A CONSEQUENCE OF THE RELATIVE BOGOMOLOV CONJECTURE

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Abstract. We propose a formulation of the relative Bogomolov conjecture and show that it gives an affirmative answer to a question of Mazur's concerning the uniformity of the Mordell–Lang conjecture for curves. In particular we show that the relative Bogomolov conjecture implies the uniform Manin–Mumford conjecture for curves. The proof is built up on our previous work [DGH20].

Contents

1. Introduction

Let F be a field of characteristic 0. A smooth curve C defined over F is a geometrically irreducible, smooth, projective curve defined over F. We denote by $\mathrm{Jac}(C)$ the Jacobian of C. The following conjecture is a question posed by Mazur [Maz86, top of page 234].

Conjecture 1.1. *Let $g \geq 2$ be an integer. Then there exists a constant $c(g) \geq 1$ with the following property. Let C be a smooth curve of genus g defined over F, let $P_0 \in C(F)$, and let Γ be a subgroup of $\mathrm{Jac}(C)(F)$ of finite rank $\mathrm{rk}(\Gamma)$. Then*

$$\#(C(F) - P_0) \cap \Gamma \leq c(g)^{1+\mathrm{rk}(\Gamma)} \tag{1.1}$$

where $C - P_0$ is viewed as a curve in $\mathrm{Jac}(C)$ via the Abel–Jacobi map based at P_0.

When $F = \overline{\mathbb{Q}}$, based on Vojta's method, Rémond [Rém00a] has proved an explicit upper bound of $\#(C(\overline{\mathbb{Q}}) - P_0) \cap \Gamma$. Apart from g and $\mathrm{rk}(\Gamma)$, Rémond's bound depends also on a suitable height of $\mathrm{Jac}(C)$.

Two particular consequences of Conjecture 1.1 are:

(i) Take F a number field and $\Gamma = \mathrm{Jac}(C)(F)$. By the Mordell–Weil Theorem $\mathrm{Jac}(C)(F)$ is a finitely generated abelian group. Then (1.1) becomes a bound on the number of rational points $\#C(F) \leq c(g)^{1+\mathrm{rk}\,\mathrm{Jac}(C)(F)}$.

(ii) If $F = \mathbb{C}$ and $\Gamma = \mathrm{Jac}(C)_{\mathrm{tor}}$, then (1.1) becomes $\#(C(\mathbb{C}) - P_0) \cap \mathrm{Jac}(C)_{\mathrm{tor}} \leq c(g)$, the uniform Manin–Mumford conjecture for curves in their Jacobians.

2000 *Mathematics Subject Classification.* 11G30, 11G50, 14G05, 14G25.

In a recent work, we proved (1.1) provided that the modular height of the curve in question is large in terms of g; see [DGH20, Theorem 1.2]. Prior to our work and for torsion points, *i.e.* $F = \mathbb{C}$ and $\Gamma = \mathrm{Jac}(C)_{\mathrm{tor}}$, the desired bound (1.1) was proved by DeMarco–Krieger–Ye [DKY20] for any genus 2 curve admitting a degree-two map to an elliptic curve when the Abel–Jacobi map is based at a Weierstrass point.

The goal of this note is to give a precise statement for the folklore *relative Bogomolov conjecture*, and prove that it implies the full Conjecture 1.1.

1.1. **The Relative Bogomolov Conjecture.** We start by proposing a formulation for the relative Bogomolov conjecture.

Let S be a regular, irreducible, quasi-projective variety. Let $\pi\colon \mathcal{A} \to S$ be an abelian scheme of relative dimension $g \geq 1$, namely a proper smooth group scheme whose fibers are abelian varieties. Let $\mathcal{L}$ be a symmetric relatively ample line bundle on $\mathcal{A}/S$. Assume that S, π and $\mathcal{L}$ are all defined over $\overline{\mathbb{Q}}$.

For each $s \in S(\overline{\mathbb{Q}})$, the line bundle $\mathcal{L}_s$ on the abelian variety $\mathcal{A}_s = \pi^{-1}(s)$ is symmetric and ample; note that $\mathcal{A}_s$ is defined over $\overline{\mathbb{Q}}$. Tate's Limit Process provides a fiberwise Néron–Tate height $\hat{h}_{\mathcal{A}_s,\mathcal{L}_s}\colon \mathcal{A}_s(\overline{\mathbb{Q}}) \to [0,\infty)$; it vanishes precisely on the torsion points in $\mathcal{A}_s(\overline{\mathbb{Q}})$. Finally define $\hat{h}_{\mathcal{L}}\colon \mathcal{A}(\overline{\mathbb{Q}}) \to [0,\infty)$ to be $P \mapsto \hat{h}_{\mathcal{A}_{\pi(P)},\mathcal{L}_{\pi(P)}}(P)$.

Let η be the generic point of S and fix an algebraic closure of the function field of S. For any subvariety X of $\mathcal{A}$ that dominates S, denote by $X_{\overline{\eta}}$ the geometric generic fiber of X. In particular, $\mathcal{A}_{\overline{\eta}}$ is an abelian variety over an algebraically closed field.

Conjecture 1.2 (Relative Bogomlov Conjecture)**.** *Let X be an irreducible subvariety of $\mathcal{A}$ defined over $\overline{\mathbb{Q}}$ that dominates S. Assume that $X_{\overline{\eta}}$ is irreducible and not contained in any proper algebraic subgroup of $\mathcal{A}_{\overline{\eta}}$. If $\mathrm{codim}_{\mathcal{A}} X > \dim S$, then there exists $\epsilon > 0$ such that*

$$X(\epsilon;\mathcal{L}) := \{x \in X(\overline{\mathbb{Q}}) : \hat{h}_{\mathcal{L}}(x) \leq \epsilon\}$$

is not Zariski dense in X.

The name *Relative Bogomolov Conjecture* is reasonable: the same statement with $\epsilon = 0$ is precisely the relative Manin–Mumford conjecture proposed by Pink [Pin05, Conjecture 6.2] and Zannier [Zan12], which is proved when $\dim X = 1$ in a series of papers [MZ12, MZ14, MZ15, CMZ18, MZ20].

The classical Bogomolov conjecture, proved by Ullmo [Ull98] and S. Zhang [Zha98a], is precisely Conjecture 1.2 for $\dim S = 0$. When $\dim S = 1$ and X is the image of a section, Conjecture 1.2 is equivalent to S. Zhang's conjecture in his 1998 ICM note [Zha98b, §4] if $\mathcal{A}_{\overline{\eta}}$ is simple and is proved by DeMarco–Mavraki [DM20, Theorem 1.4] if $\mathcal{A} \to S$ is isogenous to a fibered power of an elliptic surface. In general Conjecture 1.2 is still open.

1.2. **Main result.** Our main result, which is built up on [DGH20], is the following theorem.

Theorem 1.3. *Assume that the Relative Bogomolov Conjecture, i.e., Conjecture 1.2, holds true. Then Conjecture 1.1 holds true.*

We emphasize that what the Relative Bogomolov Conjecture does is to *complement* [DGH20] to prove the full Conjecture 1.1. More precisely, [DGH20, Thm.1.2] proves Conjecture 1.1 for curves C whose modular height is larger than a number $\delta = \delta(g)$

depending only on the genus g, and the Relative Bogomolov Conjecture can handle curves with small modular height.

The proof of Theorem 1.3 is as follows. First we reduce Conjecture 1.1 to the case $F = \overline{\mathbb{Q}}$ by using a specialization result of Masser [Mas89]. This is executed in §3. When $F = \overline{\mathbb{Q}}$, our proof follows closely and uses our previous work [DGH20], where we proved (1.1) for all curves whose modular height is bounded below by a constant depending only on g; see [DGH20, Theorem 1.2]. The key point in [DGH20] is to prove a height inequality, which we cite as Theorem 2.2 in the current paper. As is shown by the proof of [DGH20, Proposition 7.1], the extra hypothesis on the modular height of curves required in [DGH20, Theorem 1.2] is necessary because of the constant term in this height inequality. In this paper, we show that this constant term can be removed if we assume the Relative Bogomolov Conjecture; see Proposition 2.3 for a precise statement. Then by following the framework presented in [DGH20] we can prove our Theorem 1.3 for $F = \overline{\mathbb{Q}}$.

Acknowledgements. We would like to thank the referee for their comments. Vesselin Dimitrov has received funding from the European Union's Seventh Framework Programme (FP7/2007–2013) / ERC grant agreement n° 617129. Ziyang Gao has received fundings from the French National Research Agency grant ANR-19-ERC7-0004, and the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement n° 945714). Philipp Habegger has received funding from the Swiss National Science Foundation (grant n° 200020_184623).

2. Proof of the main result for $F = \overline{\mathbb{Q}}$

In this section we prove Theorem 1.3 when $F = \overline{\mathbb{Q}}$.

Theorem 2.1. *The Relative Bogomolov Conjecture, i.e., Conjecture 1.2, implies Conjecture 1.1 for* $F = \overline{\mathbb{Q}}$.

The proof follows closely and is based on our previous work [DGH20].

2.1. **Basic setup.** Fix an integer $g \geq 2$. Let $\mathbb{M}_g$ be the fine moduli space of smooth projective curves of genus g with level-4-structure, *cf.* [ACG11, Chapter XVI, Theorem 2.11 (or above Proposition 2.8)], [DM69, (5.14)], or [OS80, Theorem 1.8]. It is known that $\mathbb{M}_g$ is an irreducible regular quasi-projective variety defined over $\overline{\mathbb{Q}}$, and $\dim \mathbb{M}_g = 3g - 3$. This variety solves the underlying moduli problem. There exists a universal curve $\mathfrak{C}_g$ over $\mathbb{M}_g$, it is smooth and proper over $\mathbb{M}_g$ with fibers that are smooth curves of genus g. Moreover, it is equipped with level 3-structure.

Let $\mathrm{Jac}(\mathfrak{C}_g)$ be the relative Jacobian of $\mathfrak{C}_g \to \mathbb{M}_g$. It is an abelian scheme equipped with a natural principal polarization and with level-3-structure; see [MFK94, Proposition 6.9].

Let $\mathbb{A}_g$ be the fine moduli space of principally polarized abelian varieties of dimension g with level-3-structure. It is known that $\mathbb{A}_g$ is an irreducible regular quasi-projective variety defined over $\overline{\mathbb{Q}}$; see [MFK94, Theorem 7.9 and below] or [OS80, Theorem 1.9]. Here too we have a universal object, the universal abelian scheme $\pi\colon \mathfrak{A}_g \to \mathbb{A}_g$ of fiber dimension g. There exists a relatively very ample line bundle $\mathcal{L}$ on $\mathfrak{A}_g/\mathbb{A}_g$ satisfying $[-1]^*\mathcal{L} = \mathcal{L}$; see [Ray70, Théorème XI 1.4]. By [Gro61, Proposition 4.4.10(ii) and

Proposition 4.1.4], we then have a closed immersion $\mathfrak{A}_g \to \mathbb{P}^n_{\overline{\mathbb{Q}}} \times \mathbb{A}_g$ over $\mathbb{A}_g$ arising from $\mathcal{L} \otimes \pi^*\mathcal{M}^{\otimes p}$, where $\mathcal{M}$ is an ample line bundle on $\mathbb{A}_g$, for some integer $p \geq 1$.

Attaching the Jacobian to a smooth curve induces the Torelli morphism $\tau\colon \mathbb{M}_g \to \mathbb{A}_g$. The famous Torelli theorem states that, absent level structure, the Torelli morphism is injective on $\mathbb{C}$-points. In out setting, τ is finite-to-1 on $\mathbb{C}$-points, *cf.* [OS80, Lemma 1.11]. As $\mathbb{A}_g$ is a fine moduli space we have the following Cartesian diagram

$$\begin{array}{ccc} \mathrm{Jac}(\mathfrak{C}_g) & \longrightarrow & \mathfrak{A}_g \\ \big\downarrow & \lrcorner & \big\downarrow \pi \\ \mathbb{M}_g & \xrightarrow[\tau]{} & \mathbb{A}_g \end{array} \tag{2.1}$$

Suppose we have an immersion $\mathbb{A}_g \subseteq \mathbb{P}^N_{\overline{\mathbb{Q}}}$ defined over $\overline{\mathbb{Q}}$, such an immersion exists. We write $\overline{\mathbb{A}_g}$ for the Zariski closure of $\mathbb{A}_g$ in $\mathbb{P}^N_{\overline{\mathbb{Q}}}$. Then the absolute logarithmic Weil height on $\mathbb{P}^N_{\overline{\mathbb{Q}}}(\overline{\mathbb{Q}})$ restricts to a height function $h_{\overline{\mathbb{A}_g}}\colon \overline{\mathbb{A}_g}(\overline{\mathbb{Q}}) \to \mathbb{R}$. Thus $h_{\overline{\mathbb{A}_g}}$ represents the Weil height attached to the ample line bundle obtained by restricting $\mathcal{O}(1)$ on $\mathbb{P}^N_{\overline{\mathbb{Q}}}$ to $\overline{\mathbb{A}_g}$. Moreover, $h_{\overline{\mathbb{A}_g}}$ takes values in $[0,\infty)$ as the Weil height function is non-negative.

For $M \in \mathbb{N} = \{1,2,3,\ldots\}$ we write $\mathfrak{A}_g^{[M]}$ for the M-fold fibered power $\mathfrak{A}_g \times_{\mathbb{A}_g} \cdots \times_{\mathbb{A}_g} \mathfrak{A}_g$ over $\mathbb{A}_g$. Then $\mathfrak{A}_g^{[M]} \to \mathbb{A}_g$ is an abelian scheme.

Similarly, for any morphism of schemes $S \to \mathbb{M}_g$, the base change is $\mathfrak{C}_S = \mathfrak{C}_g \times_{\mathbb{M}_g} S$. Furthermore, the M-fold fibered power $\mathfrak{C}_S \times_S \cdots \times_S \mathfrak{C}_S$ is denoted by $\mathfrak{C}_S^{[M]}$. The morphism $\mathfrak{C}_S \to S$ is smooth and therefore open, thus so is $\mathfrak{C}_S^{[M]} \to S$. Each fiber of the latter morphism is a product of curves and thus irreducible. We conclude that $\mathfrak{C}_S^{[M]}$ is irreducible if S is.

Suppose C is a smooth curve defined over a field and $A = \mathrm{Jac}(C)$. The difference morphism $C^{M+1} \to A^M$ determined by

$$(P_0, P_1, \ldots, P_M) \mapsto (P_1 - P_0, \ldots, P_M - P_0). \tag{2.2}$$

is well-defined; we do not need to specify a base point for the Abel–Jacobi map. It is an astonishingly powerful tool in diophantine geometry.

Let us make this more precise in our relative setting. For each morphism of schemes $S \to \mathbb{M}_g \xrightarrow{\tau} \mathbb{A}_g$, we briefly recall the construction of the proper S-morphism

$$\mathscr{D}_M\colon \mathfrak{C}_S^{[M+1]} \to \mathfrak{A}_g^{[M]} \times_{\mathbb{A}_g} S \tag{2.3}$$

from [DGH20, §6.1]. Indeed by the proof of [MFK94, Proposition 6.9] there is a morphism $\iota\colon \mathfrak{C}_S \to \mathrm{Pic}^1(\mathfrak{C}_S/S)$ to the line bundles of degree 1. Let f_1, f_2 be any two morphisms from an S-scheme T with target $\mathfrak{C}_S$. Then the difference of $\iota \circ f_1$ and $\iota \circ f_2$ is a morphism $T \to \mathrm{Pic}^0(\mathfrak{C}_S/S) = \mathrm{Jac}(\mathfrak{C}_S)$. So we get a morphism $\mathfrak{C}_S^{[2]} \to \mathrm{Jac}(\mathfrak{C}_S)$. We compose with the natural morphism $\mathrm{Jac}(\mathfrak{C}_S) \to \mathfrak{A}_g \times_{\mathbb{A}_g} S$ coming from the Torelli morphism. This construction extends to $M+1$ section and yields (2.3). Fiberwise the morphism $\mathscr{D}_M$ behaves on points as (2.2).

The morphism $\mathscr{D}_M$ in (2.3) is called the *M-th Faltings–Zhang* map. Note that if S is irreducible, then $\mathscr{D}_M(\mathfrak{C}_S^{[M+1]})$ is an irreducible subvariety of the abelian scheme $\mathfrak{A}_g^{[M]} \times_{\mathbb{A}_g} S \to S$.

We will use the following theorem, which we proved in [DGH20], by applying [DGH20, Theorem 1.6] to [Gao20, Theorem 1.2'].

Theorem 2.2. *Let S be an irreducible variety with a (not necessarily dominant) quasi-finite morphism $S \to \mathbb{M}_g$. Assume $g \geq 2$ and $M \geq 3g-2$. Then there exist constants $c > 0$ and $c' \geq 0$ and a Zariski open dense subset U of $\mathscr{D}_M(\mathfrak{C}_S^{[M+1]})$ with*

$$\hat{h}_{\mathcal{L}}(P) \geq c h_{\overline{\mathbb{A}_g}}(\pi(P)) - c' \quad \text{for all} \quad P \in U(\overline{\mathbb{Q}}). \tag{2.4}$$

2.2. **A strengthend height inequality.** We use the notation in the previous subsection. Here we prove that the Relative Bogomolov Conjecture allows us to furthermore strengthen the height inequality given by Theorem 2.2. Let $g \geq 2$.

Proposition 2.3. *Let S be an irreducible variety with a (not necessarily dominant) quasi-finite morphism $S \to \mathbb{M}_g$. Let M be an integer satisfying $M \geq 3g-1$ if $g = 2$ and $M \geq 3g-2$ if $g \geq 3$.*

Assume that the Relative Bogomolov Conjecture, Conjecture 1.2, holds true. Then there exist a constant $c > 0$ and a Zariski open dense subset U of $\mathscr{D}_M(\mathfrak{C}_S^{[M+1]})$ with

$$\hat{h}_{\mathcal{L}}(P) \geq c \max\{1, h_{\overline{\mathbb{A}_g}}(\pi(P))\} \quad \text{for all} \quad P \in U(\overline{\mathbb{Q}}). \tag{2.5}$$

Proof. The fiber of $\mathfrak{C}_S^{[M+1]} \to S$ above $s \in S(\overline{\mathbb{Q}})$, is a product of $M+1$ curves of genus g. The Faltings–Zhang map is generically finite on this product. So we have

$$\operatorname{codim}_{\mathfrak{A}_g^{[M]} \times_{\mathbb{A}_g} S} \mathscr{D}_M(\mathfrak{C}_S^{[M+1]}) = Mg - (M+1) = (g-1)M - 1 > 3g-3 \geq \dim S. \tag{2.6}$$

Let $\overline{\eta}$ be the geometric generic point of S. Each fiber of $\mathscr{D}_M(\mathfrak{C}_S^{[M+1]}) \to S$ is the image of the $(M+1)$-fold power of a smooth curve under the Faltings–Zhang map. So all fibers are irreducible and thus so is the geometric generic fiber $\mathscr{D}_M(\mathfrak{C}_S^{[M+1]})_{\overline{\eta}}$. As a smooth curve generates its Jacobian we find that $\mathscr{D}_M(\mathfrak{C}_S^{[M+1]})_{\overline{\eta}}$ is not contained in a proper algebraic subgroup of $\mathfrak{A}_g^{[M]} \times_{\mathbb{A}_g} \overline{\eta}$. By (2.6) the image $X := \mathscr{D}_M(\mathfrak{C}_S^{[M+1]})$ satisfies the assumptions of the Relative Bogomolov Conjecture. Thus there exists $\epsilon > 0$ such that $\overline{X(\epsilon; \mathcal{L})}^{\mathrm{Zar}}$, the Zariski closure of $X(\epsilon; \mathcal{L}) = \{x \in X(\overline{\mathbb{Q}}) : \hat{h}_{\mathcal{L}}(x) \leq \epsilon\}$, is not equal to X.

Let $c > 0$, $c' \geq 0$, and U be as in Theorem 2.2. The paragraph above implies that

$$U \setminus \overline{X(\epsilon; \mathcal{L})}^{\mathrm{Zar}}$$

is still Zariski open and dense in X.

It suffices to prove that (2.5) holds true with c a positive constant that is independent of P and with U replaced $U \setminus \overline{X(\epsilon; \mathcal{L})}^{\mathrm{Zar}}$.

Take any $P \in (U \setminus \overline{X(\epsilon; \mathcal{L})}^{\mathrm{Zar}})(\overline{\mathbb{Q}})$. So $\hat{h}_{\mathcal{L}}(P) \geq \epsilon$ and $\hat{h}_{\mathcal{L}}(P) \geq c h_{\overline{\mathbb{A}_g}}(\pi(P)) - c'$.

We split up into the two cases depending on whether $\max\{1, h_{\overline{\mathbb{A}_g}}(\pi(P))\} \leq \max\{1, 2c'/c\}$ holds or does not hold.

In the first case we have

$$\hat{h}_{\mathcal{L}}(P) \geq \frac{\epsilon}{\max\{1, 2c'/c\}} \max\{1, h_{\overline{\mathbb{A}_g}}(\pi(P))\}$$

and (2.5) follows with $\epsilon / \max\{1, 2c'/c\}$ for the constant c.

In the second case we have $h_{\overline{\mathbb{A}_g}}(\pi(P)) > \max\{1, 2c'/c\}$ and hence $ch_{\overline{\mathbb{A}_g}}(\pi(P)) - c' \geq ch_{\overline{\mathbb{A}_g}}(\pi(P))/2$. Thus

$$\hat{h}_{\mathcal{L}}(P) \geq \frac{c}{2}\max\{1, h_{\overline{\mathbb{A}_g}}(\pi(P))\}.$$

Again (2.5) holds with $c/2$ for the constant c. □

Remark 2.4. *The proof of Proposition 2.3 is the only place where the Relative Bogomolov Conjecture is used in the proof of Theorem 1.3.*

2.3. **Dichotomy on the Néron–Tate distance between points on curves.** We use the notation of Subsection 2.1.

Proposition 2.5. *Let S be an irreducible closed subvariety of $\mathbb{M}_g$.*

Assume that the Relative Bogomolov Conjecture, Conjecture 1.2, holds true. Then there exist positive constants c_2, c_3, c_4 depending on S with the following property. For all $s \in S(\overline{\mathbb{Q}})$ there is a subset $\Xi_s \subseteq \mathfrak{C}_s(\overline{\mathbb{Q}})$ with $\#\Xi_s \leq c_2$ such that any $P \in \mathfrak{C}_s(\overline{\mathbb{Q}})$ satisfies one of the following cases.

(i) *We have $P \in \Xi_s$;*
(ii) *or $\#\{Q \in \mathfrak{C}_s(\overline{\mathbb{Q}}) : \hat{h}_{\mathcal{L}}(Q - P) \leq c_3^{-1}\max\{1, h_{\overline{\mathbb{A}_g}}(\tau(s))\}\} < c_4$.*

We start the numbering of the constants from c_2 to make the statement comparable to [DGH20, Proposition 7.1]. The proof of this proposition is similar to the proof of [DGH20, Proposition 7.1]. The main difference is that the height inequality Theorem 2.2 is replaced by the strengthend version Proposition 2.3, and this allows us to remove the extra hypothesis on $h_{\overline{\mathbb{A}_g}}(\tau(s))$ in [DGH20, Proposition 7.1].

Proof. We prove the proposition by induction on $\dim S$. The induction start $\dim S = 0$ is treated as part of the induction step.

We fix M as in Proposition 2.3. Then there exist a constant $c > 0$ and a Zariski open dense subset U of $X = \mathscr{D}_M(\mathfrak{C}_S^{[M+1]})$ satisfying the following property. For all $s \in S(\overline{\mathbb{Q}})$ and all $P, Q_1, \ldots, Q_M \in \mathfrak{C}_s(\overline{\mathbb{Q}})$, we have

(2.7)
$$c\max\{1, h_{\overline{\mathbb{A}_g}}(\tau(s))\} \leq \hat{h}(Q_1 - P) + \cdots + \hat{h}(Q_M - P) \text{ if } (Q_1 - P, \ldots, Q_M - P) \in U(\overline{\mathbb{Q}}).$$

Observe that $\pi_S(X) = S$, where $\pi_S\colon \mathfrak{A}_g^{[M]} \times_{\mathbb{M}_g} S \to S$ is the structure morphism. Thus $S \setminus \pi_S(U)$ is not Zariski dense in S. Let $S_1, \ldots, S_r$ be the irreducible components of the Zariski closure of $S \setminus \pi_S(U)$ in S. Then $\dim S_j \leq \dim S - 1$ for all j.

Note that if $\dim S = 0$, then $\pi_S(U) = S$ and $r = 0$. If $\dim S \geq 1$, then this proposition holds for all, if any, S_j by the induction hypothesis. So it remains to prove the conclusion of this proposition for curves above

$$s \in S(\overline{\mathbb{Q}}) \setminus \bigcup_{j=1}^{r} S_j(\overline{\mathbb{Q}}). \tag{2.8}$$

The image of s under the Torelli map is $\tau(s)$. For any $P \in \mathfrak{C}_s(\overline{\mathbb{Q}})$, we will consider $\mathfrak{C}_s - P$ as a curve inside $(\mathfrak{A}_g)_{\tau(s)}$ via the Abel–Jacobi map based at P.

The set $W = X \setminus U$ is a Zariski closed proper subset of X. The fiber W_s of W above s satisfies $W_s \subsetneq X_s = \mathscr{D}_M(\mathfrak{C}_s^{[M+1]})$ as (2.8) implies $s \in \pi_S(U(\overline{\mathbb{Q}}))$.

We define

$$\Xi_s = \{P \in \mathfrak{C}_s(\overline{\mathbb{Q}}) : (\mathfrak{C}_s - P)^M \subseteq W_s\}. \tag{2.9}$$

Note that $\Xi_s = \bigcup_Z \Xi_Z$ where Z ranges over the irreducible components of W_s and $\Xi_Z := \{P \in \mathfrak{C}_s(\overline{\mathbb{Q}}) : (\mathfrak{C}_s - P)^M \subseteq Z\}$. We can thus apply [DGH20, Lemma 6.4] to $A = (\mathfrak{A}_g)_{\tau(s)}, C = \mathfrak{C}_s - P \subseteq A$, and each Z. As $Z \subsetneq \mathscr{D}_M(\mathfrak{C}_s^{[M+1]})$ we have $\#\Xi_Z \leq 84(g-1)$. The precise value of $84(g-1)$, the classical upper bound for the automorphism group of a curve of genus g, is not so important; but its uniformity in s is. As W_s appears as a fiber of W considered as a family over S, we see that number of irreducible components of W_s is bounded from above by a constant c_2' that is independent of s. Our estimates imply $\#\Xi_s \leq c_2$ where $c_2 = 84(g-1)c_2'$ is also independent of s.

We have constructed Ξ_s which serves as the exceptional set in part (i). We will assume that (i) fails, *i.e.*, $P \in \mathfrak{C}_s(\overline{\mathbb{Q}}) \setminus \Xi_s$ and conclude (ii). So $(\mathfrak{C}_s - P)^M \not\subseteq W_s$ by (2.9). Our next task is to apply [DGH20, Lemma 6.3] to $\mathfrak{C}_s - P$ and W_s.

Recall that $\mathfrak{A}_g$ can be embedded in $\mathbb{P}^n_{\overline{\mathbb{Q}}} \times \mathbb{A}_g$; see §2.1 above (2.1). The image of $s \in \mathbb{M}_g(\overline{\mathbb{Q}})$ under the Torelli morphism is $\tau(s) \in \mathbb{A}_g(\overline{\mathbb{Q}})$. We may take $\mathfrak{C}_s - P$ as a smooth curve in $\mathbb{P}^n_{\overline{\mathbb{Q}}}$. The degree of $\mathfrak{C}_s$ as a subvariety of $\mathbb{P}^n_{\overline{\mathbb{Q}}}$ is bounded from above independently of s. As translating inside an abelian variety does not affect the degree, we see that the degree of $\mathfrak{C}_s - P$ is bounded from above independently of s; see [DGH20, Lemma 6.1(i)]. Recall that W_s is a Zariski closed subset of $X_s \subseteq (\mathfrak{A}_g^{[M]})_{\tau(s)}$ we may identify it with a Zariski closed subset of $(\mathbb{P}^n_{\overline{\mathbb{Q}}})^M$. The degree of W_s is bounded from above independently of s as it is the fiber above $\tau(s)$ of a subvariety of $(\mathbb{P}^n_{\overline{\mathbb{Q}}})^M \times \mathbb{A}_g$. From [DGH20, Lemma 6.3] we thus obtain a number c_4, depending only on these bounds but not on s with the following property. Any subset $\Sigma \subseteq \mathfrak{C}_s(\overline{\mathbb{Q}})$ with cardinality $\geq c_4$ satisfies $(\Sigma - P)^M \not\subseteq W_s(\overline{\mathbb{Q}})$.

Finally set $\Sigma = \left\{Q \in \mathfrak{C}_s(\overline{\mathbb{Q}}) : \hat{h}(Q-P) \leq c_3^{-1} \max\{1, h_{\overline{\mathbb{A}_g}}(\tau(s))\}\right\}$ with $c_3 = 2M/c$.

It remains to prove $\#\Sigma < c_4$, in which case we are in case (ii) of the proposition and hence we are done. Suppose $\#\Sigma \geq c_4$. Then $(\Sigma - P)^M \not\subseteq W_s(\overline{\mathbb{Q}})$. So there exist $Q_1, \ldots, Q_M \in \Sigma$ such that $(Q_1 - P, \ldots, Q_M - P) \in U(\overline{\mathbb{Q}})$. Thus we can apply (2.7) and obtain

$$c \max\{1, h_{\overline{\mathbb{A}_g}}(\tau(s))\} \leq M \frac{c}{2M} \max\{1, h_{\overline{\mathbb{A}_g}}(\tau(s))\} = \frac{c}{2} \max\{1, h_{\overline{\mathbb{A}_g}}(\tau(s))\},$$

a contradiction. □

2.4. **Completion of the proof of Theorem 1.3 for $F = \overline{\mathbb{Q}}$.** We follow the argumentation in [DGH19], or more precisely [DGH20, §8]. We will assume that Conjecture 1.2 holds true.

Let C be a smooth genus $g \geq 2$ defined over $\overline{\mathbb{Q}}$, and let Γ be a subgroup of $\mathrm{Jac}(C)(\overline{\mathbb{Q}})$ of finite rank ρ. Let $P_0 \in C(\overline{\mathbb{Q}})$.

The curve C corresponds to a $\overline{\mathbb{Q}}$-point s_c of $\mathbb{M}_{g,1}$, the coarse moduli space of smooth curves of genus g without level structure.

The fine moduli space $\mathbb{M}_g$ of smooth curves of genus g with level-4-structure admits a finite and surjective morphism of $\mathbb{M}_{g,1}$. So there exists an $s \in \mathbb{M}_g(\overline{\mathbb{Q}})$ that maps to s_c. Thus C is isomorphic, over $\overline{\mathbb{Q}}$, to the fiber $\mathfrak{C}_s$ of the universal curve $\mathfrak{C}_g \to \mathbb{M}_g$ above s. We thus view Γ as a finite rank subgroup of $\mathrm{Jac}(\mathfrak{C}_s)(\overline{\mathbb{Q}})$, and $P_0 \in \mathfrak{C}_s(\overline{\mathbb{Q}})$.

A standard application of Rémond's explicit formulation of the Vojta and Mumford inequalities [Rém00b, Rém00a] yields the following bound. There exists a constant $c = c(g) \geq 1$ with the following property. There is $P_s \in \mathfrak{C}_s(\overline{\mathbb{Q}})$ such that

$$\#\left\{P \in \mathfrak{C}_s(\overline{\mathbb{Q}}) : P - P_s \in \Gamma,\ \hat{h}_{\mathcal{L}}(P - P_s) > c\max\{1, h_{\overline{\mathbb{A}_g}}(\tau(s))\}\right\} \leq c^{\rho};$$

we refer to Lemma 8.2 and the proof of Proposition 8.1, both [DGH20], for details.

Let us start by conditionally verifying Conjecture 1.1 for $P_0 = P_s$. In this case it suffices to prove

$$\#\left\{P \in \mathfrak{C}_s(\overline{\mathbb{Q}}) : P - P_s \in \Gamma,\ \hat{h}_{\mathcal{L}}(P - P_s) \leq c\max\{1, h_{\overline{\mathbb{A}_g}}(\tau(s))\}\right\} \leq c'^{1+\rho} \tag{2.10}$$

for some c' independent of s. To do this we apply Proposition 2.5 to $S = \mathbb{M}_g$. Let c_2, c_3, c_4 be from this proposition; they are independent of s. If $P \in \mathfrak{C}_s(\overline{\mathbb{Q}})$, then either $P \in \Xi_s$ for some $\Xi_s \subseteq \mathfrak{C}_s(\overline{\mathbb{Q}})$ with $\#\Xi_s \leq c_2$ or

$$\#\left\{Q \in \mathfrak{C}_s(\overline{\mathbb{Q}}) : \hat{h}_{\mathcal{L}}(Q - P) \leq c_3^{-1}\max\{1, h_{\overline{\mathbb{A}_g}}(\tau(s))\}\right\} < c_4. \tag{2.11}$$

So to prove the desired bound (2.10) we may assume $P \in \mathfrak{C}_s(\overline{\mathbb{Q}}) \setminus \Xi_s$ and thus (2.11).

Now we apply the ball packing argument as in [DGH20] to prove the bound (2.10). Consider

$$\#\left\{P - P_s \in \Gamma : P \in \mathfrak{C}_s(\overline{\mathbb{Q}}) \setminus \Xi_s,\ \hat{h}_{\mathcal{L}}(P - P_s) \leq c\max\{1, h_{\overline{\mathbb{A}_g}}(\tau(s))\}\right\}. \tag{2.12}$$

Set $R = (c\max\{1, h_{\overline{\mathbb{A}_g}}(\tau(s))\})^{1/2}$. We start by doing ball packing in the ρ-dimensional $\mathbb{R}$-vector space $\Gamma \otimes \mathbb{R}$. It is well-known that $\hat{h}^{1/2}$ defines an Euclidean norm on $\Gamma \otimes \mathbb{R}$. The image in $\Gamma \otimes \mathbb{R}$ of the set in (2.12) is contained in the closed ball of radius R centered at the image of P_s. Let $r \in (0, R]$. By [Rém00a, Lemme 6.1] a subset of $\Gamma \otimes \mathbb{R}$ that is contained in a closed ball of radius R is covered by at most $(1 + 2R/r)^{\rho}$ closed balls of radius r centered at elements of the given set. The bound in (2.11) suggests the choice $r = (c_3^{-1}\max\{1, h_{\overline{\mathbb{A}_g}}(\tau(s))\})^{1/2}$. By possibly increasing c_3 we may assume that the quotient $R/r = (cc_3)^{1/2}$ lies in $[1, \infty)$. The crucial observation is that R/r is independent of s. So we can cover the image of the set in (2.12) in $\Gamma \otimes \mathbb{R}$ with at most c_5^{ρ} balls of radius r where $c_5 \geq 1$ is independent of s.

Say $P - P_s \in \Gamma$ maps to the center of a ball of radius r in the covering. If $Q - P_s \in \Gamma$ maps to the same ball, then $\hat{h}(Q - P) \leq r^2$. As $P \notin \Xi_s$, by (2.11) the number of the $Q - P_s$'s that map to this closed ball of radius r is less than c_4. Thus the number of points in (2.12) is at most $c_4 c_5^{\rho}$. So the number of points in the set from (2.10) is at most $\#\Xi_s + c_4 c_5^{\rho} \leq c_2 + c_4 c_5^{\rho}$, which is at most $c'^{1+\rho}$ for a suitable c'. This completes the proof of the proposition in the case $P_0 = P_s$.

Now we turn to a general $P_0 \in \mathfrak{C}_s(\overline{\mathbb{Q}})$. The subgroup Γ' of $\mathrm{Jac}(\mathfrak{C}_s)(\overline{\mathbb{Q}})$ generated by Γ and $P_0 - P_s$ has rank most $\rho + 1$. Now if $Q \in \mathfrak{C}_s(\overline{\mathbb{Q}}) - P_0$ lies in Γ, then $Q + P_0 - P_s \in \mathfrak{C}_s(\overline{\mathbb{Q}}) - P_s$ lies in Γ'. We have just proved that the number of such Q is at most $c^{1+\mathrm{rk}(\Gamma')} \leq c^{2+\rho} \leq (c^2)^{1+\rho}$ for a $c \geq 1$ that is independent of s. □

3. From $\overline{\mathbb{Q}}$ to an arbitrary base field in characteristic 0

In this section we perform a specialization argument to reduce Conjecture 1.1 to the case $F = \overline{\mathbb{Q}}$.

Lemma 3.1. *If Conjecture 1.1 holds true for $F = \overline{\mathbb{Q}}$, then it holds true for an arbitrary field F of characteristic* 0.

Proof. Without loss of generality we may and do assume that $F = \overline{F}$.

Let C, $P_0 \in C(F)$, and Γ be as in Conjecture 1.1 of rank ρ. By the definition of a finite rank group, there exists a finitely generated subgroup Γ_0 of $\mathrm{Jac}(C)(F)$ with rank ρ such that

$$\Gamma \subseteq \{x \in \mathrm{Jac}(C)(F) : [n]x \in \Gamma_0 \text{ for some } n \in \mathbb{N}\}.$$

For each $n \in \mathbb{N}$, define

$$\frac{1}{n}\Gamma_0 := \{x \in \mathrm{Jac}(C)(F) : [n]x \in \Gamma_0\}.$$

Then $\frac{1}{n}\Gamma_0$ is again a finitely generated subgroup of $\mathrm{Jac}(C)(F)$ of rank ρ. Note that $\{\frac{1}{n}\Gamma_0\}_{n\in\mathbb{N}}$ is a filtered system and $\Gamma \subseteq \bigcup_{n\in\mathbb{N}} \frac{1}{n}\Gamma_0$. So in order to prove the desired bound (1.1), it suffices to prove that there exists a constant $c = c(g) > 0$ such that

$$\#(C(F) - P_0) \cap \frac{1}{n}\Gamma_0 \leq c^{1+\rho} \tag{3.1}$$

for each $n \in \mathbb{N}$.

Let $n \in \mathbb{N}$ and let $\gamma_1, \ldots, \gamma_r \in \mathrm{Jac}(C)(F)$ be generators of $\frac{1}{n}\Gamma_0$ such that $\gamma_{\rho+1}, \ldots, \gamma_r$ are torsion points, we allow r to depend on n. There exists a field K_n, finitely generated over $\overline{\mathbb{Q}}$, such that C, P_0, and the $\gamma_1, \ldots, \gamma_r$ are defined over K_n. Then K_n is the function field of some regular, irreducible quasi-projective variety V_n defined over $\overline{\mathbb{Q}}$.

Up to replacing V_n by a Zariski open dense subset, C extends to a smooth family $\mathfrak{C} \to V_n$ (*i.e.*, C is the generic fiber of $\mathfrak{C} \to V_n$) with each fiber being a smooth curve of genus g, the point P_0 extends to a section of $\mathfrak{C} \to V_n$, and $\gamma_1, \ldots, \gamma_r$ extend to sections of the relative Jacobian $\mathrm{Jac}(\mathfrak{C}/V_n) \to V_n$. We retain the symbols $P_0, \gamma_1, \ldots, \gamma_r$ for these sections.

Let $v \in V_n(\overline{\mathbb{Q}})$. We may specialize $\gamma_1, \ldots, \gamma_r$ at v and obtain elements $\gamma_1(v), \ldots, \gamma_r(v)$ of the fiber $\mathrm{Jac}(\mathfrak{C}/V_n)_v$ above v.

We define the specialization of $\frac{1}{n}\Gamma_0$ at v, which we denote with $(\frac{1}{n}\Gamma_0)_v$, to be the subgroup of $\mathrm{Jac}(\mathfrak{C}/V_n)_v(\overline{\mathbb{Q}}) = \mathrm{Jac}(\mathfrak{C}_v)(\overline{\mathbb{Q}})$ generated by $\gamma_1(v), \ldots, \gamma_r(v)$. Note that $\mathrm{rk}(\frac{1}{n}\Gamma_0)_v \leq \rho$.

Suppose $\dim V_n \geq 1$, by [Mas89, Main Theorem and Scholium 1] there exists $v \in V_n(\overline{\mathbb{Q}})$ such that the specialization homomorphism $\frac{1}{n}\Gamma_0 \to (\frac{1}{n}\Gamma_0)_v$ is injective. If $\dim V_n = 0$, then V_n is a point $\{v\}$ and $K_n = \overline{\mathbb{Q}}$. Here specialization is the identity and the same conclusion holds. Thus if we denote by $\mathfrak{C}_v - P_0(v)$ the curve in $\mathrm{Jac}(\mathfrak{C}_v) = \mathrm{Jac}(\mathfrak{C}/V_n)_v$ obtained via the Abel–Jacobi map based at $P_0(v)$, the specialization of P_0 at v, then we have

$$\#(C - P_0)(F) \cap \frac{1}{n}\Gamma_0 \leq \#\big(\mathfrak{C}_v - P_0(v)\big)(\overline{\mathbb{Q}}) \cap \left(\frac{1}{n}\Gamma_0\right)_v.$$

By hypothesis, Conjecture 1.1 holds true for $F = \overline{\mathbb{Q}}$. So the right-hand side of the inequality above has an upper bound $c^{1+\rho}$ for some $c = c(g) \geq 1$. Observe that this bound is independent of n. Thus we have established (3.1). □

Proof of Theorem 1.3. By hypothesis and Theorem 2.1 Conjecture 1.1 holds for $F = \overline{\mathbb{Q}}$. So it suffices to apply Lemma 3.1. □

4. Relative Bogomolov for isotrivial abelian schemes

In this section, we prove that the relative Bogomolov conjecture holds true for isotrivial abelian schemes as a consequence of S. Zhang's Theorem [Zha98a]. An abelian scheme $\mathcal{A} \to S$ defined over $\overline{\mathbb{Q}}$ is said to be isotrivial if there exists a finite and surjective morphism $S' \to S$ with S' irreducible such that $\mathcal{A} \times_S S'$ is isomorphic to $A \times S'$ with A an abelian variety defined over $\overline{\mathbb{Q}}$.

Let $\mathcal{A} \to S$ and $\mathcal{L}$ be as above Conjecture 1.2.

Proposition 4.1. *Conjecture 1.2 holds true if $\mathcal{A} \to S$ is isotrivial.*

Proof. Let X be an irreducible subvariety of $\mathcal{A}$ defined over $\overline{\mathbb{Q}}$ that dominates S such that $X_{\overline{\eta}}$ is irreducible and not contained in any proper algebraic subgroup of $\mathcal{A}_{\overline{\eta}}$; here $X_{\overline{\eta}}$ means the geometric generic fiber of X and $\mathcal{A}_{\overline{\eta}}$ means the geometric generic fiber of $\mathcal{A}$. Assume

$$\mathrm{codim}_{\mathcal{A}} X > \dim S. \tag{4.1}$$

Case: Trivial abelian scheme We start by proving the proposition when $\mathcal{A} \to S$ is a trivial abelian scheme, *i.e.*, $\mathcal{A} = A \times S$ for some abelian variety A over $\overline{\mathbb{Q}}$.

Denote by $p\colon \mathcal{A} = A \times S \to A$ the natural projection. Let L be an ample and symmetric line bundle on A defined over $\overline{\mathbb{Q}}$. For simplicity denote by $Y = \overline{p(X)}^{\mathrm{Zar}}$. Then $\dim Y \leq \dim X$, and Y is not contained in any proper algebraic subgroup of A by our assumption on X.

Assume that for all $\epsilon > 0$, the set

$$X(\epsilon; \mathcal{L}) = \{x \in X(\overline{\mathbb{Q}}) : \hat{h}_{\mathcal{L}}(x) \leq \epsilon\}$$

is Zariski dense in X. We will prove a contradiction to (4.1).

Since $\mathcal{L}$ is relatively ample on $\mathcal{A} \to S$, there exists an integer $N \geq 1$ such that $\mathcal{L}^{\otimes N} \otimes (p^*L)^{\otimes -1}$ is relatively ample on $\mathcal{A} \to S$. For the Néron–Tate height functions on $\mathcal{A}(\overline{\mathbb{Q}})$ we have $N\hat{h}_{\mathcal{L}} \geq \hat{h}_{p^*L}$.

Take any $\epsilon > 0$. If $x \in X(\epsilon; \mathcal{L})$, then $\hat{h}_{\mathcal{L}}(x) \leq \epsilon$, and hence

$$\hat{h}_L(p(x)) = \hat{h}_{p^*L}(x) \leq N\hat{h}_{\mathcal{L}}(x) \leq N\epsilon.$$

Letting x run over elements in $X(\epsilon; \mathcal{L})$, we then obtain

$$p(X(\epsilon; \mathcal{L})) \subseteq Y(N\epsilon; L) := \{y \in Y(\overline{\mathbb{Q}}) : \hat{h}_L(y) \leq N\epsilon\}. \tag{4.2}$$

We have assumed $\overline{X(\epsilon; \mathcal{L})}^{\mathrm{Zar}} = X$. Applying p to both sides and taking the Zariski closure, we get $\overline{p(X(\epsilon; \mathcal{L}))}^{\mathrm{Zar}} = Y$. Hence $\overline{Y(N\epsilon; L)}^{\mathrm{Zar}} = Y$ by (4.2). Recall that Y is not contained in any proper algebraic subgroup of A. As $N\epsilon$ runs over all positive real numbers, the classicial Bogomolov conjecture, proved by S. Zhang [Zha98a], implies that $Y = A$.

So $\dim X \geq \dim Y = \dim A = \dim \mathcal{A} - \dim S$, and thus $\mathrm{codim}_{\mathcal{A}} X \leq \dim S$. This contradicts (4.1). Hence we are done in this case.

Case: General isotrivial abelian scheme Now we go back to an arbitrary isotrivial abelian scheme $\mathcal{A} \to S$.

There exists a finite and surjective morphism $\rho\colon S' \to S$, with S' irreducible, such that the base change $\mathcal{A}' := \mathcal{A} \times_S S' \to S'$ is a trivial abelian scheme. Denote by $\rho_{\mathcal{A}}\colon \mathcal{A}' \to \mathcal{A}$

the natural projection. Then $\rho_{\mathcal{A}}$ is finite and surjective, so $\dim \mathcal{A}' = \dim \mathcal{A}$. Moreover, there is an irreducible component X' of $\rho_{\mathcal{A}}^{-1}(X)$ with $\rho_{\mathcal{A}}(X') = X$ and $\dim X' = \dim X$. So X' dominates S' and $X'_{\overline{\eta}}$ is irreducible, as $X_{\overline{\eta}}$ is. Moreover, $X'_{\overline{\eta}}$ is not contained in any proper algebraic subgroup of $\mathcal{A}'_{\overline{\eta}} = \mathcal{A}_{\overline{\eta}}$, and $\mathrm{codim}_{\mathcal{A}'} X' = \mathrm{codim}_{\mathcal{A}} X > \dim S = \dim S'$. Finally, $\rho_{\mathcal{A}}^* \mathcal{L}$ is relatively ample on $\mathcal{A}' \to S'$.

We have proved the relative Bogomolov conjecture for the trivial abelian scheme $\mathcal{A}' \to S'$. So there exists $\epsilon > 0$ such that

$$X'(\epsilon; \rho_{\mathcal{A}}^* \mathcal{L}) = \{x' \in X'(\overline{\mathbb{Q}}) : \hat{h}_{\rho_{\mathcal{A}}^* \mathcal{L}}(x') \leq \epsilon\}$$

is not Zariski dense in X'. In particular

$$\dim \overline{X'(\epsilon; \rho_{\mathcal{A}}^* \mathcal{L})}^{\mathrm{Zar}} < \dim X' = \dim X. \tag{4.3}$$

It is not hard to check $\rho_{\mathcal{A}}(X'(\epsilon; \rho_{\mathcal{A}}^* \mathcal{L})) = X(\epsilon; \mathcal{L})$ using $\hat{h}_{\rho_{\mathcal{A}}^* \mathcal{L}}(x') = \hat{h}_{\mathcal{L}}(\rho_{\mathcal{A}}(x'))$ and $\rho_{\mathcal{A}'}(X') = X$. Therefore, and as $\rho_{\mathcal{A}}$ is a closed morphism,

$$\rho_{\mathcal{A}}(\overline{X'(\epsilon; \rho_{\mathcal{A}}^* \mathcal{L})}^{\mathrm{Zar}}) = \overline{\rho_{\mathcal{A}}(X'(\epsilon; \rho_{\mathcal{A}}^* \mathcal{L}))}^{\mathrm{Zar}} = \overline{X(\epsilon; \mathcal{L})}^{\mathrm{Zar}}.$$

So we have $\dim \overline{X(\epsilon; \mathcal{L})}^{\mathrm{Zar}} = \dim \rho_{\mathcal{A}}(\overline{X'(\epsilon; \rho_{\mathcal{A}}^* \mathcal{L})}^{\mathrm{Zar}}) = \dim \overline{X'(\epsilon; \rho_{\mathcal{A}}^* \mathcal{L})}^{\mathrm{Zar}}$ because $\rho_{\mathcal{A}}$ is finite. By (4.3) we then have $\dim \overline{X(\epsilon; \mathcal{L})}^{\mathrm{Zar}} < \dim X$. Hence $X(\epsilon; \mathcal{L})$ is not Zariski dense in X. We are done. □

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