THE GEOMETRIC BOGOMOLOV CONJECTURE

SERGE CANTAT, ZIYANG GAO, PHILIPP HABEGGER, AND JUNYI XIE

ABSTRACT. We prove the geometric Bogomolov conjecture over a function field of characteristic zero.

1. INTRODUCTION

1.1. The geometric Bogomolov conjecture.

1.1.1. Abelian varieties and heights. Let \( k \) be an algebraically closed field. Let \( B \) be an irreducible normal projective variety over \( k \) of dimension \( d_B \geq 1 \). Let \( K := k(B) \) be the function field of \( B \). Let \( A \) be an abelian variety defined over \( K \) of dimension \( g \). Fix an ample line bundle \( M \) on \( B \), and a symmetric ample line bundle \( L \) on \( A \).

Denote by \( \hat{h} : A(K) \to [0, +\infty) \) the canonical height on \( A \) with respect to \( L \) and \( M \) where \( K \) is an algebraic closure of \( K \) (see Section 3.1). For any irreducible subvariety \( X \) of \( A_K \) and any \( \varepsilon > 0 \), we set

\[ X_\varepsilon := \{ x \in X(K) | \hat{h}(x) < \varepsilon \}. \]

(1.1)

Set \( A_K = A \otimes_k \overline{K} \), and denote by \( (A_K^\times, tr) \) the \( \overline{K}/k \)-trace of \( A_K \): it is the final object of the category of pairs \( (C, f) \), where \( C \) is an abelian variety over \( k \) and \( f \) is a morphism from \( C \otimes_k \overline{K} \) to \( A_K \) (see [13]). If \( \text{char } k = 0 \), \( tr \) is a closed immersion and \( A_K^\times \otimes_k \overline{K} \) can be naturally viewed as an abelian subvariety of \( A_K \). By definition, a torsion coset of \( A \) is a translate \( a + C \) of an abelian subvariety \( C \subset A \) by a torsion point \( a \). An irreducible subvariety \( X \) of \( A_K \) is said to be special if

\[ X = \text{tr}(Y \otimes_k \overline{K}) + T \]

(1.2)

for some torsion coset \( T \) of \( A_K \) and some subvariety \( Y \) of \( A_K^\times \). When \( X \) is special, \( X_\varepsilon \) is Zariski dense in \( X \) for all \( \varepsilon > 0 \) ([13, Theorem 5.4, Chapter 6]).

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1.1.2. **Bogomolov conjecture.** The following conjecture was proposed by Yamaki [20, Conjecture 0.3], but particular instances of it were studied earlier by Gubler in [10]. It is an analog over function fields of the Bogomolov conjecture which was proved by Ullmo [16] and Zhang [26].

**Geometric Bogomolov Conjecture.**—Let $X$ be an irreducible subvariety of $\mathbb{A}_K^n$. If $X$ is not special there exists $\varepsilon > 0$ such that $X_{\varepsilon}$ is not Zariski dense in $X$.

The aim of this paper is to prove the geometric Bogomolov conjecture over a function field of characteristic zero.

**Theorem A.** Assume that $k$ is an algebraically closed field of characteristic 0. Let $X$ be an irreducible subvariety of $A_K^n$. If $X$ is not special then there exists $\varepsilon > 0$ such that $X_{\varepsilon}$ is not Zariski dense in $X$.

1.1.3. **Historical note.** Gubler proved the geometric Bogomolov conjecture in [10] when $A$ is totally degenerate at some place of $K$. When $\dim B = 1$ and $X \subset A$ is a curve in its Jacobian, Yamaki proved it for nonhyperelliptic curves of genus $3$ in [18] and for any hyperelliptic curve in [19]. If moreover $\text{char } k = 0$, Faber [6] proved it if $X$ is a curve of genus at most 4 and Cinkir [3] covered the case of arbitrary genus. Later on Yamaki proved the cases $(\co) \dim X = 1$ [24] and $\dim (A_K^n/k) \geq \dim (A) - 5$ [23]; in [22], he reduced the conjecture to the case of abelian varieties with trivial $K/k$-trace and good reduction everywhere. In [12], the third-named author gave a new proof of this conjecture in characteristic 0 when $A$ is the power of an elliptic curve and $\dim B = 1$, introducing the original idea of considering the Betti map and its monodromy. Recently, the second and the third-named authors [7] proved the conjecture in the case $\text{char } k = 0$ and $\dim B = 1$.

1.2. **An overview of the proof of Theorem A.**

1.2.1. **Notation.** From now on, the algebraically closed field $k$ has characteristic 0. There exists an algebraically closed subfield $k'$ of $k$ such that $B$, $A$, $X$, $M$ and $L$ are defined over $k'$ and the transcendental degree of $k'$ over $\mathbb{Q}$ is finite. In particular, $k'$ can be embedded in the complex field $\mathbb{C}$. Thus, in the rest of the paper, we assume $k = \mathbb{C}$ and we denote by $K$ the function field $\mathbb{C}(B)$.

Let $\pi : \mathcal{A} \to B$ be an irreducible projective scheme over $B$ whose generic fiber is isomorphic to $A$. We may assume that $\mathcal{A}$ is normal, and we fix an ample line bundle $L$ on $\mathcal{A}$ such that $L|_A = L$. For $b \in B$, we set $\mathcal{A}_b := \pi^{-1}(b)$. We denote by $e : B \dashrightarrow \mathcal{A}$ the zero section and by $[n]$ the multiplication by $n$ on $A$; it defines a rational mapping $\mathcal{A} \dashrightarrow \mathcal{A}$.

We may assume that $M$ is very ample, and we fix an embedding of $B$ in a projective space such that the restriction of $O(1)$ to $B$ coincides with $M$. The restriction of the Fubini-Study form to $B$ is a Kähler form $\nu$. 

Fix a Zariski dense open subset $B^o$ of $B$ such that $B^o$ is smooth and $\pi|_{\pi^{-1}(B^o)}$ is smooth; then, set $A^o := \pi^{-1}(B^o)$.

Let $X$ be a geometrically irreducible subvariety of $A$ such that $X_\varepsilon$ is Zariski dense in $X$ for every $\varepsilon > 0$. We denote by $X$ its Zariski closure in $A$, by $X^o$ its Zariski closure in $A^o$, and by $X^{o,\text{reg}}$ the regular locus of $X^o$. Our goal is to show that $X$ is special.

1.2.2. The main ingredients. One of the main ideas of this paper is to consider the Betti foliation (see Section 2.1). It is a smooth foliation of $A^o$ by holomorphic leaves, which is transverse to $\pi$. Every torsion point of $A$ gives local sections of $\pi|_{\pi^{-1}(B^o)}$; these sections are local leaves of the Betti foliation, and this property characterizes it.

To prove Theorem A, the first step is to show that $X^o$ is invariant under the foliation when small points are dense in $X$. In other words, at every smooth point $x \in X^o$, the tangent space to the Betti foliation is contained in $T_xX^o$. For this, we introduce a semi-positive closed $(1,1)$-form $\omega$ on $A^o$ which is canonically associated to $L$ and vanishes along the foliation. An inequality of Gubler implies that the canonical height $\hat{h}(X)$ of $X$ is 0 when small points are dense in $X$; Theorem B asserts that the condition $\hat{h}(X) = 0$ translates into

$$\int_{X^o} \omega^{\dim X+1} \wedge (\pi^*\nu)^{m-1} = 0$$

where $\nu$ is any Kähler form on the base $B^o$. From the construction of $\omega$, we deduce that $X$ is invariant under the Betti foliation.

The first step implies that the fibers of $\pi|_{X^o}$ are invariant under the action of the holonomy of the Betti foliation; the second step shows that a subvariety of a fiber $A_b$ which is invariant under the holonomy is the sum of a torsion coset and a subset of $A^{R/k}$. The conclusion easily follows from these two main steps. The second step already appeared in [12] and [7], but here, we make use of a more efficient dynamical argument which may be derived from a result of Muchnik and is independent of the Pila-Zannier’s counting strategy. A recent paper of André, Corvaja and Zannier [1] also studied the Betti foliation but for a different purpose. They used this foliation to show, as Theorem 2.3.2 of loc.cit., that given a Hodge generic subvariety $S$ of the moduli space of abelian varieties of dimension $g$, any section of the universal abelian variety restricted to $S$ contains a dense subset of torsion points if $\dim S \geq g$.

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2. The Betti form

In this section, we define a foliation, and a closed $(1,1)$-form on $\mathcal{A}^o$ which is naturally associated to the line bundle $L$.

2.1. The local Betti maps. Let $b$ be a point of $B^o$, and $U \subseteq B^o(\mathbf{C})$ be a connected and simply connected open neighbourhood of $b$ in the euclidean topology. Fix a basis of $H_1(\mathcal{A}_b;\mathbf{Z})$ and extend it by continuity to all fibers above $U$. There is a natural real analytic diffeomorphism $\phi_U : \pi^{-1}(U) \to U \times \mathbf{R}^{2g}/\mathbf{Z}^{2g}$ such that

(1) $\pi_1 \circ \phi_U = \pi$ where $\pi_1 : U \times \mathbf{R}^{2g}/\mathbf{Z}^{2g} \to U$ is the projection to the first factor;
(2) for every $b \in U$, the map $\phi_U|_{\mathcal{A}_b} : \mathcal{A}_b \to \pi_1^{-1}(b)$ is an isomorphism of real Lie groups that maps the basis of $H_1(\mathcal{A}_b;\mathbf{Z})$ onto the canonical basis of $\mathbf{R}^{2g}$.

For $b$ in $U$, denote by $i_b : \mathbf{R}^{2g}/\mathbf{Z}^{2g} \to U \times \mathbf{R}^{2g}/\mathbf{Z}^{2g}$ the inclusion $y \mapsto (b, y)$. The Betti map is the $C^\infty$-projection $\beta^b_U : \pi^{-1}(U) \to \mathcal{A}_b$ defined by

$$\beta^b_U := (\phi_U|_{\mathcal{A}_b})^{-1} \circ i_b \circ \pi_2 \circ \phi_U \quad (2.1)$$

where $\pi_2 : U \times \mathbf{R}^{2g}/\mathbf{Z}^{2g} \to \mathbf{R}^{2g}/\mathbf{Z}^{2g}$ is the projection to the second factor.

Changing the basis of $H_1(\mathcal{A}_b;\mathbf{Z})$, we obtain another trivialization $\phi'_U$ that is given by post-composing $\phi_U$ with a constant linear transformation

$$i_{b'} \circ h \circ \phi_U \quad (2.2)$$

for some element $h$ of the group $\text{GL}_{2g}(\mathbf{Z})$; thus, $\beta^b_U$ does not depend on $\phi_U$.

Note that $\beta^b_U$ is the identity on $\mathcal{A}_b$. In general, $\beta^b_U$ is not holomorphic. However, for every $p \in \mathcal{A}_b$, $(\beta^b_U)^{-1}(p)$ is a complex submanifold of $\mathcal{A}^o$. (For instance, every section of $\pi|_{\pi^{-1}U}$ which is given by a torsion point provides a fiber of $\beta^b_U$, and continous limits of holomorphic sections are holomorphic.)

2.2. The Betti foliation. The local Betti maps determine a natural foliation $\mathcal{F}$ on $\mathcal{A}^o$: for every point $p$, the local leaf $\mathcal{F}_{U,p}$ through $p$ is the fiber $(\beta^b_U)^{-1}(p)$. We call $\mathcal{F}$ the Betti foliation. The leaves of $\mathcal{F}$ are holomorphic, in the following sense: for every $p \in \mathcal{A}^o$, the local leaf $\mathcal{F}_{U,p}$ is a complex submanifold of $\pi^{-1}(U) \subseteq \mathcal{A}^o$. But a global leaf $\mathcal{F}_p$ can be dense in $\mathcal{A}^o$ for the euclidean topology. Moreover, $\mathcal{F}$ is everywhere transverse to the fibers of $\pi$, and $\pi|_{\mathcal{F}_p} : \mathcal{F}_p \to B^o$ is a regular holomorphic covering for every point $p$ (it may have finite or infinite degree, and this may depend on $p$).

Remark 2.1. The foliation $\mathcal{F}$ is characterized as follows. Let $q$ be a torsion point of $\mathcal{A}_b$; it determines a multisection of the fibration $\pi$, obtained by analytic
continuation of \( q \) as a torsion point in nearby fibers of \( \pi \). This multisection coincides with the leaf \( F_q \). There is a unique foliation of \( \mathcal{A}^o \) which is everywhere transverse to \( \pi \) and whose set of leaves contains all those multisections.

**Remark 2.2.** One can also think about \( F \) dynamically. The endomorphism \([n]\) determines a rational transformation of the model \( \mathcal{A} \) and induces a regular transformation of \( \mathcal{A}^o \). It preserves \( F \), mapping leaves to leaves. Preperiodic leaves correspond to preperiodic points of \([n]\) in the fiber \( \mathcal{A}_b \); they are exactly the leaves given by the torsion points of \( A \).

**Remark 2.3.** Assume that the family \( \pi: \mathcal{A}^o \to B^o \) is trivial, i.e. \( \mathcal{A}^o = B^o \times A_C \) where \( A_C \) is an abelian variety over \( \mathbb{C} \) and \( \pi \) is the first projection. Then, the leaves of \( F \) are exactly the fibers of the second projection.

### 2.3. The Betti form

The Betti form is introduced by Mok in [14, pp. 374] to study the Mordell-Weil group over function fields. We hereby sketch the construction of this \((1,1)\)-form. For \( b \in B^o \), there exists a unique smooth \((1,1)\)-form \( \omega_b \in c_1(L|_{\mathcal{A}_b}) \) on \( \mathcal{A}_b \) which is invariant under translations. If we write \( \mathcal{A}_b = C^g/\Lambda \) and denote by \( z_1, \ldots, z_g \) the standard coordinates of \( C^g \), then

\[
\omega_b = \sum_{1 \leq i, j \leq g} a_{i,j} dz_i \wedge d\bar{z}_j
\]

for some complex numbers \( a_{i,j} \). This form \( \omega_b \) is positive, because \( L|_{\mathcal{A}_b} \) is ample.

Now, we define a smooth 2-form \( \omega \) on \( \mathcal{A}^o \). Let \( p \) be a point of \( \mathcal{A}^o \). First, define \( P_p: T_p \mathcal{A}^o \to T_p \mathcal{A}_{\pi(p)} \) to be the projection onto the first factor in

\[
T_p \mathcal{A}^o = T_p \mathcal{A}_{\pi(p)} \oplus T_p F.
\]

Since the tangent spaces \( T_p F \) and \( T_p \mathcal{A}_{\pi(p)} \) are complex subspaces of \( T_p \mathcal{A}^o \), the map \( P_p \) is a complex linear map. Then, for \( v_1 \) and \( v_2 \in T_p \mathcal{A}^o \) we set

\[
\omega(v_1, v_2) := \omega_{\pi(p)}(P_p(v_1), P_p(v_2)).
\]

We call \( \omega \) the **Betti form**. By construction, \( \omega|_{\mathcal{A}_b} = \omega_b \) for every \( b \). Since \( \omega_b \) is of type \((1,1)\) and \( P_p \) is \( \mathbb{C} \)-linear, \( \omega \) is an antisymmetric form of type \((1,1)\). Since \( \omega_b \) is positive, \( \omega \) is semi-positive.

Let \( U \) and \( \phi_U \) be as in Section 2.1. Let \( y_i, i = 1, \ldots, 2g \), denote the standard coordinates of \( \mathbb{R}^{2g} \). Then there are real numbers \( b_{i,j} \) such that

\[
(\phi_U^{-1})^* \omega = \sum_{1 \leq i < j \leq 2g} b_{i,j} dy_i \wedge dy_j.
\]

It follows that \( d((\phi_U^{-1})^* \omega) = 0 \) and that \( \omega \) is closed. Moreover, \([n]^* \omega = n^2 \omega\). Thus, we get the following lemma.
Lemma 2.4. The Betti form $\omega$ is a real analytic, closed, semi-positive $(1,1)$-form on $\mathcal{A}^o$ such that $\omega|_{\mathcal{A}_b} = \omega_b$ for every point $b \in B^o$. In particular, the cohomology class of $\omega|_{\mathcal{A}_b}$ coincides with $c_1(L|_{\mathcal{A}_b})$ for every $b \in B^o$.

Since the monodromy of the foliation preserves the polarization $L_{\mathcal{A}_b}$, it preserves $\omega_b$ and is contained in a symplectic group.

3. The canonical height and the Betti form

3.1. The canonical height. Recall that $K = C(B)$. Let $X$ be any subvariety of $A_K$. There exists a finite field extension $K'$ over $K$ such that $X$ is defined over $K'$; in other words, there exists a subvariety $X'$ of $A_{K'}$ such that $X = X' \otimes_{K'} K$. Let $\rho' : B' \to B$ be the normalization of $B$ in $K'$. Set $\mathcal{A}' := \mathcal{A} \times_B B'$ and denote by $\rho : \mathcal{A}' \to \mathcal{A}$ the projection to the first factor; then, denote by $X'$ the Zariski closure of $X$ in $\mathcal{A}'$. The naive height of $X$ associated to the model $\pi : \mathcal{A} \to B$ and the line bundles $L$ and $M$ is defined by the intersection number

$$h(X) = \frac{1}{[K' : K]} \left( X' \cdot c_1(\rho^* L)^{d_X + 1} \cdot \rho^* \pi^*(c_1(M))^{d_B - 1} \right) \quad (3.1)$$

where $d_X = \dim X$ and $d_B = \dim B$. It depends on the model $\mathcal{A}$ and the extension $L$ of $L$ to $\mathcal{A}$ but it does not depend on the choice of $K'$.

The canonical height is the limit

$$\hat{h}(X) = \lim_{n \to +\infty} \frac{h([n]X)}{n^{2(d_X + 1)}} = \lim_{n \to +\infty} \frac{\deg([n]X)h([n]X)}{n^{2(d_X + 1)}}. \quad (3.2)$$

It depends on $L$ but not on the model $(\mathcal{A}, L)$; we refer to Gubler’s work [9] for more details. By [13, Theorem 5.4, Chapter 6], the condition $\hat{h}(X) = 0$ does not depend on $L$. In particular, we may modify $L$ on special fibers to assume that $L$ is ample. See also [10, Section 3].

Now we reformulate the canonical height in differential geometric terms. For simplicity, assume that $X$ is already defined over $K$. Set $\mathcal{A}_1 := \mathcal{A}$, $\pi_1 := \pi$ and $L_1 := L$. Pick a Kähler form $\alpha_1$ in $c_1(L)$ (such a form exists because we choose $L$ ample). For every $n \geq 1$, there exists an irreducible smooth projective scheme $\pi_n : \mathcal{A}_n \to B$ over $B$, extending $\pi|_{\mathcal{A}_o} : \mathcal{A}_o \to B'$, such that the rational map $[n] : \mathcal{A}_o \to \mathcal{A}_o$ lifts to a morphism $f_n : \mathcal{A}_n \to \mathcal{A}$ over $B$. Write $L_n := f_n^* L$ and $\alpha_n := f_n^* \alpha_1$. Denote by $X_n$ the Zariski closure of $X_o$ in $\mathcal{A}_n$. Since the Kähler form $\nu$ introduced in Section 1.2.1 represents the class $c_1(M)$, the projection
formula gives
\[
\hat{h}(X) = \lim_{n \to \infty} n^{-2(d_X + 1)}(X_n \cdot \mathcal{L}_n^{d_X+1} \cdot (\pi_n^* M)^{d_B-1})
\] (3.3)
\[
= \lim_{n \to \infty} n^{-2(d_X + 1)} \int_{X_n \alpha_n^{d_X+1} \wedge (\pi_n^* \nu)^{d_B-1}}
\] (3.4)
\[
= \lim_{n \to \infty} n^{-2(d_X + 1)} \int_{X^0} ([H]^* \alpha)^{d_X+1} \wedge (\pi^* \nu)^{d_B-1}
\] (3.5)
because the integral on $X_n$ is equal to the integral on the dense Zariski open subset $X^0$ (and even on the regular locus $X^{0, \text{reg}}$).

3.2. **Gubler-Zhang inequality.** By definition, the essential height $\text{ess}(X)$ of a subvariety $X \subset A$ is the real number
\[
\text{ess}(X) = \sup_Y \inf_{x \in X(Y) \setminus Y} \hat{h}(x),
\] (3.6)
where $Y$ runs through all proper Zariski closed subsets of $X$. The following inequality is due to Gubler in [10, Lemma 4.1]; it is an analogue of Zhang’s inequality [25, Theorem 1.10] over number fields.
\[
0 \leq \frac{\hat{h}(X)}{(d_X + 1) \deg_L(X)} \leq \text{ess}(X).
\] (3.7)
The converse inequality $\text{ess}(X) \leq \hat{h}(X)/\deg_L(X)$ also holds, but we shall not use it in this article.

**Definition 3.1.** We say that $X$ is small, if $X_\varepsilon$ is Zariski dense in $X$ for all $\varepsilon > 0$.

The above inequalities comparing $\hat{h}(X)$ to $\text{ess}(X)$ show that $X$ is small if, and only if $\hat{h}(X) = 0$.

**Proposition 3.2.** Let $g : A \to A'$ be a morphism of abelian varieties over $K$, and let $a \in A(K)$ be a torsion point. Let $X$ be an absolutely irreducible subvariety of $A$ over $K$.

1. If $X$ is small, then $g(X)$ is small.
2. If $g$ is an isogeny and $g(X)$ is small, then $X$ is small.
3. $X$ is small if and only if $a + X$ is small.

**Proof.** Assertions (1) and (2) follow from [21, Proposition 2.6.]. To prove the third one fix an integer $n \geq 1$ such that $na = 0$. By assertions (1) and (2), $a + X$ is small if and only if $[n](a + X) = [n](X)$ is small, if and only if $X$ is small. □
3.3. **Smallness and the Betti Form.** Here is the key relationship between the
density of small points and the Betti form.

**Theorem B.** Let $X$ be an absolutely irreducible subvariety of $A$ over $C(B)$. If
$X$ is small, then
\[
\int_{X^o} \omega^{d_X + 1} \wedge (\pi^* \nu)^{d_B - 1} = 0,
\]
with $\omega$ the Betti form associated to $L$ and $\nu$ the Kähler form on $B$ representing
the class $c_1(M)$.

**Proof.** Since $X$ is small, $\hat{h}(X) = 0$ and equation (3.5) shows that
\[
0 = \hat{h}(X) = \lim_{n \to \infty} n^{-2(d_X + 1)} \int_{X^o} (n)^* \alpha^{d_X + 1} \wedge (\pi^* \nu)^{d_B - 1}.
\] (3.8)

Let $U \subset B^o$ be any relatively compact open subset of $B^o$ in the euclidean
topology. There exists a constant $C_U > 0$ such that $C_U \alpha - \omega$ is semi-positive
on $\pi^{-1}(U)$. Since $[n] : A^o \to A^o$ is regular, the $(1,1)$-form $n^{-2}[n]^*(C_U \alpha - \omega) =
C_U n^{-2}[n]^*\alpha - \omega$ is semi-positive. Since $\omega$ and $\nu$ are semi-positive, we get
\[
0 \leq \int_{\pi^{-1}(U) \cap X^o} \omega^{d_X + 1} \wedge (\pi^* \nu)^{d_B - 1} \leq \left(\frac{C_U}{n^2}\right)^{d_X + 1} \int_{X^o} (n)^* \alpha^{d_X + 1} \wedge (\pi^* \nu)^{d_B - 1}
\]
for all $n \geq 1$. Letting $n$ go to $+\infty$, equation (3.8) gives
\[
\int_{\pi^{-1}(U) \cap X^o} \omega^{d_X + 1} \wedge (\pi^* \nu)^{d_B - 1} = 0.
\] (3.9)

Since this holds for all relatively compact subsets $U$ of $B^o$, the theorem is proved.

**Corollary 3.3.** Assume that $X$ is small. Let $U$ and $V$ be open subsets of $B^o$ and
$X^o$ with respect to the euclidean topology such that $U$ contains the closure of
$\pi(V)$. Let $\mu$ be any smooth real semi-positive $(1,1)$-form on $U$. We have
\[
\int_V \omega^{d_X + 1} \wedge (\pi^* \mu)^{d_B - 1} = 0.
\]

**Proof of the Corollary.** Since $\omega$ and $\mu$ are semi-positive, the integral is non-
negative. Since $\nu$ is strictly positive on $U$, there is a constant $C > 0$ such that
$C \nu - \mu$ is semi-positive. From Theorem B we get
\[
0 \leq \int_V \omega^{d_X + 1} \wedge (\pi^* \mu)^{d_B - 1} \leq C^{d_B - 1} \int_V \omega^{d_X + 1} \wedge (\pi^* \nu)^{d_B - 1} = 0,
\] (3.10)
and the conclusion follows.

**Theorem B’.** Assume that $X$ is small. Then at every point $p \in X^o$, we have
$T_p F \subseteq T_p X^o$. In other words, $X^o$ is invariant under the Betti foliation: for
every $p \in X^o$, the leaf $F_p$ is contained in $X^o$.
Proof. We start with a simple remark. Let $P: \mathbb{C}^{N+1} \to \mathbb{C}^N$ be a complex linear map of rank $N$. Let $\omega_0$ be a positive $(1,1)$-form on $\mathbb{C}^N$. If $V$ is a complex linear subspace of $\mathbb{C}^{N+1}$ of dimension $N$, then $\ker(P) \subset V$ if and only if $P|V$ is not onto, if and only if $(P^*\omega_0^*)|V = 0$. Now, assume that $B$ has dimension 1. Then, the integral of $\omega^x + 1$ on $X_0$ vanishes; since the form $\omega$ is non-negative, the remark implies that the kernel of $P_p$ from Section 2.3 is contained in $T_pX_0$ at every smooth point $p$ of $X_0$. This proves the proposition when $d_B = 1$.

The general case reduces to $d_B = 1$ as follows. Let $U$ and $U'$ be open subsets of $B'(\mathbb{C})$ such that: (i) $U \subset U'$ in the euclidean topology and (ii) there are complex coordinates $(z_j)$ on $U'$ such that $U = \{|z_j| < 1, j = 1, \ldots, d_B\}$. Set
\[
\mu := i(dz_2 \wedge dz_2 + \cdots + dz_{d_B} \wedge d\overline{z}_{d_B}).
\]
It is a smooth real non-negative $(1,1)$-form on $U'$. By Corollary 3.3, we have
\[
\int_{\pi^{-1}(U) \cap X} \omega^{d_B - 1} - (\pi^*\mu)^{d_B - 1} = 0.
\]
For $(w_2, \ldots, w_{d_B})$ in $\mathbb{C}^{d_B - 1}$ with norm $|w_i| < 1$ for all $i$, consider the slice
\[
X(w_2, \ldots, w_{d_B}) = X \cap \pi^{-1}(U \cap \{z_2 = w_2, \ldots, z_{d_B} = w_{d_B}\});
\]
is this slice provides a family of subsets of $\mathcal{A}$ over the one-dimensional disk $\{(z_1, w_2, \ldots, w_{d_B}); |z_1| < 1\}$. Then, the integral of $\omega^x + 1$ over $X(w_2, \ldots, w_{d_B})$ vanishes for almost every point $(w_2, \ldots, w_{d_B})$; from the case $d_B = 1$, we deduce that, at every point $p$ of $X_0 \cap \pi^{-1}U$, the tangent $T_pX_0$ intersects $T_p\mathcal{F}$ on a line whose projection in $T_{\pi(p)}B$ is the line $\{z_2 = \cdots = z_{d_B} = 0\}$. Doing the same for all coordinates $z_i$, we see that $T_p\mathcal{F}$ is contained in $T_pX_0$. \hfill $\Box$

As a direct application of Theorem B’ and Remark 2.3, we prove Theorem A in the isotrivial case.

Corollary 3.4. If $A_{R} = A_{R}/C \otimes_{C} R$ and $X$ is small, then there exists a subvariety $Y \subseteq A_{R}/C$ such that $X \otimes_{K} R = Y \otimes_{C} R$.

Proof. Replacing $K$ by a suitable finite extension $K'$ and then $B$ by its normalization in $K'$, we may assume that $\mathcal{A} = B' \times A_{R}/C$ and that $\pi: B' \to B$ is the projection to the first factor. By Remark 2.3, the leaves of the Betti foliation are exactly the fibers of the projection $\pi_2$ onto the second factor. Since $X$ is small, Theorem B’ shows that $X = \pi_2^{-1}(Y)$, with $Y := \pi_2(X)$. \hfill $\Box$

4. Invariant analytic subsets of real and complex tori

Let $m$ be a positive integer. Let $M = \mathbb{R}^m/\mathbb{Z}^m$ be the torus of dimension $m$ and $\pi: \mathbb{R}^m \to M$ be the natural projection. The group $GL_m(\mathbb{Z})$ acts by real analytic homomorphisms on $M$. In this section, we study analytic subsets of $M$ which
are invariant under the action of a subgroup $\Gamma \subset \text{SL}_m(\mathbb{Z})$. The main ingredient is a result of Muchnik and of Guivarc’h and Starkov.

4.1. Zariski closure of $\Gamma$. We denote by
\[ G = \text{Zar}(\Gamma)^{irr} \] (4.1)
the neutral component, for the Zariski topology, of the Zariski closure of $\Gamma$ in $\text{GL}_m(\mathbb{R})$. We shall assume that $G$ is semi-simple. The real points $G(\mathbb{R})$ form a real Lie group, and the neutral component in the euclidean topology is denoted $G(\mathbb{R})^+$. Let $\Gamma_0$ be the intersection of $\Gamma$ with $G(\mathbb{R})^+$; then $\Gamma_0$ is both contained in $\text{GL}_m(\mathbb{Z})$ and Zariski dense in $G$: every polynomial equation that vanishes identically on $\Gamma_0$ vanishes also on $G$. But the Zariski closure of $\Gamma_0$ in $\text{GL}_m(\mathbb{R})$ may be larger than $G(\mathbb{R})^+$ (it may include other connected components).

We shall denote by $V$ the vector space $\mathbb{R}^m$; the lattice $\mathbb{Z}^m$ determines an integral, hence a rational structure on $V$. The Zariski closures $\text{Zar}(\Gamma)$ and $\text{Zar}(\Gamma_0)$ are $\mathbb{Q}$-algebraic subgroups of $\text{SL}_m$ for this rational structure.

We shall say that $\Gamma$ (or $G$) has no trivial factor if every $G$-invariant vector $u \in V$ is equal to 0. Note that this notion depends only on $G$, not on $\Gamma$.

4.2. Results of Muchnik and Guivarc’h and Starkov. Assume that $V$ is an irreducible representation of $G$ over $\mathbb{Q}$; this means that every proper $\mathbb{Q}$-subspace of $V$ which is $G$-invariant is the trivial subspace $\{0\}$. We decompose $V$ into irreducible subrepresentations of $G$ over $\mathbb{R}$,
\[ V = W_1 \oplus W_2 \oplus \cdots \oplus W_s. \] (4.2)
To each $W_i$ corresponds a subgroup $G_i$ of $\text{GL}(W_i)$ given by the restriction of the action of $G$ to $W_i$. Some of the groups $G_i(\mathbb{R})$ may be compact, and we denote by $V_c$ the sum of the corresponding subspaces: $V_c$ is the maximal $G$-invariant subspace of $V$ on which $G(\mathbb{R})$ acts by a compact factor. It is a proper subspace of $V$; indeed, if $V_c$ were equal to $V$ then $G(\mathbb{R})$ would be compact, $\Gamma$ would be finite, and $G$ would be trivial (contradicting the non-existence of trivial factor).

**Theorem 4.1** (Muchnik [15]; Guivarc’h and Starkov [11]). Assume that $G$ is semi-simple, and its representation on $\mathbb{Q}^m$ is irreducible. Let $x$ be an element of $M$. Then, one of the following two exclusive properties occur

1. the $\Gamma$-orbit of $x$ is dense in $M$;
2. there exists a torsion point $a \in M$ such that $x \in a + \pi(V_c)$.

In the second assertion, the torsion point $a$ is uniquely determined by $x$, because otherwise $V_c$ would contain a non-zero rational vector and the representation $V$ would not be irreducible over $\mathbb{Q}$. As a corollary, if $F \subset M$ is a closed, proper, connected and $\Gamma$-invariant subset, then $F$ is contained in a translate of
\pi(V_c) by a (unique) torsion point. Also, if \( x \) is a point of \( M \) with a finite orbit under the action of \( \Gamma \), then \( x \) is a torsion point.

**Remark 4.2.** Theorem 4.1 will be used to describe \( \Gamma \)-invariant real analytic subsets \( Z \subset M \). If it is infinite, such a set contains the image of a non-constant real analytic curve. The existence of such a curve in \( Z \) is the main difficulty in Muchnik’s argument, but in our situation it is given for free.

**Remark 4.3.** Assume that \( m = 2g \) for some \( g \geq 1 \) and \( M \) is in fact a complex torus \( \mathbb{C}^g/\Lambda \), with \( \Lambda \simeq \mathbb{Z}^{2g} \). Suppose that \( F \) is a complex analytic subset of \( M \). The inclusion \( F \to M \) factors through the Albanese torus \( F \to A_F \) of \( F \), via a morphism \( A_F \to M \), and the image of \( A_F \) is the quotient of a subspace \( W \) in \( \mathbb{C}^g \) by a lattice \( W \cap \Lambda \). So, if \( F \subset a + \pi(V_c) \), the subspace \( V_c \) contains a subspace \( W \subset \mathbb{R}^m \) which is defined over \( \mathbb{Q} \), contradicting the irreducibility assumption.

To separate clearly the arguments of complex geometry from the arguments of dynamical systems, we shall not use this type of idea before Section 4.4.

**Remark 4.4.** Theorem 2 of [11] should assume that the group \( G \) has no compact factor (this is implicitly assumed in [11, Proposition 1.3]).

### 4.3. Invariant real analytic subsets

Let \( F \) be an analytic subset of \( M \). We say that \( F \) does not **fully generate** \( M \) if there is a proper subspace \( W \) of \( V \) and a non-empty open subset \( \mathcal{U} \) of \( F \) such that \( T_x F \subset W \) for every regular point \( x \) of \( F \) in \( \mathcal{U} \). Otherwise, we say that \( F \) fully generates \( M \).

**Proposition 4.5.** Let \( \Gamma \) be a subgroup of \( \text{GL}_m(\mathbb{Z}) \). Assume that the neutral component \( \text{Zar}(\Gamma)^{irr} \subset \text{GL}_m(\mathbb{R}) \) is semi-simple, and has no trivial factor. Let \( F \) be a real analytic and \( \Gamma \)-invariant subset of \( M \). If \( F \) fully generates \( M \), it is equal to \( M \).

To prove this result, we decompose the linear representation of \( G = \text{Zar}(\Gamma)^{irr} \) on \( V \) into a direct sum of irreducible representations over \( \mathbb{Q} \):

\[
V = V_1 \oplus \cdots \oplus V_s. \tag{4.3}
\]

Since there is no trivial factor, none of the \( V_i \) is the trivial representation. For each index \( i \), we denote by \( V_{i,c} \) the compact factor of \( V_i \). The projection \( \pi \) is a diffeomorphism from \( V_{i,c} \) onto its image in \( M_i \), because otherwise \( V_{i,c} \) would contain a non-zero vector in \( \mathbb{Z}^m \) and \( V_i \) would not be an irreducible representation over \( \mathbb{Q} \). Set

\[
M_i = V_i / (\mathbb{Z}^m \cap V_i). \tag{4.4}
\]

Then, each \( M_i \) is a compact torus of dimension \( \dim(V_i) \), and \( M \) is isogenous to the product of the \( M_i \). We may, and we shall assume that \( M \) is in fact equal to this product:

\[
M = M_1 \times \cdots \times M_s; \tag{4.5}
\]
lytic too. We also assume, with no loss of generality, that \( \Gamma \) cause the image and the pre-image of a real analytic set by an isogeny is analytic too. We also assume, with no loss of generality, that \( \Gamma \) is contained in \( G \). For every index \( 1 \leq i \leq s \), we denote by \( \pi_i \) the projection on the \( i \)-th factor \( M_i \).

**Lemma 4.6.** If \( F \) fully generates \( M \), the projection \( F_i := \pi_i(F) \) is equal to \( M_i \) for every \( 1 \leq i \leq s \).

**Proof.** By construction, \( F_i \) is a closed, \( \Gamma \)-invariant subset of \( M_i \). Fix a connected component \( F_i^0 \) of \( F_i \). If it were contained in a translate of \( \pi(V_{i,c}) \), then \( F \) would not fully generate \( M \). Thus, Theorem 4.1 implies \( F_i^0 = M_i \). \( \square \)

We do an induction on the number \( s \) of irreducible factors. For just one factor, this is the previous lemma. Assuming that the proposition has been proven for \( s-1 \) irreducible factors, we now want to prove it for \( s \) factors. To simplify the exposition, we suppose that \( s = 2 \), which means that \( M \) is the product of just two factors \( M_1 \times M_2 \). The proof will only use that \( \pi_1(f) = M_1 \) and \( F \) fully generates \( M \); thus, changing \( M_1 \) into \( M_1 \times \ldots \times M_{s-1} \), this proof also establishes the induction in full generality.

There is a closed subanalytic subset \( Z_1 \) of \( M_1 \) with empty interior such that \( \pi_1 \) restricts to a locally trivial analytic fibration from \( F \setminus \pi_1^{-1}(Z_1) \) to \( M_1 \setminus Z_1 \). If \( F \) does not coincide with \( M \), the fiber \( F_c \) is a proper, non-empty analytic subset of \( \{x\} \times M_2 \) for every \( x \) in \( M_1 \setminus Z_1 \). We shall derive a contradiction from the fact that \( F \) fully generates \( M \).

Theorem 4.1 tells us that, for every torsion point \( x \) in \( M_1 \setminus Z_1 \), there is a finite set of points \( a_j(x) \) in \( M_2 \) such that

\[
F_x \subset \bigcup_{j=1}^{J} a_j(x) + \pi(V_{2,c});
\]

the number of such points \( a_j(x) \) is bounded from above by the number of connected components of \( F_x \). Since torsion points are dense in \( M_1 \), this property holds for every point \( x \) in \( M_1 \setminus Z_1 \) (the \( a_j(x) \) are not torsion points a priori). Since there are points with a dense \( \Gamma \)-orbit in \( M_1 \), we can assume that the number \( J \) of points \( a_j(x) \) does not depend on \( x \).

Assume temporarily that \( J = 1 \), so that \( F_x \) is contained in \( a(x) + \pi(V_{2,c}) \) for some point \( a(x) \) of \( M_2 \). The point \( a(x) \) is not uniquely defined by this property (one can replace it by \( a(x) + \pi(v) \) for any \( v \in V_{2,c} \), but there is a way to choose \( a(x) \) canonically. First, the action of \( G(\mathbb{R}) \) on \( V_{2,c} \) factors through a compact subgroup of \( \text{GL}(V_{2,c}) \), so we can fix a \( G(\mathbb{R}) \)-invariant euclidean metric \( \text{dist}_2 \) on \( V_{2,c} \). Then, any compact subset \( K \) of \( V_{2,c} \) is contained in a unique ball of smallest radius for the metric \( \text{dist}_2 \); we denote by \( c(K) \) and \( r(K) \) the center and radius of this ball. Since the projection \( \pi \) is a diffeomorphism from \( V_{2,c} \) onto
its image in $M_2$, the center of $F_x$ inside the translate of $\pi(V_{2,c})$ containing $F_x$ is a well defined point

$$c(x) := c(F_x)$$

of $M_2$ such that $F_x$ is contained in $c(x) + \pi(V_{2,c})$. When $J > 1$, this procedure gives a finite set of centers $\{c_j(x)\}_{1 \leq j \leq J}$.

The centers $c_j(x)$ and the radii $r_j(x)$ are (restricted) sub-analytic functions of $x$. Thus, there is a proper, closed analytic subset $D_1$ of $M_1$, containing $Z_1$, such that all $r_j(x)$ and $c_j(x)$ are smooth and analytic on its complement (see [2, 4, 17]). Let $\tilde{G}$ be the subset of $\pi_{1}^{-1}(M_1 \setminus D_1)$ given by the union of the graphs of the centers: $\tilde{G} = \{(x,y) \in M_1 \times M_2; x \in M_1 \setminus D_1, y = c_j(x) \text{ for some } j\}$.

**Lemma 4.7.** The set $\tilde{G}$ is contained in finitely many translates of subtori of $M_1 \times M_2$, each of dimension $\dim M_1$.

This lemma concludes the proof of Proposition 4.5, because if $\tilde{G}$ is locally contained in $a + \pi(W)$ for some proper subset $W$ of $V$ of dimension $\dim M_1$, then $F$ is locally contained in $a + \pi(W + V_{2,c})$, and $F$ does not fully generate $M$ because $\dim(W + V_{2,c}) < \dim V$.

**Proof.** By construction, $\tilde{G}$ is a smooth analytic subset of $\pi_{1}^{-1}(M_1 \setminus D_1)$ and it is invariant by $\Gamma$. For $x$ in $M_1 \setminus D_1$, we denote by $\tilde{G}_x$ the finite fiber $\pi_{1}^{-1}(x) \cap \tilde{G}$. Fix one of these torsion points $z = (x,y)$ with $x$ in $M_1 \setminus D_1$, and consider the tangent subspace $T_z \tilde{G}$. It is the graph of a linear morphism $\varphi_z: T_zM_1 \to T_zM_2$. Identifying the tangent spaces $T_zM_1$ and $T_zM_2$ with $V_1$ and $V_2$ respectively, $\varphi_z$ becomes a morphism that interlaces the representations $\rho_1$ and $\rho_2$ of $\Gamma_x'$ on $V_1$ and $V_2$; since $\Gamma_x'$ is Zariski dense in $G$, we get

$$\rho_2(g) \circ \varphi_z = \varphi_z \circ \rho_1(g)$$

for every $g$ in $G$. In other words, $\varphi_z \in \operatorname{End}(V_1;V_2)$ is a morphism of $G$-spaces. This holds for every torsion point $z$ of $\tilde{G}$; by continuity of tangent spaces and density of torsion points, this holds everywhere on $\tilde{G}$.

Since $\tilde{G}$ is $\Gamma$-invariant, we also have

$$\varphi_{g(z)} \circ \rho_1(g) = \rho_2(g) \circ \varphi_z$$

for all $g \in \Gamma$ and $z \in \tilde{G}$. Then equation (4.8) shows that $\varphi_{g(z)} = \varphi_z$, which means that the tangent space $T_z \tilde{G}$ is constant along the orbits of $\Gamma$. Taking a point $z$ in $\tilde{G}$ whose first projection has a dense $\Gamma$-orbit in $M_1$, we see that the tangent space $w \in \tilde{G} \mapsto T_w \tilde{G}$ takes only finitely many values, at most $|\tilde{G}_{\pi_1(z)}|$.
Let \((W_j)_{1 \leq j \leq k}\) be the list of possible tangent spaces \(T_xG\). Locally, near any point \(z \in G\), \(G\) coincides with \(z + \pi(W_j)\) for some \(j\). By analytic continuation \(G\) contains the intersection of \(z + \pi(W_j)\) with \(\pi^{-1}_1(M_1 \setminus D_1)\); thus, \(W_j\) is a rational subspace of \(V\) and \(\pi(W_j)\) is a subtorus of \(M\). Then \(G\) is contained in a finite union of translates of the tori \(\pi(W_j)\).

4.4. **Complex analytic invariant subsets.** Let \(J\) be a complex structure on \(V = \mathbb{R}^m\), so that \(M\) is now endowed with a structure of complex torus. Then, \(m = 2g\) for some integer \(g\), \(\mathbb{R}^m\) can be identified to \(\mathbb{C}^g\), and \(M = \mathbb{C}^g/\Lambda\) where \(\Lambda\) is the lattice \(\mathbb{Z}^m\); to simplify the exposition, we denote by \(A\) the complex torus \(\mathbb{C}^g/\Lambda\) and by \(M\) the real torus \(\mathbb{R}^m/\mathbb{Z}^m\). Thus, \(A\) is just \(M\), together with the complex structure \(J\). Let \(X\) be an irreducible complex analytic subset of \(A\), and let \(X^{\text{reg}}\) be its smooth locus.

**Lemma 4.8.** Let \(W\) be the real subspace of \(V\) generated by the tangent spaces \(T_xX\), for \(x \in X^{\text{reg}}\). Then \(W\) is both complex and rational, and \(X\) is contained in a translate of the complex torus \(\pi(W)\).

**Proof.** Since \(X\) is complex, its tangent space is invariant under the complex structure: \(1T_xX = T_xX\) for all \(x \in X^{\text{reg}}\). So, the sum \(W := \sum_x T_xX\) of the \(T_xX\) over all points \(x \in X^{\text{reg}}\) is invariant by \(J\) and \(W\) is a complex subspace of \(V \simeq \mathbb{C}^g\). Observe that if \(V'\) is any real subspace of \(V\) such that \(\pi(V')\) contains some translate of \(X^{\text{reg}}\), then \(W \subseteq V'\).

Let \(a\) be a point of \(X^{\text{reg}}\), and \(Y\) be the translate \(X - a\) of \(X\). It is an irreducible complex analytic subset of \(A\) that contains the origin \(0\) of \(A\) and satisfies \(T_yY \subset W\) for every \(y \in Y^{\text{reg}}\). Thus, \(Y^{\text{reg}}\) is contained in the projection \(\pi(W) \subset A\). Set \(Y^{(1)} = Y, Y^{(0)}_o = Y^{\text{reg}}\) and then
\[
Y^{(\ell+1)} = Y^{(\ell)} - Y^{(\ell)}, \quad Y^{(\ell+1)}_o = Y^{(\ell)}_o - Y^{(\ell)}_o
\] (4.10)
for every integer \(\ell \geq 1\). Since \(Y^{(1)}\) is irreducible, and \(Y^{(2)}\) is the image of \(Y^{(1)} \times Y^{(1)}\) by the complex analytic map \((y_1, y_2) \mapsto y_1 - y_2\), we see that \(Y^{(2)}\) is an irreducible complex analytic subset of \(A\). Moreover \(Y^{(2)}_o\) is a connected, dense, and open subset of \(Y^{(2),\text{reg}}\). Observe that \(Y^{(2)}_o\) is contained in \(\pi(W)\) and contains \(Y^{(1)}_o\) because \(0 \in Y^{(1)}_o\). By a simple induction, the sets \(Y^{(\ell)}\) form an increasing sequence of irreducible complex analytic subsets of \(A\), and \(Y^{(\ell)}_o\) is a connected, dense and open subset of \(Y^{(\ell),\text{reg}}\) that is contained in \(\pi(W)\). By the Noether property, there is an index \(\ell_0 \geq 1\) such that \(Y^{(\ell)} = Y^{(\ell_0)}\) for every \(\ell \geq \ell_0\). This complex analytic set is a subgroup of \(A\), hence it is a complex subtorus. Write \(Y^{(\ell_0)} = \pi(V')\) for some rational subspace \(V'\) of \(V\). Since \(Y \subset \pi(V')\), we get \(W \subseteq V'\). Since \(Y^{(\ell_0)}_o \subseteq \pi(W)\), we derive \(V' = T_xY^{(\ell_0)}_o \subseteq W\) for every \(x \in Y^{(\ell_0)}_o\). This implies \(W = V'\), and shows that \(W\) is rational.
Thus, $\pi(W)$ is a complex subtorus of $A$. Since $T_xX$ is contained in $W$ for every regular point, $X$ is locally contained in a translate of $\pi(W)$. Being irreducible, $X$ is connected, and it is contained in a unique translate $a + \pi(W)$.

Lemma 4.9. Let $X$ be an irreducible complex analytic subset of $A$. The following properties are equivalent:

(i) $X$ is contained in a translate of a proper complex subtorus $B \subset A$;
(ii) $X$ does not fully generate $M$;
(iii) there is a proper real subspace $V'$ of $V$ that contains $T_xX$ for every $x \in X^{\text{reg}}$.

Proof. Obviously (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii). We now prove that (ii) implies (i). If $X$ does not fully generate $M$, then (iii) is satisfied on some non-empty open subset $U$ of $X^{\text{reg}}$. Since $X^{\text{reg}}$ is connected and locally analytic, we deduce from analytic continuation that $T_xX \subset V'$ for every regular point of $X$. From Lemma 4.8, $X$ is contained in a complex subtorus $B = \pi(W) \subset A$ for some complex subspace $W$ of $V'$.

Theorem 4.10. Let $\Gamma$ be a subgroup of $\text{SL}_m(\mathbb{Z})$. Assume that the neutral component for the Zariski topology of the Zariski closure of $\Gamma$ in $\text{SL}_m(\mathbb{R})$ is semi-simple and has no trivial factor. Let $\mathfrak{i}$ be a complex structure on $M = \mathbb{R}^m/\mathbb{Z}^m$ and let $X$ be an irreducible complex analytic subset of the complex torus $A = (M, \mathfrak{i})$. If $X$ is $\Gamma$-invariant, it is equal to a translate of a complex subtorus $B \subset A$ by a torsion point.

Proof. Set $W := \sum_{x \in X^{\text{reg}}} T_xX$. Lemma 4.8 shows that $W$ is complex and rational. Since $X$ is $\Gamma$-invariant, so is $W$. Its projection $B = \pi(W) \subset A$ is a complex subtorus of $A$ such that

1. $B$ is $\Gamma$-invariant;
2. $B$ contains a translate $Y = X - a$ of $X$;
3. $Y$ fully generates $B$.

The group $\Gamma$ acts on the quotient torus $A/B$ and preserves the image of $X$, i.e. the image $\overline{a}$ of $a$. Since $V$ has no trivial factor, $\overline{a}$ is a torsion point of $A/B$. Then there exists a torsion point $a'$ in $A$ such that $X \subseteq a' + B$. Replacing $a$ by $a'$ and $\Gamma$ by a finite index subgroup $\Gamma'$ which fixes $a'$, we may assume that $a$ is torsion and $Y = X - a$ is invariant by $\Gamma$. We apply Proposition 4.5 to $B$, the restriction $\Gamma_B$ of $\Gamma$ to $B$, and the complex analytic subset $Y$: we conclude via Lemma 4.9 that $Y$ coincides with $B$. Thus, $X = a + B$. 

5. PROOF OF THEOREM A

By base change, we may suppose that \( X \) is an absolutely irreducible subvariety of \( A \). We assume that \( X \) is small (\( X_\varepsilon \) is dense in \( X \) for all \( \varepsilon > 0 \)), and prove that \( X \) is a torsion coset of \( A \).

5.1. Monodromy and invariance. Let \( b \in B^o \) be any point. The monodromy \( \rho : \pi_1(B^o) \to \text{GL}_{2g}(\mathbb{Z}) \) of the Betti foliation maps the fundamental group of \( \pi_1(B^o) \) onto a subgroup \( \Gamma := \text{Im}(\rho) \) of \( \text{GL}_{2g}(\mathbb{Z}) \) that acts by linear diffeomorphisms on the torus \( \mathcal{A}_b \cong \mathbb{R}^{2g}/\mathbb{Z}^{2g} \). As in Section 4.1, we denote by \( G \) the neutral component \( \text{Zar}(\Gamma)^{irr} \). We let \( V^G \) denote the subspace of elements \( \nu \in \mathbb{R}^{2g} \) which are fixed by \( G \). By Deligne’s semi-simplicity theorem, the group \( G \) is semi-simple (see [5, Corollary 4.2.9]). Theorem B’ implies that \( X \) is invariant under the Betti foliation, so that \( \mathcal{X}_b \) is invariant under the action of \( \Gamma \).

5.2. Trivial trace. We first treat the case when \( A^{\mathbb{K}/\mathbb{C}} \) is trivial. According to [22, Theorem 1.5], this is the only case we need to treat. However we shall also treat the case of a non-trivial trace below for completeness.

By [5, Corollary 4.1.2] and [8] (see also [5, 4.1.3.2]), we have \( V^G = \{0\} \) and Theorem 4.10 implies that \( \mathcal{X}_b \) is a translation of an abelian subvariety of \( \mathcal{A}_b \) by some torsion point \( y_b \in \mathcal{A}_b \). Observe that the leaf \( \mathcal{F}_{y_b} \) is an algebraic muti-section of \( \mathcal{A}^o \) (see Remark 2.1). By base change, we may assume that \( \mathcal{F}_{y_b} \) is a section and is the Zariski closure of a torsion point \( y \in A(\mathbb{K}) \) in \( \mathcal{A}^o \).

Theorem B’ shows that \( y \in X \), and replacing \( X \) by \( X - y \) we may suppose that \( 0 \in X \); then \( \mathcal{X}_b \) is an abelian subvariety of \( \mathcal{A}_b \) for all \( b \in B^o \). It follows that \( X^o \) is a subscheme of the abelian scheme \( \mathcal{A}^o \) over \( B^o \) which is stable under the group laws. So \( X \) is an abelian subvariety of \( A \).

5.3. The general case. We do not assume anymore that \( A^{\mathbb{K}/\mathbb{C}} \) is trivial. Set \( \mathcal{A}' = A^{\mathbb{K}/\mathbb{C}} \otimes_{\mathbb{C}} K \). Replacing \( K \) by a finite extension and \( A \) by a finite cover, we assume that \( A = A' \times A'' \) where \( A'' \) is an abelian variety over \( K \) with trivial trace. We also choose the model \( \mathcal{A} \) so that \( \mathcal{A}^o = (\mathcal{A}')^o \times_{B^o} (\mathcal{A}'')^o \) where \( (\mathcal{A}')^o \) and \( (\mathcal{A}'')^o \) are the Zariski closures of \( \mathcal{A}' \) and \( \mathcal{A}'' \) in \( \mathcal{A}^o \) respectively. Denote by \( \pi' : \mathcal{A}^o \to (\mathcal{A}')^o \) the projection to the first factor and \( \pi'' : \mathcal{A}^o \to (\mathcal{A}'')^o \) the projection to the second factor. After replacing \( K \) by a further finite extension and \( B \) by its normalization, we may assume that \( (\mathcal{A}')^o = A^{\mathbb{K}/\mathbb{C}} \times B^o \). Note that \( \pi'|_{\mathcal{A}'_b} : \mathcal{A}'_b \to A^{\mathbb{K}/\mathbb{C}} \) is an isomorphism for every fiber \( \mathcal{A}'_b \) with \( b \in B^o \).

By Proposition 3.2-(i), the generic fibers of \( \pi'(X^o) \) and \( \pi''(X^o) \) are small. Corollary 3.4 shows that \( \pi' (X^o) = Y \times B^o \) for some subvariety \( Y \) of \( A^{\mathbb{K}/\mathbb{C}} \). Section 5.2 shows that the geometric generic fiber of \( \pi''(X^o) \) is a torsion coset \( a + \mathcal{A}' \) for some torsion point \( a \in A''(K) \) and some abelian subvariety \( \mathcal{A}' \). Replacing \( K \) by a finite extension, we may assume that \( a \) and \( \mathcal{A}' \) are defined over
$K$. We have that $X^o \subseteq \pi'(X) \times_{B^o} \pi^{\text{nt}}(X) = \pi'(X) + \pi^{\text{nt}}(X)$ and we only need to show that $X^o = \pi'(X) \times_{B^o} \pi^{\text{nt}}(X)$.

For every $b \in B^o$, $A_b = A_b' \times A_b^{\text{nt}}$. The monodromy on $A_b$ is the diagonal product of the monodromies on each factor. It is trivial on the first one so, for every $x \in A_b'$, the fiber $\pi^1_{A_b'}(x) \simeq A_b^{\text{nt}}$ is invariant under $\Gamma$. It follows that $\pi^1_{A_b'}(x) \cap \mathcal{X}_b$ is also $\Gamma$-invariant. By Theorem 4.10, $\pi^{\text{nt}}(\pi^1_{A_b'}(x) \cap \mathcal{X}_b) \subseteq \pi^{\text{nt}}(\mathcal{X}_b)$ is a torsion coset of the abelian variety $A_b^{\text{nt}}$. Since the set of all torsion cosets of $\pi^{\text{nt}}(\mathcal{X}_b)$ is countable, $\pi^{\text{nt}}(\pi^1_{A_b'}(x) \cap \mathcal{X}_b)$ does not depend on $x \in \pi'(\mathcal{X}_b)$. Hence, $\mathcal{X}_b = \pi'(\mathcal{X}_b) \times \pi^{\text{nt}}(\mathcal{X}_b)$ for all $b \in B^o$. Then $X^o = \pi'(X) \times_{B^o} \pi^{\text{nt}}(X)$ which concludes the proof.

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SERGE CANTAT, IRMAR, CAMPUS DE BEAULIEU, BÂTIMENTS 22-23 263 AVENUE DU GÉNÉRAL LECLERC, CS 74205 35042 RENNES CÉDEX

E-mail address: serge.cantat@univ-rennes1.fr

ZIYANG GAO, CNRS, IMJ-PRG, 4 PLACE DE JUSSIEU, 75005 PARIS, FRANCE; DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, NJ 08544, USA

E-mail address: ziyang.gao@imj-prg.fr

PHILIPP HABEGGER, DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, UNIVERSITY OF BASEL, SPIEGELGASSE 1,4051 BASEL, SWITZERLAND

E-mail address: philipp.habegger@unibas.ch

JUNYI XIE, IRMAR, CAMPUS DE BEAULIEU, BÂTIMENTS 22-23 263 AVENUE DU GÉNÉRAL LECLERC, CS 74205 35042 RENNES CÉDEX

E-mail address: junyi.xie@univ-rennes1.fr