\mathcal{B}$ of $A_X \to B$ (whose relative dimension we denote by $g_{\mathcal{B}}$) such that for the map $i/B \circ p_B$ constructed above, we have $\dim(i/B \circ p_B)(X) < \dim X - g_B + t$ and that $i/B \circ p_B$ is not generically finite.

Note that when $t \leq 0$, the condition $i/B \circ p_B$ is not generically finite is redundant. This is because in this case, $\dim(i/B \circ p_B)(X) < \dim X - g_B + t$ implies $\dim(i/B \circ p_B)(X) < \dim X$, and hence $i/B \circ p_B$ is not generically finite.

8.1. **Auxiliary proposition.** Let $M$ be the smallest special subvariety of $\mathfrak{A}_g$ which contains $X$. Assume that $M$ is associated with the mixed Shimura subdatum of Kuga type $(Q, Y^+)$ of $(P_{2g,a}, X_{2g,a})$.

We start by proving the following auxiliary proposition.

**Proposition 8.2.** We have $X = X_{\text{deg}}(t)$ if and only if there exist

- a special subvariety $M_*$ of $\mathfrak{A}_g$, associated with $(Q_*, Y^{+}_*)$, which contains $X$;
- a non-trivial connected normal subgroup $N_*$ of $Q_*$ whose reductive part is semi-simple;
such that the following condition holds: For the operation of taking quotient mixed Shimura datum \( \tilde{\mathcal{N}}_*: (Q, \mathcal{Y}^+) \rightarrow (Q, \mathcal{Y}_r^+)/N_* \) discussed in \( \S 5.4 \) and the induced morphism on the corresponding mixed Shimura varieties of Kuga type

\[
(8.2) \quad \mathcal{Y}^+ \xrightarrow{\text{dim}} \mathcal{Y}_r^+.
\]

we have \( \dim X - \dim p_{N_*}(X) > g_{N_*} - t \), where \( g_{N_*} = \frac{1}{2} \dim(V_{2g} \cap N_*) \).

**Proof.** We prove \( \Leftarrow \). Let \( h_* = g_{N_*} + 1 - t \) and \( X_{M_*} = X \cap M_* \). Then one can prove directly

\[
X = \{ x \in X : (p_{N_*}|_{X_{M_*}})^{-1}(p_{N_*}(x)) \text{ has an irreducible component of dimension } \geq h_* \text{ that contains } x \}.
\]

Note that (7.1) and (8.4) have the same shape. Hence by (ii) of Proposition 7.6, we have \( X = X^{\text{deg}}(t) \).

Now let us prove \( \Rightarrow \). By the proof of Theorem 7.1 (\( \S 7.3 \)), we have that \( X^{\text{deg}}(t) \) is a finite union of some Zariski closed subsets \( E_{h_*} \)'s of \( X \). Since \( X = X^{\text{deg}}(t) \), we have \( X = E_{h_*} \) for some \( E_{h_*} \).

This subset \( E_{h_*} \) is defined in the following way (see above Proposition 7.6). For some mixed Shimura subdatum of Kuga type \((Q, \mathcal{Y}^+)_*\) of \((P_{2g, a}, \mathcal{X}^+_{2g, a})\) and a non-trivial connected normal subgroup \( N_* \) of \( Q_* \) whose reductive part is semi-simple, Set \( g_{N_*} = \frac{1}{2} \dim(V_{2g} \cap N_*), \ h_* = g_{N_*} + 1 - t \) and \( X_{M_*} = X \cap M_* \), then \( E_{h_*} \) is defined to be

\[
\{ x \in X : (p_{N_*}|_{X_{M_*}})^{-1}(p_{N_*}(x)) \text{ has an irreducible component of dimension } \geq h_* \text{ that contains } x \}.
\]

By definition of \( E_{h_*} \), it is contained in \( M_* \). Hence \( X = E_{h_*} \subseteq M_* \). In particular \( X_{M_*} = X \).

Then we have

\[
(8.3) \quad \dim X - \dim p_{N_*}(X) > g_{N_*} - t.
\]

Otherwise \( h_* = g_{N_*} + 1 - t > \dim X - \dim p_{N_*}(X) \), and then \( E_{h_*} \) is proper Zariski closed in \( X \), contradicting \( X = E_{h_*} \). \( \square \)

**8.2. Theorem 8.1 in terms of mixed Shimura variety.** In this subsection, we prove Theorem 8.1 in terms of mixed Shimura variety. Then we translate it into the desired geometric description in the next subsection.

Let \( M \) be the smallest special subvariety of \( \mathfrak{A}_g \) which contains \( X \). Assume that \( M \) is associated with the mixed Shimura subdatum of Kuga type \((Q, \mathcal{Y}^+)_*\) of \((P_{2g, a}, \mathcal{X}^+_{2g, a})\).

**Proposition 8.3.** Assume either \( t \leq 0 \), or \( t = 1 \) and \( M = \mathfrak{A}_{g|\pi(M)} \). Then \( X = X^{\text{deg}}(t) \) if and only if there exists a non-trivial connected normal subgroup \( N \) of \( Q \) whose reductive part is semi-simple, such that the following condition holds: For the operation of taking quotient Shimura datum \( \tilde{\mathcal{N}}_*: (Q, \mathcal{Y}^+) \rightarrow (Q, \mathcal{Y}_r^+)/N \) discussed in \( \S 5.4 \) and the induced morphism on the corresponding mixed Shimura varieties of Kuga type

\[
(8.4) \quad \mathcal{Y}^+ \xrightarrow{\text{dim}} \mathcal{Y}_r^+,
\]

we have \( \dim X - \dim p_{N}(X) > g_N - t \), where \( g_N = \frac{1}{2} \dim(V_{2g} \cap N) \).
Proof of Proposition 8.3. We use Proposition 8.2. First \( \iff \) of Proposition 8.3 follows directly from \( \iff \) of Proposition 8.2.

Let us prove \( \Rightarrow \) of Proposition 8.3. Let \( M_\ast, (Q_\ast, Y^\ast_\ast) \) and \( N_\ast \) be as in \( \Rightarrow \) of Proposition 8.2.

Set \( N \) to be the identity component of \( Q \cap N_\ast \). Since \( X \subseteq M \) and \( M \) is the smallest special subvariety of \( \mathfrak{A}_g \) which contains \( X \), we have \( M \subseteq M_\ast \). Hence we may assume \((Q_\ast, Y^\ast_\ast) \subseteq (Q_\ast, Y^\ast_\ast) \). Let \( N = Q \cap N_\ast \), then \( N \vartriangleleft Q \). Replacing \( N \) by \( N \cap (\pi(N)^{\text{der}}) \), we may and do assume that the reductive part of \( N \) is semi-simple; here \( \pi : P_{2g,a} \to \text{GSp}_{2g} \).

Let us temporarily assume

\[
\text{dim } p_N(X) = \text{dim } p_N(X)
\]

and finish the proof by showing that this \( N \) can be taken as the desired connected normal subgroup of \( Q \). First for \( g_N = \frac{1}{2} \text{dim}(V_{2g} \cap N) \), we have \( g_N \leq g_\ast \) since \( N < N_\ast \). Hence (8.3) implies \( \text{dim } X - \text{dim } p_N(X) > g_N - t \). Next suppose \( N \) is trivial. Then \( p_N = \text{id}_M \), and \( \text{dim } p_N(X) = \text{dim } X \). Hence \( \text{dim } p_N(X) = \text{dim } X \), namely \( p_N|_{\tilde{X}} \) is generically finite. On the other hand we have shown

\[
X = \{ x \in X : (p_N|_X)^{-1}(p_N(x)) \text{ has an irreducible component of dimension } \geq h_\ast \text{ that contains } x \}.
\]

So \( h_\ast \leq 0 \). Thus \( g_N + 1 - t \leq 0 \). Hence \( g_N \leq t - 1 \leq 0 \) for \( t \leq 1 \). But then \( g_N = 0 \) and \( t = 1 \). Hence \( N_\ast \) is reductive. It can be viewed as a subgroup of \( \text{Sp}_{2g} \) via \( \pi : P_{2g,a} \to \text{GSp}_{2g} \).

Now \( N_\ast \vartriangleleft Q_\ast \) implies that the subgroup \( N_\ast \) of \( \text{Sp}_{2g} \) acts trivially on \( V_{2g} \cap Q_\ast \). By our hypothesis on \( X \) (note that \( t = 1 \) now) and Proposition 5.7, we have \( M = \mathfrak{A}_g|_{\pi(M)} \). Hence \( V_{2g} \cap Q = V_{2g} \).

Thus \( V_{2g} \cap Q_\ast = V_{2g} \) since \( Q \subseteq Q_\ast \). But the only connected subgroup of \( \text{Sp}_{2g} \) acting trivially on \( V_{2g} \) is 1. Hence \( N_\ast \) is trivial, contradicting to the choice of \( N_\ast \). Thus \( N \) is non-trivial. Hence this \( N \) can be taken as the desired connected normal subgroup of \( Q \).

Now it remains to prove (8.5). Denote by \( \tilde{X} \) a complex analytic irreducible component of \( u^{-1}(X) \) which is contained in \( Y^+ \). Each fiber of \( \tilde{p}_N \) is of the form \( N(\mathbb{R}) \tilde{y} \) for some \( \tilde{y} \in Y^+ \), and each fiber of \( \tilde{p}_N \) is of the form \( N(\mathbb{R}) \tilde{y} \) for some \( \tilde{y} \in Y^+ \). Hence each fiber of \( \tilde{p}_N|_{\tilde{X}} \) is of the form \( N(\mathbb{R}) \tilde{y} \cap \tilde{X} \) for some \( \tilde{y} \in Y^+ \).

Since \( \tilde{X} \subseteq Y^+ \subseteq Y^+ \), we have that each fiber of \( \tilde{p}_N|_{\tilde{X}} \) is of the form \( N_\ast(\mathbb{R}) \tilde{y} \cap \tilde{X} \) for some \( \tilde{y} \in Y^+ \). By definition of \( N \), we have \( N(\mathbb{R}) \tilde{y} = Q(\mathbb{R}) \tilde{y} \cap N_\ast(\mathbb{R}) \tilde{y} = Y^+ \cap N_\ast(\mathbb{R}) \tilde{y} \). Hence

\[
N(\mathbb{R}) \tilde{y} \cap \tilde{X} = Y^+ \cap N(\mathbb{R}) \tilde{y} \cap \tilde{X} = N_\ast(\mathbb{R}) \tilde{y} \cap \tilde{X}
\]

since \( \tilde{X} \subseteq Y^+ \). In particular the fibers of \( \tilde{p}_N|_{\tilde{X}} \) and \( \tilde{p}_N|_{\tilde{X}} \) have the same dimensions. Hence the fibers of \( p_N|_{\tilde{X}} \) and \( p_N|_{\tilde{X}} \) have the same dimensions. Thus (8.5) holds. Now we are done. \( \square \)

8.3. Proof of Theorem 8.1. Let \( M \) be the smallest special subvariety of \( \mathfrak{A}_g \) which contains \( X \). Assume that \( M \) is associated with the mixed Shimura subdatum of Kuga type \( (Q, Y^+) \) of \( (P_{2g,a}, X^\ast_{2g,a}) \). Denote by \( V_Q = V_{2g} \cap Q \) and \( G_Q = \pi(Q) \) the reductive part of \( Q \). Denote by \( g_Q = \frac{1}{2} \text{dim } V_Q \).

Special subvarieties of \( \mathfrak{A}_g \) are described by Proposition 5.7. Hence the smallest special subvariety \( M \) of \( \mathfrak{A}_g \) which contains \( X \) can be described as follows:

- Let \( M_G \) be the smallest special subvariety of \( \mathfrak{A}_g \) which contains \( \pi(X) \);
- Let \( M \) be the translate of an abelian subscheme of \( \mathfrak{A}_g|_{M_G} \to M_G \) by a torsion section which contains \( X \), minimal for this property.

Then \( M \to M_G \) itself is an abelian scheme of relative dimension \( g_Q \).

Apply \( \Rightarrow \) of Proposition 8.3. Then we obtain a non-trivial connected normal subgroup \( N \) of \( Q \) whose reductive part is semi-simple, such that for the quotient Shimura morphism \( p_N : M \to M' \), we have \( (g_N = \frac{1}{2} \text{dim}(V_{2g} \cap N)) \)

\[
\text{dim } X - \text{dim } p_N(X) > g_N - t.
\]
Recall the geometric meaning of $p_N$ (5.4)

$$
\begin{array}{c}
\xymatrix{
M \ar[r]^-{p_0} & M_0 := \Gamma_0 \setminus Y_0 \ar[d]^-{\pi_0} \ar[r]^-{p'} & M' \ar[r]^-{i} & \mathcal{A}_{gQ-gN} \\
M_G \ar[r]^-{id_{M_G}} \ar[u]^-{\pi|_M} & M_G \ar[r]^-{p_{G_N}} & M'_G \ar[r]^-{i_G} & \mathcal{A}_{gQ-gN} \ar[u]^-{\pi'}
}
\end{array}
$$

The morphism $p_0$ is taking the quotient of $M \to M_G$ by the abelian subscheme $\operatorname{Ker}(p_0)^\circ$ (which has relative dimension $g_N$). For each $\mathfrak{B} \in M'_G$, the abelian scheme $M_0|_{p_{G_N}^{-1}(\mathfrak{B})} \to p_{G_N}^{-1}(\mathfrak{B})$ is isotrivial.

Take $\mathcal{B} = \operatorname{Ker}(p_0)^\circ \cap A_X$. Then $\mathcal{B}$ is an abelian subscheme of $A_X \to B$ of relative dimension $g_N$, namely $g_B = g_N$. Now we can construct the maps in (8.1). We have $A_X = M|_B, p_B = p_0|A_X, \pi/B = \pi_0|_{M|_B}, \iota/B = (i \circ p')|_{M|_B}$ and $\iota/B,G = (i_G \circ p_{G_N})|_{B}$. Thus

$$
dim(\iota/B \circ p_B)(X) \leq dim p_N(X) < dim X - g_N + t = dim X - g_B + t$$

where the first inequality follows from $X \subseteq M|_B$ and the second inequality follows from (8.6). Thus it suffices to prove that $\iota/B \circ p_B$ is not generically finite.

Suppose $\iota/B \circ p_B$ is generically finite. Then $g_N = 0, p_B = id_{A_X}$ and $\iota/B$ is generically finite. Hence $dim(\iota/B \circ p_B)(X) = dim X$. But then $dim X < dim X - g_B + t = dim X + t$. When $t \leq 0$ this cannot hold. Hence $t = 1$ and $A_X = A_y|_B$ by our hypothesis. Hence $M = A_y|_{M_G}$.

Since $g_N = 0$, we have $p_0 = id_M$ and $M_0 = M = A_y|_{M_G}$. For each $\mathfrak{B} \in M'_G$, the abelian scheme $A_y|_{p_{G_N}^{-1}(\mathfrak{B})} \to p_{G_N}^{-1}(\mathfrak{B})$ is isotrivial. So $dim p_{G_N}^{-1}(\mathfrak{B}) = 0$; see the end of §2. Hence $p_{G_N}$ is generically finite, and so is $\iota'$. Thus $dim M = dim M'$. This contradicts our choice of $N$ (non-trivial connected).

\begin{footnote}{of Theorem 8.1} Before moving on, let me point out that if $t \leq 0$, then this implication follows rather easily from Proposition 6.1 and Theorem 7.1 because we can translate it into studying the generic rank of the Betti map. See the first paragraph of the proof Theorem 10.2 for more details. However the argument below works for $t \leq 0$ and $t = 1$.

Hodge theory says that: (1) every abelian subscheme of $A_X \to B$ is the intersection of an abelian subscheme of $M \to M_G$ with $A_y|_B$ (in particular $A_X = M \cap A_y|_B = M|_B$ and $g_X = g_N$); (2) the abelian subschemes of $M \to M_G$ are in 1-to-1 correspondence to $g_Q$-submodules of $V_Q$. See [Del71, 4.4.1-4.4.3].

Assume that $\mathcal{B} = \mathcal{B} \cap A_y|_B$ where $\mathcal{B}$ is an abelian subscheme of $M \to M_G$, and that $\mathcal{B}$ corresponds to the $g_Q$-submodule $V_N$ of $V_Q$. Let $g_N = \frac{1}{2} \dim V_N$, then $\mathcal{B} \to M_G$ has relative dimension $g_N$. Hence $g_B = g_N$.

Taking the quotient, we get

$$
\begin{array}{c}
\xymatrix{
M \ar[r]^-{p_B} & M/\mathcal{B} \\
M_G \ar[r]^-{id_{M_G}} & M_G
}
\end{array}
$$

In particular $\pi/B$ is an abelian scheme of relative dimension $g_Q - g_N$. It induces canonically a modular map

$$
\begin{array}{c}
\xymatrix{
M/\mathcal{B} \ar[r]^-{\iota/B} & A_{gQ-gN} \\
M_G \ar[r]^-{\iota/B,G} & A_{gQ-gN}
}
\end{array}
$$
By definition of \( \iota_{/B, G} \), \( (M/\mathbb{B})|_{\iota_{/B, G}(a)} \) is an isotrivial abelian scheme for each \( a \in \mathbb{A}_{gQ - gN} \).

Modular interpretation of Shimura varieties implies that \( \iota_{/B, G} \) is a Shimura morphism, namely \( \iota_{/B, G} \) is induced from some \( \tilde{\iota}_{/B, G} : (G_Q, \tilde{\pi}(Y^+)) \to (G_{\text{Sp}_{2(gQ - gN)}}, \tilde{\pi}^{+}_{gQ - gN}) \). Denote by \( H \) the kernel of \( \tilde{\iota}_{/B, G} \) on the underlying groups, then \( H \triangleleft G_Q \). Replace \( H \) by \( H^{\text{der}} \), then \( H \triangleleft G_Q^{\text{der}} \).

Each fiber of \( \iota_{/B, G} \) on the underlying spaces is of the form \( H(\mathbb{R})^+ \tilde{y}_G \) for some \( \tilde{y}_G \in \tilde{\pi}(Y^+) \). Hence each fiber of \( \iota_{/B, G} \) is of the form \( u_G(H(\mathbb{R})^+ \tilde{y}_G) \) for some \( \tilde{y}_G \in \tilde{\pi}(Y^+) \), where \( u_G : \tilde{\pi}^+_{gQ - gN} \to \mathbb{A}_g \) is the uniformization. Thus the connected algebraic monodromy group of \( \mathcal{M} \) is fiberwise defined by \( (\iota_{/B, G})^* \).

Let \( \mu \in \mathbb{A}_{gQ - gN} \). Consider the isomorphism \( G_{\text{Sp}_{2(gQ - gN)}}(\mathbb{R})^+ \tilde{y}_G \to \tilde{\pi}(Y^+) \) for each \( \mu \in \mathbb{A}_{gQ - gN} \). By Deligne’s Theorem of the Fixed Part [Del71, Corollaire 4.1.2], \( H \) acts trivially on \( V_Q/V_N \cong V_{2(gQ - gN)} \) because \( (M/\mathbb{B})|_{\iota_{/B, G}(a)} \) is isotrivial.[8]

The zero section of the abelian scheme \( M \to M_G \) gives rise to a Levi decomposition \( Q = V_Q \rtimes H \). Let \( N = V_N \rtimes H \). Then \( N \) is a connected normal subgroup of \( Q \) whose reductive part is semi-simple. Moreover the quotient morphism \( p_N \) is precisely \( \iota_{/B} \circ p_{2B} \), namely the geometric interpretation (5.4) for this \( N \) becomes

\[
\begin{align*}
M & \xrightarrow{p_N} M/\mathbb{B} \xrightarrow{\iota_{/B}} \iota(M/\mathbb{B}) , \\
\pi|_{M_G} & \circ \iota_{/B} \circ p_{2B} \circ \pi' \circ \iota|_{M} \circ \pi|_{M_G} \circ \iota_{/B, G} \circ \pi' \circ \iota|_{M_G}.
\end{align*}
\]

Note that (8.1) is precisely (8.7) restricted to (fibers over) \( B \subseteq M_G \).

If \( N = 1 \), then \( p_N = \text{id}_M \). This contradicts \( \iota_{/B} \circ p_{2B} \) being not generically finite. Hence \( N \) is non-trivial.

Note that \( X \subseteq M_{1,B} \). Thus the inequality in \( \Leftarrow \) of Theorem 8.1 becomes \( \dim p_N(X) < \dim X - g_N + t \). Now it suffices to apply \( \Leftarrow \) of Proposition 8.3 to this \( N \).

9. APPLICATION OF THE CRITERION TO FIBERED POWERS

In this section we present an application of the criterion Theorem 8.1 to the fiber product and the Faltings-Zhang locus of some (general enough) subvariety of an abelian scheme.

Let \( S \) be an irreducible subvariety over \( \mathbb{C} \) and let \( \pi_S : \mathcal{A} \to S \) be an abelian scheme of relative dimension \( g_0 \geq 1 \). Recall the modular map (4.2)

\[
\begin{align*}
\mathcal{A} & \xrightarrow{\varphi} \mathbb{A}_{g_0} , \\
\pi_S & \circ \varphi, \\
S & \xrightarrow{\varphi} \mathbb{A}_{g_0}.
\end{align*}
\]

Let \( m \geq 1 \) be an integer. By abuse of notation denote also by \( \mathbb{A}_{g_0} \) its image under the diagonal embedding \( \mathbb{A}_{g_0} \to \mathbb{A}_{mg_0} \). Consider the \( m \)-th Faltings-Zhang map

\[
\mathcal{D}^m : \mathbb{A}_{g_0} \times \mathbb{A}_{g_0} \times \mathbb{A}_{g_0} \times \ldots \times \mathbb{A}_{g_0} \to \mathbb{A}_{g_0} \times \mathbb{A}_{g_0} \times \ldots \times \mathbb{A}_{g_0} \mathbb{A}_{g_0} =: \mathbb{A}_{mg_0}|_{\mathbb{A}_{g_0}}
\]

which is fiberwise defined by \( (P_0, P_1, \ldots, P_m) \mapsto (P_1 - P_0, \ldots, P_m - P_0) \).

Let \( X \) be a closed irreducible subvariety of \( \mathcal{A} \) such that \( \pi_S(X) = S \). Set \( r = \dim X - \dim \varphi(X) \). Denote by \( X^{[m]} = X \times S \ldots \times S X \) \((m\text{-copies})\). The morphism \( \varphi \) induces a morphism \( \varphi^{[m]} : \mathcal{A}^{[m]} \to \) \( X^{[m]} \).

---

[8] It is known that the only connected subgroup of \( \text{Sp}_{2(gQ - gN)} \) which acts trivially on \( V_{2(gQ - gN)} \) is the trivial group. Thus \( H \) is the largest connected normal subgroup of \( G_Q^{\text{der}} \) which acts trivially on \( V_Q/V_N \).
Denote by \( X_\varphi^{[m]} = \varphi^{[m]}(X^{[m]}) \). It is a closed subvariety of \( \varphi(X) \times_{\varphi_S(S)} \cdots \times_{\varphi_S(S)} \varphi(X) \), which may be proper.

**Theorem 9.1.** Assume that \( X \) satisfies the following conditions:

(i) We have \( \dim X > \dim S \).

(ii) The subvariety \( X \) is not contained in the translate of any proper abelian subscheme of \( \mathcal{A} \to S \) by a torsion section.

(iii) We have \( X + \mathcal{A}' \not\subseteq X \) for any non-isotrivial abelian subscheme \( \mathcal{A}' \) of \( \mathcal{A} \to S \).

Then we have

1. For \( t \leq 0 \), we have \( X_\varphi^{[m]} \neq (X_\varphi^{[m]})^{\deg(t)} \) for all \( m \geq (\dim S + t)/(r + 1) \).

2. For \( t \leq 0 \), if furthermore \( X \) contains the image of \( S \) under a constant section \( \sigma \) of \( \mathcal{A} \to S \), then \( \mathcal{D}^m(X_\varphi^{[m+1]}) \neq (\mathcal{D}^m(X_\varphi^{[m+1]}))^{\deg(t)} \) for all \( m \geq (\dim X + t)/(r + 1) \).

3. We have \( X_\varphi^{[m]} \neq (X_\varphi^{[m]})^{\deg(1)} \) for all \( m \geq (\dim S + 1)/(r + 1) \) if hypothesis (iii) above is strengthened to (iii'): For any abelian subscheme \( \mathcal{A}' \) of \( \mathcal{A}_{g_0}|_{\varphi_S(S)} \to \varphi_S(S) \) with \( \dim \mathcal{A}' - \dim \varphi_S(S) > 0 \), we have \( \varphi(X) + \mathcal{A}' \not\subseteq \varphi(X) \).

**Proof.** Denote by \( B = \varphi_S(S) \). Up to shrinking \( S \), we may assume that \( \varphi(X) \to B \) is flat.

Let \( d = \dim X - \dim S > 0 \). For any \( m \geq 1 \), we have \( \dim X^{[m]} - \dim S = md \) and \( \dim X^{[m]} - \dim X_\varphi^{[m]} = mr \). In particular \( \dim X_\varphi^{[m]} = md + \dim S - mr \).

For notation let \( \mathcal{A}_{X^{[m]}} \), resp. \( \mathcal{A}_{X_\varphi^{[m]}} \), be the translate of an abelian subscheme of \( \mathcal{A}^{[m]} \to S \), resp. of \( \mathcal{A}_{g_0}|_B \to B \), by a torsion section which contains \( X^{[m]} \), resp. contains \( X_\varphi^{[m]} \), minimal for this property. Hypothesis (ii) then implies \( \mathcal{A}_{X^{[m]}} = \mathcal{A} \). Then by hypothesis (i) on \( X \), we have \( \mathcal{A}_{X_\varphi^{[m]}} = \mathcal{A}_{g_0} \) for all \( m \geq 1 \). So \( \mathcal{A}_{X_\varphi^{[m]}} = \mathcal{A}_{g_0}|_B := \mathcal{A}_{g_0} \times_{\mathcal{A}_{g_0}} B \) for all \( m \geq 1 \).

To ease notation let \( g = g_0 \).

We start by proving part (1) and (3). So \( t \leq 1 \). Suppose \( X_\varphi^{[m]} = (X_\varphi^{[m]})^{\deg(t)} \), then by Theorem 8.1, there exists an abelian subscheme \( \mathcal{B} \) of \( \mathcal{A}_g|_B \to B \) (whose relative dimension we denote by \( gb \)) such that for the quotient abelian scheme \( p_B : \mathcal{A}_g|_B \to (\mathcal{A}_g|_B)/\mathcal{B} \) and the modular map \( \iota_B : (\mathcal{A}_g|_B)/\mathcal{B} \to \mathcal{A}_{g - gb} \) (see (8.1)), we have that \( \iota_B \circ p_B \) is not generically finite and

\[
\dim(\iota_B \circ p_B)(X_\varphi^{[m]}) < \dim X_\varphi^{[m]} - gb + t.
\]

To make the morphisms more clear, let us redraw the diagram (8.1)

\[
\begin{align*}
\mathcal{A}_g|_B & \xrightarrow{p_B} (\mathcal{A}_g|_B)/\mathcal{B} & \xrightarrow{\iota_B} \mathcal{A}_{g - gb} \\
\pi|_{(\mathcal{A}_g|_B)} & \downarrow \pi_B & \downarrow \pi' \\
B & \xrightarrow{id_B} B & \xrightarrow{\iota_B, G} \mathcal{A}_{g - gb}
\end{align*}
\]

Under a possibly new decomposition (up to isogeny)

\[
\mathcal{A}_g|_{\mathcal{A}_{g_0}} \simeq \mathcal{A}_{g_0} \times_{\mathcal{A}_{g_0}} \cdots \times_{\mathcal{A}_{g_0}} \mathcal{A}_{g_0},
\]

we may write \( B = \mathcal{B}_1 \times_B \cdots \times_B \mathcal{B}_m \) for some abelian subschemes \( \mathcal{B}_1, \ldots, \mathcal{B}_m \) of \( \mathcal{A}_{g_0}|_B \). This new decomposition of \( \mathcal{A}_g|_{\mathcal{A}_{g_0}} \) induces a new decomposition of \( \mathcal{A}^{[m]} \simeq \mathcal{A} \times_S \cdots \times_S \mathcal{A} \) (\( m \)-copies) up to isogeny. By hypothesis (i) on \( X \), we still have \( X^{[m]} = X \times_S \cdots \times_S X \) under this new decomposition.

Assume \( \iota_B \) is generically finite, and hence

\[
\dim X_\varphi^{[m]} - \dim(\iota_B \circ p_B)(X_\varphi^{[m]}) = \dim X_\varphi^{[m]} - \dim p_B(X_\varphi^{[m]}) \leq gb.
\]
Recall that $t \leq 1$. Thus by (9.1), we have $t = 1$ and $\dim X_{\varphi}^{[m]} - \dim p_B(X_{\varphi}^{[m]}) = g_B$. So $X_{\varphi}^{[m]} + B = X_{\varphi}^{[m]}$. Projecting to each factor under the decomposition (9.3), we get $\varphi(X) + B_i = \varphi(X)$ for each $i \in \{1, \ldots, m\}$. Then hypothesis (iii) on $X$ implies that $\dim B_i = \dim B$ for each $i \in \{1, \ldots, m\}$. Thus $\dim B - \dim B = \sum_{i=1}^{m} (\dim B_i - \dim B) = 0$. But then $\dim \mathfrak{A}_g|B = \dim((\mathfrak{A}_g|B)/B) = \dim((\mathfrak{A}_g|B)/B)$, contradicting to $\iota_B \circ p_B$ being not generically finite.

Hence $\iota_B$ is not generically finite. Equivalently $\iota_B^{-1}(B)$ is of positive dimension for each $a \in \iota_B^{-1}(B)$.

We prove that $B_i \to B$ is non-isotrivial for each $i \in \{1, \ldots, m\}$. By construction of $\iota_B^{-1}$, $((\mathfrak{A}_g|B)/B)|_{\iota_B^{-1}(a)} \to \iota_B^{-1}(a)$ is an isotrivial abelian scheme. Hence $\mathfrak{A}_{g_0}|B_i$, being an abelian subscheme of $(\mathfrak{A}_g|B)/B$ via the decomposition (9.3), is isotrivial when restricted to $\iota_B^{-1}(a)$. Suppose that $B_i \to B$ is isotrivial for some $i_0$, then $\mathfrak{A}_{g_0}|\iota_B^{-1}(a) \to \iota_B^{-1}(a)$ is an isotrivial abelian scheme. This cannot happen since $\dim \iota_B^{-1}(a) > 0$; see the end of §2.

Assume that exactly $k$ among the $B_i$’s are $\mathfrak{A}_{g_0}|B$. Denote by $\Sigma$ the set of all the $B_i$’s which are not $\mathfrak{A}_{g_0}|B$. Then $\# \Sigma = m - k$. Hence

$$g_B = kg_0 + \sum_{B_i \in \Sigma} (\dim B_i - \dim B).$$

We also have the following naive lower bound for $\dim(\iota_B \circ p_B)(X_{\varphi}^{[m]})$

$$\dim(\iota_B \circ p_B)(X_{\varphi}^{[m]}) \geq (m - k)d - \sum_{B_i \in \Sigma} (\dim B_i - \dim B - 1).$$

Let us explain more details for it. Use the notation in (9.2). Take a point $b \in B$ such that $(\iota_B \circ p_B)(X_{\varphi}^{[m]}) = \pi((\iota_B \circ p_B)(X_{\varphi}^{[m]}))$ is flat over $b' := \iota_B^{-1}(b)$. Then $\dim(\iota_B \circ p_B)(X_{\varphi}^{[m]}) = \dim \pi((\iota_B \circ p_B)(X_{\varphi}^{[m]})) + \dim((\pi')^{-1}(b') \cap (\iota_B \circ p_B)(X_{\varphi}^{[m]})) \geq \dim((\pi')^{-1}(b') \cap (\iota_B \circ p_B)(X_{\varphi}^{[m]}))$.

Now it suffices to give a lower bound for $\dim((\pi')^{-1}(b') \cap (\iota_B \circ p_B)(X_{\varphi}^{[m]}))$. On the other hand $(\iota_B \circ p_B)(\mathfrak{A}_g|B \cap X_{\varphi}^{[m]}) \subseteq (\pi')^{-1}(b') \cap (\iota_B \circ p_B)(X_{\varphi}^{[m]})$ by the commutative diagram (9.2). Hence it suffices to give a lower bound for $\dim(\iota_B \circ p_B)((\mathfrak{A}_g|B \cap X_{\varphi}^{[m]})).$ Back to $\pi_S: A \to S$. Take any $s \in S$ with $\varphi_S(s) = b$. Then $\varphi^{[m]}(\mathfrak{A}_s|B \cap X^{[m]}) \subseteq (\mathfrak{A}_g|B \cap X_{\varphi}^{[m]}).$ As $(p_B \circ \varphi^{[m]})(X_s \times \ldots \times X_s)$ is contained in one fiber of $\pi_B,$ we have

$$\dim(\iota_B \circ p_B)(\varphi^{[m]}(\mathfrak{A}_s^{[m]} \cap X^{[m]})) = \dim \iota_B \left( (p_B \circ \varphi^{[m]})(\mathfrak{A}_s^{[m]} \cap X^{[m]}) \right) = \dim(p_B \circ \varphi^{[m]})(\mathfrak{A}_s^{[m]} \cap X^{[m]}) = \dim(\mathfrak{A}_s|B \cap X^{[m]}).$$

So it suffices to give a lower bound for $\dim(p_B \circ \varphi^{[m]})(\mathfrak{A}_s^{[m]} \cap X^{[m]}).$

The pullback $(\varphi^{[m]})^{-1}(B)$ is an abelian subscheme of $\mathfrak{A}_s^{[m]} \to S$ of relative dimension $g_B$. We have the following commutative diagram

$$\begin{array}{ccc}
\mathfrak{A}^{[m]} & \xrightarrow{\varphi^{[m]}} & \mathfrak{A}_g|B \\
p \downarrow & & \downarrow p_B \\
\mathfrak{A}^{[m]} / (\varphi^{[m]})^{-1}(B) & \xrightarrow{\varphi^{[m]}} & (\mathfrak{A}_g|B)/B
\end{array}$$

with $p$ being the quotient abelian scheme over $S$. Now that $(p_B \circ \varphi^{[m]})(\mathfrak{A}_s^{[m]} \cap X^{[m]}) = (\varphi^{[m]} \circ p)(\mathfrak{A}_s^{[m]} \cap X^{[m]}),$ it suffices to give a lower bound for $\dim(p_B \circ \varphi^{[m]})(\mathfrak{A}_s^{[m]} \cap X^{[m]}).$
Since $B = B_1 \times_B \ldots \times_B B_m$, we have $(\varphi[m])^{-1}(B) = \varphi^{-1}(B_1) \times_S \ldots \times_S \varphi^{-1}(B_m)$. Moreover $\dim \varphi^{-1}(B_i) - \dim S = \dim B_i - \dim B$, and $\varphi^{-1}(B_i) \to S$ is non-isotrivial because $B_i \to B$ is non-isotrivial. Denote by $p_i: A \to A/\varphi^{-1}(B_i)$ the quotient abelian scheme. Then $\varphi = (p_1, \ldots, p_m)$.

Note that $A_s[m] \cap X[m] = X_s \times \ldots \times X_s$ (m-copies) where $X_s$ is the fiber of $X$ over $s$. So

$$p(A_s[m] \cap X[m]) = p(X_s \times \ldots \times X_s) = p_1(X_s) \times \ldots \times p_m(X_s).$$

By hypothesis (iii) on $X$, a generic $s \in S$ satisfies $\dim p_i(X_s) \geq d - (\dim B_i - \dim B - 1)$ for each $i \in \Sigma$. Thus we obtain (9.5).

Now by (9.4) and (9.5), we have

$$g_B - (\dim X^r[m] - \dim(\iota_B \circ p_B)(X^r[m])) = g_B + \dim(\iota_B \circ p_B)(X^r[m]) - \dim X^r[m]$$

$$\geq k \sigma g_0 + \sum_{B_i \in \Sigma} (\dim B_i - \dim B) + (m - k)d - \sum_{B_i \in \Sigma} (\dim B_i - \dim B - 1) - (md + \dim S - mr)$$

$$= k \sigma g_0 + (m - k)d + (m - k) - md - \dim S + mr$$

$$= k \sigma (g_0 - d - 1) + m - \dim S + mr$$

$$\geq m - \dim S + mr$$

by hypothesis (iii) on $X$ (applied to $A' = A$).

This contradicts (9.1) for $m \geq (\dim S + t)/(r + 1)$. So we must have $X^{[m]}_\varphi \neq (X^{[m]}_\varphi)^{\deg(t)}$.

The proof of part (2), namely the statement regarding $\mathcal{D}^m(X^{[m+1]}_\varphi)$ is similar. Let us sketch it. Denote by $j_1: \mathfrak{A}_g / A_{\mathfrak{g}_{g_0}} = \mathfrak{A}_g \times_{A_{g_0}} \ldots \times_{A_{g_0}} \mathfrak{A}_g$ be fiberwise defined by $P \mapsto (P, 0, \ldots, 0)$. Then by definition of $\mathcal{D}^m(X^{[m+1]}_\varphi)(t)$ (Definition 1.6), we know that in order to prove $\mathcal{D}^m(X^{[m+1]}_\varphi) \neq (\mathcal{D}^m(X^{[m+1]}_\varphi))^{\deg(t)}(t)$, it suffices to prove $Y \neq Y^{\deg(t)}(t)$ where $Y = \mathcal{D}^m(X^{[m+1]}_\varphi) - (j_1 \circ \varphi \circ \sigma)(S)$. We have

$$\mathcal{D}^m(X^{[m+1]}_\varphi) \supseteq \mathcal{D}^m(\varphi^{[m+1]}(\sigma(S) \times_S X \times_S \ldots \times_S X)) = X^{[m]}_\varphi + (j_1 \circ \varphi \circ \sigma)(S).$$

So $X^{[m]}_\varphi \subseteq Y$. Denote by $\pi: \mathfrak{A}_g \to A_{\mathfrak{g}}$. Since $\pi(Y) = \pi(X^{[m]}_\varphi) = B$ and $A_{X^{[m]}_\varphi} = A_{\mathfrak{g}}|_B$, we have $\mathcal{A}_Y = \mathfrak{A}_Y|_B$.

Suppose $Y = Y^{\deg(t)}$. Applying Theorem 8.1 to $Y$, we get an abelian subscheme $B$ of $\mathfrak{A}_g|_B \to B$ (whose relative dimension we denote by $g_B$) such that for the map $\iota_B \circ p_B$ in (9.2), we have $\dim(\iota_B \circ p_B)(Y) < \dim Y - g_B + t$. If $\iota_B$ is generically finite, then $\dim(\iota_B \circ p_B)(Y) = \dim p_B(Y) \geq \dim Y - g_B$. Thus we get a contradiction to our hypothesis $t \leq 0$. Hence $\iota_B$ is not generically finite. But then

$$g_B - (\dim Y - \dim(\iota_B \circ p_B)(Y)) = g_B + \dim(\iota_B \circ p_B)(Y) - \dim Y$$

$$\geq g_B + \dim(\iota_B \circ p_B)(Y) - ((m + 1)d + \dim S - mr)$$

$$\geq g_B + \dim(\iota_B \circ p_B)(X^r[m]) - (m + 1)d - \dim S + mr$$

$$\geq m - d - \dim S + mr$$

by (9.4) and (9.5)

$$= m - \dim X + mr$$

This contradicts $\dim Y - \dim(\iota_B \circ p_B)(Y) > g_B - t$ when $m \geq (\dim X + t)/(r + 1)$. Thus we have proven $Y \neq Y^{\deg(t)}$, and hence $\mathcal{D}^m(X^{[m+1]}_\varphi) \neq (\mathcal{D}^m(X^{[m+1]}_\varphi))^{\deg(t)}(t)$.  

□
10. Generic rank of the Betti map

Let $S$ be an irreducible quasi-projective variety over $\mathbb{C}$, and let $\pi_S : A \to S$ be an abelian scheme of relative dimension $g \geq 1$. Recall the modular map (4.2)

\[
\begin{array}{c}
A \\
\varphi \\
\pi_S \\
\downarrow \\
S \\
\varphi_S \\
\pi \\
\mathbb{A}_g \\
\end{array}
\]

Let $b_\Delta : A_\Delta \to \mathbb{T}^{2g}$ as defined in (4.3) where $\Delta$ is some open subset of $S^\an$.

10.1. Summary. Let $X$ be a closed irreducible subvariety of $A$ dominant to $S$. Denote for simplicity by $\text{rank}_R(d\Delta|_X) = \max_{x \in X^\an(\mathbb{C}) \cap A_\Delta} (\text{rank}_R(d\Delta|_{X \cap A_\Delta})_{x})$. In this subsection we summarized results in the previous sections to conclude when we have

\[
\text{rank}_R(d\Delta|_X) = 2l
\]

for each $l \in \{1, \ldots, \min(\text{dim } \varphi(X), g)\}$ or $l = \text{dim } X$. An equivalent way to formulate the result is to characterize when $\text{rank}_R(d\Delta|_X) < 2l$ for all such $l$.

The following theorem follows easily from Proposition 6.1 applied to $\varphi(X)$ (and for part (1) also the trivial upper bound $\text{rank}_R(d\Delta|_X) \leq 2 \dim \varphi(X)$).

**Theorem 10.1.** Let $x \in X^\an(\mathbb{C}) \cap A_\Delta$. Then

1. We have

$$\text{rank}_R(d\Delta|_X)_{x} = 2 \dim X \Leftrightarrow x \notin X^\deg \Leftrightarrow \varphi|_{X^\an} \text{ is injective around } x \text{ and } \varphi(x) \notin \varphi(X)^{\deg}.$$  

In particular if $x \notin \varphi^{-1}(Z \cup \varphi(X)^{\deg}) = X^\deg$ where $Z$ is the Zariski closed subset of $\varphi(X)$ on which $\varphi|_X$ has positive dimensional fibers, then $\text{rank}_R(d\Delta|_{X \cap A_\Delta})_{x} = 2 \dim X$.

2. Denote by $d = \dim \varphi(X)$. For each integer $l \in \{1, \ldots, \min(d, g)\}$, we have

$$\text{rank}_R(d\Delta|_X)_{x} = 2l \Leftrightarrow \varphi(x) \notin \varphi(X)^{\deg}(l - d).$$

Note that $\varphi(X)^{\deg}$ and $\varphi(X)^{\deg}(l - d)$ for each $l$ is a Zariski closed subset of $\varphi(X)$ by Theorem 7.1. Hence Theorem 10.1 gives a complete description of the locus of $X$ on which the Betti map has rank smaller than expected.

**Proof.** Part (2) is clearly true by Proposition 6.1 applied to $\varphi(X)$. The “In particular” part of (1) follows immediately from the main part. Let us prove part (1).

The equivalence of the first and third statements of part (1) follows easily from Proposition 6.1 (applied to $\varphi(X)$) and the trivial upper bound $\text{rank}_R(d\Delta|_X) \leq 2 \dim \varphi(X)$. It remain to prove the equivalence of second of the third statements of part (1).

Assume $x \in X^\deg$ and $\varphi|_{X^\an} \text{ is injective around } x$. Then by definition of $X^\deg$, there exists a closed irreducible subvariety $Y$ of $X$ such that $x \in Y$ and $\text{dim}(Y)_{\text{sg}} - \text{dim } \pi_S(Y) < \text{dim } Y$. Since $\varphi|_{X^\an}$ is injective around $x$, we have $\text{dim } Y = \text{dim } \varphi(Y)$. Hence $\varphi(Y) \subseteq \varphi(X)^{\deg}$ because

$$\text{dim}(\varphi(Y))_{\text{sg}} - \text{dim } \varphi_S(\pi_S(Y)) = \text{dim}(Y)_{\text{sg}} - \text{dim } \pi_S(Y) < \text{dim } Y = \text{dim } \varphi(Y).$$

Thus $\varphi(x) \in \varphi(Y) \subseteq \varphi(X)^{\deg}$. So the third statement implies the second statement in part (1).

Conversely, if $\varphi|_{X^\an}$ is not injective around $x$, then there exists a closed irreducible subvariety $Y$ of $X$ such that $x \in Y$, $\varphi(Y) = \varphi(x)$ and $\text{dim } \pi_S(Y) > 0$. But then $\langle Y \rangle_{\text{sg}} = Y$, and hence $\text{dim}(Y)_{\text{sg}} - \text{dim } Y < \text{dim } \pi_S(Y)$. So $x \in Y \subseteq X^\deg$ by definition. If $\varphi(x) \in \varphi(X)^{\deg}$, then there exists a closed irreducible subvariety $Y$ of $\varphi(X)$ such that $\varphi(x) \in Y$ and $\text{dim}(Y)_{\text{sg}} - \text{dim } \pi(Y) < \text{dim } Y$. Let $Y'$ be an irreducible component of $(\varphi|_X)^{-1}(Y)$ which contains $x$. Then
dim(Y')_{sg} - \dim \pi_S(Y') = \dim(Y')_{sg} - \dim \pi(Y) < \dim Y \leq \dim Y'. So \(x \in Y' \subseteq X^{\deg}\) by definition. So the second statement implies the third statement in part (1). 

Next in order to get \(\text{rank}_R(\mathcal{B}_X|_S) < 2l\), we apply Theorem 8.1 to \(\varphi(X)\) and \(t = l - d \leq 0\). We thus obtain:

**Theorem 10.2.** Let \(A_X\) be the translate of an abelian subscheme of \(A \to S\) by a torsion section which contains \(X\), minimal for this property. Then \(A_X \to S\) itself is an abelian scheme (up to taking a finite covering of \(S\), whose relative dimension we denote by \(g_X\).

Let \(l \in \{1, \ldots, \min(\dim \varphi(X), g)\}\) or \(l = \dim X\). Then \(\text{rank}_R(\mathcal{B}_X|_S) < 2l\) if and only if the following condition holds: up to taking a finite covering of \(S\) there exists an abelian subscheme \(\mathcal{B}\) of \(A_X \to S\) (whose relative dimension we denote by \(g_B\)) such that for the quotient abelian scheme \(p_B : A_X \to A_X/\mathcal{B}\) and the modular map

\[
P_{\mathcal{B}} : A_X/\mathcal{B} \to A_{g_X - g_B},
\]

we have \(\dim(p_B \circ p_B)(X) < l - g_B\).

**Proof.** Before moving on, let us point out that the implication \(\Leftarrow\) is clear: The generic rank of the Betti map on \((p_B \circ p_B)(X)\) has the trivial upper bound \(2 \dim(p_B \circ p_B)(X)\), thus \(\text{rank}_R(\mathcal{B}_X|_S) \leq 2g_B + 2 \dim(p_B \circ p_B)(X) < 2l\).[9]

Denote for simplicity by \(d = \dim \varphi(X)\). For \(l \in \{1, \ldots, \min(d, g)\}\), part (2) of Theorem 10.1 and Theorem 7.1 imply that \(\text{rank}_R(\mathcal{B}_X|_S) < 2l\) if and only if \(\varphi(X) = \varphi(X)^{\deg}\). Thus the conclusion holds by Theorem 8.1 (and the discussion below) applied to \(\varphi(X)\) and \(t = l - d \leq 0\).

It remains to prove for \(l = \dim X\). Part (1) of Theorem 10.1 implies that \(\text{rank}_R(\mathcal{B}_X|_S) < 2 \dim X\) if and only if

- Either \(\varphi|_X\) is not generically finite;
- Or \(\varphi(X) = \varphi(X)^{\deg}\).

If \(\varphi|_X\) is not generically finite, then one can take \(\mathcal{B}\) to be the trivial abelian scheme over \(S\). For this \(\mathcal{B}\) we have \(\dim(p_B \circ p_B)(X) = \dim \varphi(X) < \dim X\). If \(\varphi|_X\) is generically finite, then \(\dim X = \dim \varphi(X)\) and \(\varphi(X) = \varphi(X)^{\deg}\). There exists an abelian subscheme \(\mathcal{B}'\) of \(A_{\varphi(X)} \to \varphi_S(S)\) from Theorem 8.1 (applied to \(\varphi(X)\) and \(t = 0\)). Then it suffices to take \(\mathcal{B} = \varphi^{-1}(\mathcal{B}')\) because \((p_B \circ p_B)(X) = (\varphi_X)^{\deg}(\varphi(X))\). This concludes \(\Rightarrow\) when \(l = \dim X\).

We have seen that \(\Leftarrow\) holds for all \(l\). Alternatively one can give an argument via Theorem 8.1. Suppose \(l = \dim X\) and \(\varphi|_X\) is generically finite. We need to prove \(\varphi(X) = \varphi(X)^{\deg}\). But then it suffices to apply Theorem 8.1 to \(\varphi(X)\), \(t = 0\) and \(\mathcal{B}' = \varphi(\mathcal{B})\) (note that \((p_B \circ p_B)(X) = (\varphi_X)^{\deg}(\varphi(X))\)). We leave the details to the readers. \(\square\)

For any integer \(m \geq 1\), denote by \(X^{[m]} = X \times_S \ldots \times_S X\) \((\text{m}-\text{copies})\). The modular map \(\varphi(10.1)\) induces a morphism \(\varphi^{[m]} : A^{[m]} \to A_{mg}\), and the Betti map \(b_\Delta\) gives rise to \(\iota^{[m]}_\Delta : A^{[m]} \to T^{2mg}\).

Consider moreover the \(m\)-th Faltings-Zhang maps

\[
\mathcal{D}^m : A_{g_1} \times \ldots \times A_{g_m} \to A_{g_1} \times \ldots \times A_{g_m},
\]

\[
\mathcal{D}^m_A : A \times_S \ldots \times_S A \to A \times_S \ldots \times_S A
\]

[9] In other words, \(\Leftarrow\) of Theorem 8.1 is clearly true when \(t \leq 0\) if one translates the condition \(X = X^{\deg}(t)\) into studying the generic rank of the Betti map. But for \(t = 1\), we still need to go into our original proof.
which are fiberwise defined by \((P_0, P_1, \ldots, P_m) \mapsto (P_1 - P_0, \ldots, P_m - P_0)\). Then \(\varphi^{[m]} \circ D_A^m = D_A^m \circ \varphi^{[m+1]} : A^{[m+1]} \to \mathfrak{A}_{mg}\).

Now combining part (2) of Theorem 10.1, Theorem 7.1 with Theorem 9.1, we get the following theorem.

**Theorem 10.3.** Assume that \(X\) satisfies the following conditions:

(i) We have \(\dim X > \dim S\).

(ii) The subvariety \(X\) is not contained in the translate of any proper abelian sub scheme of \(A \to S\) by a torsion section.

(iii) We have \(X + A' \not\subseteq X\) for any non-isotrivial abelian sub scheme \(A'\) of \(A \to S\).

Then we have

\[
\text{(1) } \max_{x \in (X^{[m]})_{\text{sm}}(C) \cap \Delta_A} \text{rank}_R(d\varphi^{[m]}_A |_{X^{[m]} \cap \Delta_A})_x = 2 \dim \varphi^{[m]}_A(X^{[m]}) \text{ for all } m \geq \dim S.
\]

\[
\text{(2) If furthermore } X \text{ contains the image of } S \text{ under a constant section } \sigma \text{ of } A \to S, \text{ then } \max_{x \in (\Delta_A(X^{[m+1]}))_{\text{sm}}(C) \cap \Delta_A} \text{rank}_R(d\varphi^{[m]}_A |_{\Delta_A(X^{[m+1]}) \cap \Delta_A})_x = 2 \dim \varphi^{[m]}_A(\Delta_A(X^{[m+1]})) \text{ for all } m \geq \dim X.
\]

10.2. A conjecture of André-Corvaja-Zannier. In this subsection we apply the previous results to study a conjecture of André-Corvaja-Zannier.

Let \(\xi\) be a section of \(A \to S\). Denote by \(b_\Delta \circ \xi : \Delta \to \mathcal{A}_\Delta \to \mathbb{T}_g\).

**Theorem 10.4.** Assume also that \(\mathbb{Z}\xi\) is Zariski dense in \(\mathcal{A}\). Furthermore we make the following assumption:

(i) Either \(A/S\) is geometrically simple;

(ii) Or each Hodge generic curve \(C \subseteq \varphi_S(S)\) satisfies the following property: \(\varphi(A) \times \varphi_S(S)C = \pi^{-1}(C) \to C\) has no fixed part over any finite covering of \(C\).[10]

Then \(\max_{s \in \Delta} \text{rank}_R(d\phi^A_{\xi_s}) = 2 \min(\dim \varphi(\xi(S)), g)\).

**Remark 10.5.** If \(A/S\) is isotrivial, then hypothesis (ii) holds automatically since either \(S\) is a point and hence contains no curve or \(\varphi_S|_C\) is never quasi-finite for any curve \(C\). In general hypothesis (ii) can be checked in the following way. Let \(G_B\) be the connected algebraic monodromy group of \(\varphi_S(S)\). Then hypothesis (ii) is equivalent to: for any non-trivial connected normal subgroup \(H\) of \(G_B\), the only element of \(V_2\) stable under \(H\) is 0. It holds for example when \(A \to S\) has no fixed part and \(G_B\) is a simple group.

**Proof.** Denote by \(B = \varphi_S(S)\) and \(X = \varphi(\xi(S))\). Denote for simplicity by \(d = \dim X\). Since \(\mathbb{Z}\xi\) is Zariski dense in \(\mathcal{A}\), no translate of a proper abelian sub scheme of \(\mathcal{A}|_{gB} := \pi^{-1}(B) \to B\) by a torsion section contains \(X\).

Assume \(\max_{x \in (X^{[m]}(C) \cap \Delta_A)} (\text{rank}_R(d\varphi^{[m]}_A |_{X^{[m]} \cap \Delta_A})_x) < 2 \min(d, g)\). Applying Theorem 10.2 to \(l = \min(d, g)\), we get that there exists an abelian sub scheme \(B\) of \(\mathcal{A}|_{gB} \to B\) (whose relative dimension we denote by \(gb\)) such that for the quotient abelian scheme \(p_B : \mathcal{A}|_{gB} \to (\mathcal{A}|_{gB})/B\) and the modular map \(\iota_{/B} : (\mathcal{A}|_{gB})/B \to \mathcal{A}_{gB-gB}\), we have

\[
\text{(10.3) } \dim(\iota_{/B} \circ p_B)(X) < \dim X - gb + \min(d, g) - d = \min(d, g) - gb.
\]

**Case (i)** In this case \(\mathcal{A}|_{gB} \to B\) is geometrically simple. Hence \(B\) is either the whole \(\mathcal{A}|_{gB}\) or the zero section of \(\mathcal{A}|_{gB} \to B\). In the former case, \(gb = g\). But then (10.3) cannot hold. In the latter case, \(\mathcal{A}|_{\iota^{-1}_{B,G}(a)} \to \iota^{-1}_{B,G}(a)\) is an isotrivial abelian scheme for each \(a \in \iota_{/B,G}(B)\).

---

[10] We say that a curve \(C \subseteq \varphi_S(S)\) is Hodge generic if the generic Mumford-Tate group of \(C\) coincides with the generic Mumford-Tate group of \(\varphi_S(S)\).
So \(\dim \iota^{-1}_{B,G}(a) = 0\); see the end of \(\S 2\). But then \(\iota_{B,G}\) is generically finite, and so it \(\iota_B\). So \(\dim(\iota_B \circ p_B)(X) = \dim p_B(X) \geq \dim X - g_B = d - g_B \geq \min(d,g) - g_B\), contradicting (10.3).

**Case (ii)** If \(B = A_g|_B\), then \(g = g_B\), and hence (10.3) cannot hold. From now on, we may and do assume \((A_g|_B)/B \to B\) is a non-trivial abelian scheme.

By construction of \(\iota_{B,G}\), \((A_g|_B)/B\) \(\iota_{B,G}(a)\) is an isotrivial abelian scheme for each \(a \in \iota_{B,G}(B)\). Taking a splitting of \(p_B\) gives an isotrivial abelian subscheme of \(A_g|_{\iota^{-1}_{B,G}(a) \to \iota^{-1}_{B,G}(a)}\) which is non-trivial. For each a Hodge generic in \(\iota_{B,G}(B)\), we have that \(\iota^{-1}_{B,G}(a)\) is Hodge generic in \(B\). But then our hypothesis forces \(\dim \iota^{-1}_{B,G}(a) = 0\). Hence \(\iota_{B,G}\) is quasi-finite, and so is \(\iota_B\). So \(\dim(\iota_B \circ p_B)(X) = \dim p_B(X) \geq \dim X - g_B = d - g_B \geq \min(d,g) - g_B\), contradicting (10.3). Hence we are done. \(\Box\)

**Example 10.6.** The extra hypotheses (i) or (ii) in Theorem 10.4 are necessary. We illustrate this with an example with \(g = 4\). Consider \(A_4 \to A_4\). Now that \(A_2 \times A_2 \to A_2 \times A_2\) is an abelian scheme of relative dimension 4, it induces canonically the modular map

\[
\begin{array}{ccc}
A_2 \times A_2 & \longrightarrow & A_4 \\
\downarrow & & \downarrow \\
A_2 \times A_2 & \longrightarrow & A_4
\end{array}
\]

By abuse of notation we write \(A_2 \times A_2\), resp. \(A_2 \times A_2\), for the image of the morphism on the top, resp. on the bottom.

Let \(S_1\) and \(S_2\) be irreducible subvarieties of \(A_2\), then \(S := S_1 \times S_2 \subseteq A_2 \times A_2\). The above diagram implies that \(A_{4|S} \to S\) is the product of the two abelian schemes \(A_{2|S_1} \to S_1\) and \(A_{2|S_2} \to S_2\).

Let \(i \in \{1, 2\}\). Let \(\xi_i\) be a section of \(A_{2|S_i} \to S_i\) such that \(\mathbb{Z}\xi_i\) is Zariski dense in \(A_{2|S_i}\) for \(i \in \{1, 2\}\). For the Betti map \(b_{\Delta_i} : (A_2)_{\Delta_i} \to \mathbb{T}^d\), where \(\Delta_i \subseteq (S_i)^{an}\) is an open subset, we have \(\text{rank}_2(b_{\Delta_i}|_{\xi_i}) \leq 2\min(\dim S_i, 2)\) for all \(S_i \subseteq S_i^{an}\).

Now \(\xi = (\xi_1, \xi_2)\) is a section of \(A_{4|S} \to S\) such that \(\xi\xi\) is Zariski dense in \(A_{4|S}\).

Take \(S_1 = A_2\) and \(S_2\) to be a curve in \(A_2\) such that \(A_{2|S_2}\) has no fixed part (over any finite étale covering of \(S_2\)), then \(A_{4|S} \to S\) has no fixed part (over any finite étale covering of \(S\)) and \(\dim S = 4 = g\). Let \(\Delta = \Delta_1 \times \Delta_2\). Then the Betti map \(b_{\Delta}\) equals \((b_{\Delta_1}, b_{\Delta_2})\). So \(\text{rank}_2((d_{\Delta}|_{\xi})|_{\xi}) \leq \max_{s_i \in \Delta_i} \text{rank}_2((d_{\Delta_i}|_{\xi_i})|_{\xi_i}) + 2\max_{s_2 \in \Delta_2} \text{rank}_2((d_{\Delta_2}|_{\xi_2})|_{\xi_2}) \leq 2(2 + \dim S_2) = 6 < 8 = 2\min(\dim \xi(S), g)\) for any \(s \subseteq \Delta\).

**10.3. Case when \(S\) is a curve.** Let \(X\) be a closed irreducible subvariety of \(A\) dominant to \(S\). In this subsection we briefly explain how to recover [GH18, Theorem 5.1].

**Theorem 10.7.** Assume \(\dim S = 1\). Then the following statements are equivalent:

(i) We have \(\text{rank}_2((d_{\Delta}|_{X \cap A_{\Delta}})|_{\xi}) < 2\dim X\) for all \(x \in X^{an}(\mathbb{C}) \cap A_{\Delta}\).

(ii) There exists a finite covering \(S' \to S\), inducing a morphism \(\rho : A' = A \times S S' \to A\), such that \(X = x'(\sigma + Z' + B')\), where \(\sigma\) is a torsion section of \(A'/S', B'\) is an abelian subscheme of \(A'/S'\), and \(Z' = Z' \times S\) with \(C' \times S'\) being the largest constant abelian subscheme of \(A'/S'\) and \(Z' \subseteq C'\).

**Proof.** For the modular morphism (10.1), either \(\varphi_S(S) = 1\) or \(\varphi_S(S)\) is a point.

If \(\dim \varphi_S(S) = 1\), then \(\varphi_S\) is quasi-finite. Hence \(\varphi\) is also quasi-finite. So we may and do replace \(X\), \(S\) by \(\varphi(X), \varphi(S) =: B\). Apply Proposition 6.1 to \(X\) and \(l = \dim X\), we get that \(X^{\text{deg}}\) is Zariski dense in \(X\). Note that \(\dim B = 1\), so each \(Y\) in the definition of \(X^{\text{deg}}\) (Definition 1.5) satisfies \(Y = (Y)_{sg}\). Thus \(X\) equals the Zariski closure of the union of generically
special subvarieties of \( \text{sg} \) type of \( \mathcal{A} \) which are contained in \( X \). Now it suffices to apply [GH18, Proposition 1.3] to conclude.

If \( \varphi_S(S) \) is a point, then \( \mathcal{A} \to S \) is isotrivial. It is clearly true that (ii) implies (i) because \( b_\Delta \) factors through \( \mathbb{A}_g \to \mathbb{A}_q \) by definition. Conversely assume (i) holds. Now that \( \mathcal{A}/S \) is isotrivial, there exists finite covering \( S' \to S \) such that \( \mathcal{A}' := \mathcal{A} \times_S S' \) is a constant abelian scheme, namely \( \mathcal{A}' = A \times S' \) for some abelian variety \( A \). Denote by \( \rho : \mathcal{A}' \to \mathcal{A} \) the natural projection and let \( X' \) be the irreducible component of \( \rho^{-1}(X) \). Then it suffices to prove \( X' = Z' \times S' \) for some \( Z' \subseteq A \).

In this case, the Betti map \( b_\Delta : \mathcal{A}_\Delta \to \mathbb{T}^{2g} \) can be seen as (the restriction to \( \mathcal{A}_\Delta \) of) the composite of the natural projection \( \mathcal{A} = A \times S \to A \) with an identification \( A \cong \mathbb{T}^{2g} \). For any \( x \in X^{\text{sm}}(\mathbb{C}) \cap \mathcal{A}_\Delta \), consider \( (b_\Delta|_{X_\Delta})^{-1}(b_\Delta(x)) \). Since rank$_\mathbb{R}$\( (db_\Delta|_{X_\Delta})_x \leq 2 \dim X \), we have
\[
\dim_{\mathbb{R}}(b_\Delta|_{X_\Delta})^{-1}(b_\Delta(x)) > 0.
\]
But \( b^{-1}_\Delta(b_\Delta(x)) \) is complex analytic of dimension \( \dim S = 1 \). So \( \dim_{\mathbb{C}}(b_\Delta|_{X_\Delta})^{-1}(b_\Delta(x)) = 1 \). Thus \( \{t\} \times \Delta \subseteq X \) where \( t \in A(\mathbb{C}) \) is the point corresponding to \( b_\Delta(x) \). Hence \( \{t\} \times S \subseteq X \).

Letting \( x \) run over all points in \( X^{\text{sm}}(\mathbb{C}) \cap \mathcal{A}_\Delta \) and noting that \( X^{\text{sm}}(\mathbb{C}) \cap \mathcal{A}_\Delta \) is Zariski dense in \( X \), we have that \( X \) is the Zariski closure of the union of \( \{t\} \times S \) for some \( t \in A(\mathbb{C}) \). Let \( Z \) be the Zariski closure of the union of all such \( t \)'s. Then \( Z \) is a subvariety of \( A \), and \( X = Z \times S \). Hence we are done.

\[ \square \]

11. Link with Relative Manin-Mumford

In this section, we discuss about the relative Manin-Mumford conjecture. It is closely related to \( X^{\text{des}}(1) \).

Let \( S \) be an irreducible variety over \( \mathbb{C} \), and let \( \pi_S : \mathcal{A} \to S \) be an abelian scheme of relative dimension \( g \geq 1 \). Denote by \( \mathcal{A}_{\text{tor}} \) the set of points \( x \in \mathcal{A}(\mathbb{C}) \) such that \([N]x \) lies in the zero section of \( \mathcal{A} \to S \) for some integer \( N \). In other words \( x \) is a torsion point in its fiber.

Let \( X \) be a closed irreducible subvariety such that \( \pi_S(X) = S \). Denote by \( \mathcal{A}_X \) the translate of an abelian subscheme of \( \mathcal{A} \) by a torsion section which contains \( X \), minimal for this property.

**Relative Manin-Mumford Conjecture.** If \( (X \cap \mathcal{A}_{\text{tor}})^{\text{zar}} = X \), then \( \codim_{\mathcal{A}_X}(X) \leq \dim S \).

**Conjecture 11.1.** Assume \( S, \pi_S, \) and \( X \) are defined over \( \overline{\mathbb{Q}} \). If \( (X \cap \mathcal{A}_{\text{tor}})^{\text{zar}} = X \), then \( X^{\text{des}}(1) \) is Zariski dense in \( X \).

The goal is to reduce the Relative Manin-Mumford Conjecture to Conjecture 11.1.

**Proposition 11.2.** Conjecture 11.1 implies the Relative Manin-Mumford Conjecture. More precisely for \( X \subseteq \mathcal{A} \to S \) as in the Relative Manin-Mumford Conjecture, we have \( \codim_{\mathcal{A}_X}(X) \leq \dim S \) if Conjecture 11.1 holds for any irreducible subvariety \( X' \subseteq \mathcal{A}_{g'} \) defined over \( \overline{\mathbb{Q}} \) with \( 1 \leq g' \leq g \), \( \dim X' \leq \dim X \) and \( \dim \pi'(X') \leq \dim S \).

**Proof.** First let us note that by standard specialization argument, if relative Manin-Mumford holds for \( \text{all} \) \( S, \pi_S \) and \( X \) defined over \( \overline{\mathbb{Q}} \), then it holds for all \( S, \pi_S \) and \( X \) defined over \( \mathbb{C} \).

Let \( X \subseteq \mathcal{A} \to S \) be as in the relative Manin-Mumford conjecture which is defined over \( \overline{\mathbb{Q}} \). Note that \( \mathcal{A}_X \to S \) itself is an abelian scheme. Replacing \( \mathcal{A} \to S \) by \( \mathcal{A}_X \to S \), we may assume \( \mathcal{A}_X = \mathcal{A} \). It suffices to prove \( \codim_{\mathcal{A}}(X) \leq \dim S \).

Let us prove it by induction on \( \dim S, g \). When \( \dim S = 0 \) it follows from the classical Manin-Mumford conjecture (first proven by Raynaud [Ray83] and then re-proven by many others). The conclusion for the case \( g = 0 \) is easily true.

\[ [11] \text{Here } \pi'^* : \mathbb{A}_{g'} \to \mathbb{A}_g \text{ is the universal abelian variety.} \]
In general, recall the modular map (4.2)

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi} & \mathfrak{A}_g \\
\pi_S & \downarrow & \pi \\
S & \xrightarrow{\varphi_S} & \mathfrak{H}_g
\end{array}
\]

Denote by \( B = \varphi_S(S) \). Then \( \varphi(A) = \pi^{-1}(B) =: \mathfrak{A}_g|_B \). Now \((X \cap A_{\text{tor}})^{\text{Zar}} = X\) implies \((\varphi(X) \cap \varphi(A_{\text{tor}}))^{\text{Zar}} = \varphi(X)\). Applying Conjecture 11.1 to \( \varphi(X) \subseteq \varphi(A) \), we get that \( \varphi(X)^{\text{deg}}(1) \) is Zariski dense in \( \varphi(X) \). Hence by Theorem 7.1 we have \( \varphi(X)^{\text{deg}}(1) = \varphi(X) \).

Applying Theorem 8.1 to \( t = 1 \) and \( \varphi(X) \), we get an abelian subscheme \( \mathcal{B} \) of \( \mathfrak{A}_g|_B \rightarrow B \) (whose relative dimension we denote by \( g_B \)) such that for the quotient abelian scheme \( p_B: \mathfrak{A}_g|_B \rightarrow (\mathfrak{A}_g|_B)/B \) and the modular map \( \iota_B: (\mathfrak{A}_g|_B)/B \rightarrow \mathfrak{A}_g-g_B \) (see (8.1))

\[
\begin{array}{cccc}
\mathfrak{A}_g|_B & \xrightarrow{PB} & (\mathfrak{A}_g|_B)/B & \xrightarrow{\iota_B} \mathfrak{A}_g-g_B \\
\pi|_{\mathfrak{A}_g|_B} & \downarrow & \iota_B & \downarrow \pi|_{\mathfrak{A}_g-g_B} \\
B & \xrightarrow{\text{id}_B} & B & \xrightarrow{\iota_{B,G}} \mathfrak{H}_g-g_B
\end{array}
\]

we have that \( \iota_B \circ p_B \) is not generically finite and

\[
\dim(\iota_B \circ p_B)(\varphi(X)) < \dim \varphi(X) - g_B + 1.
\]

Hence we have (denote by \( \mathfrak{A}_{g-g_B}|_{\iota_{B,G}(B)} = (\pi')^{-1}(\iota_{B,G}(B)) \))

\[
\begin{align*}
& (\text{codim}_{\mathcal{A}}(X) - \dim S) - (\text{codim}_{\mathfrak{A}_{g-g_B}|_{\iota_{B,G}(B)}}(\iota_B \circ p_B)(\varphi(X)) - \dim \iota_{B,G}(B)) \\
& = (\text{codim}_{\mathcal{A}}(X) - \dim S - \dim X) - \dim(\mathfrak{A}_{g-g_B}|_{\iota_{B,G}(B)}) - \dim \iota_{B,G}(B) - \dim(\iota_B \circ p_B)(\varphi(X)) \\
& = g - \dim X - (g - g_B) + \dim(\iota_B \circ p_B)(\varphi(X)) \\
& = - \dim X + \dim(\iota_B \circ p_B)(\varphi(X)) + g_b \\
& \leq - \dim \varphi(X) + \dim(\iota_B \circ p_B)(\varphi(X)) + g_B \leq 0,
\end{align*}
\]

and so

\[
\text{codim}_{\mathfrak{A}_{g-g_B}|_{\iota_{B,G}(B)}}(\iota_B \circ p_B)(\varphi(X)) \leq \dim \iota_{B,G}(B) \Rightarrow \text{codim}_{\mathcal{A}}(X) \leq \dim S.
\]

If \( \dim \iota_{B,G}(B) < \dim B \), then we have \( \dim \iota_{B,G}(B) < \dim S \) since \( \dim B = \dim \varphi_S(S) \leq \dim S \). Now we can apply the induction hypothesis on \( \dim S \) to get

\[
\text{codim}_{\mathfrak{A}_{g-g_B}|_{\iota_{B,G}(B)}}(\iota_B \circ p_B)(\varphi(X)) \leq \dim \iota_{B,G}(B). \]

So \( \text{codim}_{\mathcal{A}}(X) \leq \dim S \) by (11.3).

If \( \dim \iota_{B,G}(B) = \dim B \), then \( \iota_{B,G} \) is generically finite. So \( \iota_B \) is generically finite. But \( \iota_B \circ p_B \) is not generically finite. So \( g_B > 0 \). Moreover

\[
\dim(\iota_B \circ p_B)(\varphi(X)) = \dim p_B(\varphi(X)) \geq \dim \varphi(X) - g_B,
\]

and hence \( \dim p_B(\varphi(X)) = \dim \varphi(X) - g_B \) by (11.2). So \( \varphi(X) + B = \varphi(X) \). Now consider the quotient \( q: \mathfrak{A}_g|_B \rightarrow (\mathfrak{A}_g|_B)/B \). The assumption \((X \cap A_{\text{tor}})^{\text{Zar}} = X \) implies

\[
(q(\varphi(X)) \cap ((\mathfrak{A}_g|_B)/B)_{\text{tor}})^{\text{Zar}} = q(\varphi(X)).
\]

\[\text{[12]} \text{Note that } \varphi \text{ and } \varphi_S \text{ are defined over } \overline{\mathbb{Q}}, \text{ dim } \varphi(X) \leq \dim X \text{ and } \dim \pi(\varphi(X)) = \dim \varphi_S(S) \leq \dim S.\]
As \((\mathfrak{A}_{g|B})/B \to B\) has relative dimension \(g - g_B < g\), we can apply the induction hypothesis on \(g\) to get \(\text{codim}(\mathfrak{A}_{g|B})/B(q(\varphi(X))) \leq \dim B\). Hence \(\text{codim}(\mathfrak{A}_{g|B})/B(\varphi(X)) \leq \dim B\). As both \(\iota/B\) and \(\iota/B,G\) are generically finite, \((11.3)\) implies \(\text{codim}(\mathcal{A}(X)) \leq \dim S\).

\[\Box\]

**APPENDIX A. DISCUSSION WHEN THE BASE TAKES SOME SIMPLE FORM**

Let \(S\) be an irreducible subvariety over \(\mathbb{C}\) and let \(\pi_S : A \to S\) be an abelian scheme of relative dimension \(g \geq 1\). Let \(X\) be a closed irreducible subvariety of \(A\) with \(\pi_S(X) = S\). Recall the modular map \((4.2)\)

\[
\begin{array}{ccc}
A & \xrightarrow{\varphi} & \mathfrak{A}_g \\
\pi_S & \downarrow & \\
S & \xrightarrow{\varphi_S} & \mathfrak{A}_g \\
\end{array}
\]

**Definition A.1.** Denote by \(\langle X \rangle_{\text{gen-sp}}\) the smallest subvariety of \(A\) of the following form which contains \(X\): Up to taking a finite covering of \(S\), we have \(\langle X \rangle_{\text{gen-sp}} = \sigma + Z + \mathcal{A}',\) where \(\mathcal{A}'\) is an abelian subscheme of \(A \to S\), \(\sigma\) is a torsion section of \(A \to B\), and \(Z = Z \times S\) where \(C \times S\) is the largest constant abelian subscheme of \(A \to S\) and \(Z \subseteq C\).

If we furthermore require the \(Z\) to be a point, then we obtain the definition of \(\langle X \rangle_{\text{sg}}\) (see Definition 1.5). For a fixed \(X\), it is easy to see \(\langle X \rangle_{\text{gen-sp}} \subseteq \langle X \rangle_{\text{sg}}\).

Let \(\mathcal{A}_X\) be the translate of an abelian subscheme of \(A \to S\) by a torsion section which contains \(X\), minimal for this property. Then \(\mathcal{A}_X \to S\) itself of an abelian scheme, whose relative dimension we denote by \(g_X\). It is easy to see \(\langle X \rangle_{\text{sg}} \subseteq \mathcal{A}_X\).

**Proposition A.2.** Assume that \(S = B\) is a locally closed irreducible subvariety of \(\mathfrak{A}_g\), and \(A = \mathfrak{A}_{g|B} := \pi^{-1}(B)\). Assume \(t \leq 1\). Then we have

(i) If \(\dim \langle X \rangle_{\text{gen-sp}} - \dim B < \dim X + t\), then \(X = X^{\text{deg}}(t)\).

(ii) Assume \(\dim B = 1\) or the connected algebraic monodromy group of \(B\) is simple. Assume furthermore \(t \leq 0\). Then the converse of (i) holds, namely \(\dim \langle X \rangle_{\text{gen-sp}} - \dim B < \dim X + t\) if \(X = X^{\text{deg}}(t)\).

(iii) Assume \(\dim B = 1\) or the connected algebraic monodromy group of \(B\) is simple. Assume furthermore that \(\mathcal{A}_X = \mathfrak{A}_{g|B}\) and no non-trivial abelian subscheme of \(\mathfrak{A}_{g|B} \to B\) stabilizes \(X\). Then the converse of (i) holds for \(t = 1\), namely \(\dim \langle X \rangle_{\text{gen-sp}} - \dim B < \dim X + 1\) if \(X = X^{\text{deg}}(1)\).

In particular if \(\dim B = 1\), then \(X = X^{\text{deg}}\) if and only if \(X = \langle X \rangle_{\text{gen-sp}}\). If furthermore \(X\) satisfies the hypotheses in (iii), then \(X = X^{\text{deg}}(1)\) if and only if \(\text{codim}(\langle X \rangle_{\text{gen-sp}}(X)) \leq 1\).

Before proving this proposition, let us see its direct corollary on the generic rank of the Betti rank (combined with Theorem 10.1 and Theorem 7.1).

**Corollary A.3.** Assume that \(\varphi_S(S)\) has dimension 1 or has simple connected algebraic monodromy group. Let \(l \in \{1, \ldots, \min(\dim(\varphi(X)), g)\}\). Then

\[
\max_{x \in X^{\text{sm}}(\mathbb{C}) \cap \mathcal{A}_g} \text{rank}_g(db_{\mathbb{E}|X})_x < 2l \Leftrightarrow \dim(\langle \varphi(X) \rangle_{\text{gen-sp}} - \varphi_S(S)) < l.
\]

**Remark A.4.** Let \(\mathcal{C}_g\) be the universal curve embedded in \(\mathfrak{A}_g\). Use notations in Theorem 1.2 and Theorem 1.2'. It is clearly true that \(\langle \mathcal{C}_S \rangle_{\text{gen-sp}} = \mathfrak{A}_{mg} \times \mathfrak{A}_{mg}\) for all \(m \geq 1\). If \(S\) has dimension 1 or has simple connected algebraic monodromy group (for example when \(S\) is the whole Torelli locus), then Corollary A.3 has the following immediate corollaries: \(\mathcal{C}_S^{[m]}\) has

\[\text{[13]}\text{Here } S \text{ is a subvariety of } \mathfrak{A}_g, \text{ and } \mathfrak{A}_g \text{ is seen as a subvariety of } \mathfrak{A}_{mg} \text{ via the diagonal embedding.} \]
maximal generic Betti rank for all \( m \geq 3 \) and \( g \geq 2 \), \( D^m(C_S^{[m+1]}) \) has maximal generic Betti rank for all \( m \geq 4 \) and \( g \geq 2 \), and \( C_S^{[m]} \rightarrow C_S^{[m]} \) has maximal generic Betti rank for all \( m \geq 4 \) when \( g \geq 5 \), for all \( m \geq 5 \) when \( g = 4 \) and for all \( m \geq 6 \) for \( g = 3 \).

**Proof of Proposition A.2.**

(i) Let \( C \) be the largest isotrivial abelian subscheme of \( A_X \rightarrow B \).

Then there exists an abelian subscheme \( B \) of \( A_X \rightarrow B \) such that \( B \cap C \) is a finite group scheme over \( B \) and \( A_X = B + C \). Denote by \( g_B \) the relative dimension of \( B \rightarrow B \).

Construct the diagram (8.1)

\[
\begin{array}{cccc}
A_X & \xrightarrow{p_B} & A_X/B & \xrightarrow{i/B} \mathfrak{A}_g - g_B \\
\pi|_{A_X} & & \pi_B & \downarrow \\
B & \xrightarrow{id_B} & B & \xrightarrow{i/B : G} \mathfrak{A}_g - g_B
\end{array}
\]

It is clear that \( i/B \circ p_B \) is not generically finite: either \( B \) is non-trivial, or \( A_X \rightarrow B \) is isotrivial, forcing \( i/B : G(B) \) to be a point.

By Theorem 8.1 it suffices to prove \( \dim(i/B \circ p_B)(X) < \dim X - g_B + t \).

Take into consideration \( \pi : \mathfrak{A}_g \rightarrow \mathfrak{A}_g \). Then by our assumption (and \( \pi(X) = B \)), it suffices to prove

\[
\dim(i/B \circ p_B)(X) + g_B = \dim(X)_{\text{gen} - \text{sp}} - \dim \pi(X).
\]

Let us prove (A.1). By choice, \( B \) is the smallest abelian subscheme of \( A_X \rightarrow X \) such that \( A_X/B \) is isotrivial. Hence in the definition of \( (X)_{\text{gen} - \text{sp}} \) (Definition A.1), the \( \sigma + A' \) is precisely our \( B \).\([14]\)

Moreover \( \pi_B \) is an isotrivial abelian scheme of choice of \( B \). Hence \( i/B : G(B) \) is a point, and thus \( (i/B \circ p_B)(X) \) is contained in one fiber of \( \pi' \). Since the morphism \( p_B|_{C} \) is finite by choice of \( B \), we have that the \( Z \) in the definition of \( (X)_{\text{gen} - \text{sp}} \) (Definition A.1) satisfies \( \dim Z = \dim p_B(Z) \) and \( p_B(Z) = i/B^{-1}((i/B \circ p_B)(X)) \).

Now we are ready to prove (A.1). By the discussion in the last two paragraphs, we have

\[
\dim(X)_{\text{gen} - \text{sp}} - \dim \pi(X) = \dim(B \rightarrow B) + \dim(Z \rightarrow B)
= g_B + \dim(p_B(Z) \rightarrow B)
= g_B + \dim(i/B \circ p_B)(X).
\]

Hence we are done.

(ii) + (iii) Assume \( X = X_{\text{deg}(t)} \). We wish to prove \( \dim(X)_{\text{gen} - \text{sp}} - \dim B < \dim X + t \).

For both (ii) and (iii), we can apply Theorem 8.1. Then there exists an abelian subscheme \( B \) of \( A_X \rightarrow B \) (whose relative dimension we denote by \( g_B \)) such that for the map \( i/B \circ p_B \) constructed in (8.1)

\[
\begin{array}{cccc}
A_X & \xrightarrow{p_B} & A_X/B & \xrightarrow{i/B} \mathfrak{A}_g - g_B \\
\pi|_{A_X} & & \pi_B & \downarrow \\
B & \xrightarrow{id_B} & B & \xrightarrow{i/B : G} \mathfrak{A}_g - g_B
\end{array}
\]

we have that \( i/B \circ p_B \) is not generically finite and

\[
\dim(i/B \circ p_B)(X) < \dim X - g_B + t.
\]

\([14]\) Recall that in the definition of \( A_X \), we already have the translation by a torsion section. Hence as a subvariety of \( \mathfrak{A}_g|_B \), our \( B \) is already the translate of an abelian subscheme of \( \mathfrak{A}_g|_B \rightarrow B \) by a torsion section.
For $B = \pi(X)$, let us temporarily assume that $\iota_{/B,G}(B)$ is a point and finish the proof. Denote by $b' = \iota_{/B,G}(B)$, and choose $z'$ to be a point in $(\pi')^{-1}(b')$. Then $(X)_{gen-sp} \subseteq p_B^{-1} \left( \iota_B^{-1} (\iota_{/B} \circ p_B)(X) \right)$. So

$$\dim(X)_{gen-sp} - \dim B \leq \dim(B \to B) + \dim(\iota_{/B} \circ p_B)(X)$$

$$< \dim(B \to B) + \dim X - g_B + t \quad \text{by (A.2)}$$

$$= \dim X + t.$$

Let $G_B$ be the connected algebraic monodromy group of $B$. Denote by $G_Q$ the Mumford-Tate group of $B \subseteq \mathbb{A}_g$. Then $G_B < G^\text{der}_Q$.

For each $a \in \iota_{/B,G}(B)$, the restriction $(A_X/B)|_{\iota_{/B,G}^{-1}(a)}$ is an isotrivial abelian scheme over $\iota_{/B,G}(a)$. Take $a$ to be Hodge generic in $\iota_{/B,G}(B)$, then by a theorem of Deligne-André [And92, §5, Theorem 1], the connected algebraic monodromy group $H$ of $\iota_{/B,G}^{-1}(a)$ is a normal subgroup of $G^\text{der}_Q$. Moreover $H < G_B$ since $\iota_{/B,G}^{-1}(a) \subseteq B$. Thus $H < G_B$.

Recall the assumption: either $\dim B = 1$ or $H$ is simple. In the first case, we have that $\dim \iota_{/B,G}(B)$ is $0$ or $\dim B$. In the second case, either $H = G_B$ or $H = 1$ by the previous paragraph. If $H = 1$, then $\dim \iota_{/B,G}^{-1}(a) = 0$. Hence $\iota_{/B,G}$ is quasi-finite, and thus $\dim \iota_{/B,G}(B) = \dim B$. If $H = G_B$, then the fact that $(A_X/B)|_{\iota_{/B,G}^{-1}(a)} \to \iota_{/B,G}^{-1}(a)$ is isotrivial implies that $A_X/B \to B$ is isotrivial. Hence $\iota_{/B,G}(B)$ is a point by definition of $\iota_{/B,G}$.

To sum it up, $\dim \iota_{/B,G}(B)$ is either $0$ or $\dim B$. The case $\dim \iota_{/B,G}(B) = 0$ is treated above. So it remains to treat the case $\dim \iota_{/B,G}(B) = \dim B$. In this case $\iota_{/B,G}$ is generically finite. So $\iota_B$ is generically finite. Thus $\dim(\iota_{/B} \circ p_B)(X) = \dim p_B(X)$.

Hence

$$\dim(\iota_{/B} \circ p_B)(X) = \dim p_B(X) \geq \dim X - \dim(B \to B) = \dim X - g_B.$$

For (ii), we have $t \leq 0$. So (A.2) and (A.3) contradict each other. Thus we are done already.

For (iii), we have $t = 1$. Hence (A.2) and (A.3) together imply $\dim(\iota_{/B} \circ p_B)(X) = \dim X - g_B$. Combined with $\dim \iota_{/B,G}(B) = \dim B$, we get that $X$ is stable under $B$. But no non-trivial abelian subscheme of $\mathfrak{g}|_B \to B$ stabilizes $X$, so $B$ is trivial. But then $\mathfrak{g}|_{\iota_{/B,G}^{-1}(a)} \to \iota_{/B,G}^{-1}(a)$ is isotrivial. But then $\iota_{/B} \circ p_B = \iota_B$ is generically finite, contradicting the choice of $B$.

\[ \square \]

**References**


