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par

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Distribution of points on varieties : various aspects and interactions

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Introduction (English version)

It is a fundamental question in math to solve equations. For example let $f(X, Y)$ be a polynomial in X and Y with coefficients in $\mathbb{Q}$. One wishes to find its $\mathbb{Q}$-solutions, namely the rational numbers x and y such that $f(x, y) = 0$.

This question is too hard to answer in general. For example, if $f(X, Y) = X^n + Y^n - 1$ for some $n \geq 3$, then the question of finding the $\mathbb{Q}$-solutions to this polynomial is equivalent to the famous *Fermat's Last Theorem*, proved by Wiles and Taylor–Wiles [Wil95, TW95] in 1995.

Instead, here is a more achievable but still fundamental question.

Question. *Is there an easy upper bound for the number of the $\mathbb{Q}$-solutions? How do these $\mathbb{Q}$-solutions distribute?*

The formulation of the question being simple, it took mathematicians decades and even centuries to answer. In modern language, the $\mathbb{Q}$-solutions becomes *rational points* on algebraic varieties.

The first steps to treat this problem are the following standard operations. First one embeds $\mathbb{C}^2$ into the complex projective space $\mathbb{P}^2(\mathbb{C})$. Then using $[X_0 : X_1 : X_2]$ to denote the projective coordinates on $\mathbb{P}^2(\mathbb{C})$, one defines the homogeneous polynomial $F(X_0, X_1, X_2) = X_0^{\deg f} f(\frac{X_1}{X_0}, \frac{X_2}{X_0})$. Now $F(X_0, X_1, X_2)$ defines a curve in $\mathbb{P}^2(\mathbb{C})$, which is the closure of the affine curve $\{(x, y) \in \mathbb{C}^2 : f(x, y) = 0\}$ in $\mathbb{P}^2(\mathbb{C})$. We call this new curve a *projective curve* defined over $\mathbb{Q}$. Note that only finitely many points are added. Next associated to each projective curve there is an intrinsic integer $g \geq 0$, called the *genus*. For example if it has only ordinary double singular points, then $g = \frac{(\deg F - 1)(\deg F - 2)}{2} - k$ with k the number of singular points. In general, it is known that any projective curve can be converted by a Cremona transformation into a projective curve whose singular points are ordinary double.

With this intrinsic integer in hand, one can already see some properties of $C(\mathbb{Q})$, the set of $\mathbb{Q}$-points on the curve C in question. When $g = 0$, $C(\mathbb{Q})$ is either empty or an infinite set, and the structure of $C(\mathbb{Q})$ is rather simple in this case. When $g = 1$, $C(\mathbb{Q})$ has a structure of abelian groups. The torsion part of $C(\mathbb{Q})$ is completely studied by Mazur [Maz77] (over number field by Merel [Mer96]), and its torsion-free part is related to the Birch and Swinnerton-Dyer conjecture.

When $g \geq 2$, the first step towards understanding $C(\mathbb{Q})$ is the *Mordell conjecture* (1922): Let C be a geometrically irreducible smooth projective curve of genus $g \geq 2$ defined over a number field K, then $C(K)$ is finite; see [Mor22]. The Mordell conjecture was proved by Faltings in 1983 [Fal83], as a consequence of his proofs of the Tate conjecture and the Shafarevich conjecture for abelian varieties.

Knowing the finiteness of $C(K)$, the next step is to look for an upper bound on $\#C(K)$. It is not hard to see that the cardinality $\#C(K)$ must depend on the genus g and the degree of the definition field $[K : \mathbb{Q}]$.

In the 90s, Vojta gave a completely new proof of the Mordell conjecture. This proof was later on simplified and generalized by Faltings [Fal91a], and further simplified by Bombieri [Bom90a]. In the proof, one first embeds the curve C into its Jacobian J by the Abel–Jacobi embedding via a rational point (if there exists any). A number associated with J, called the *Faltings height* and denoted by $h_{\mathrm{Fal}}(J)$, played a crucial role.[1] Roughly speaking, this number measures the "complexity" of the coefficients of the equations defining the curve C. Based on this new approach, Rémond [Rém00] proved an explicit upper bound

$$\#C(K) \leq c(g, d, h_{\mathrm{Fal}}(J))^{\rho+1}, \qquad (0.0.1)$$

where $c(g, d, h_{\mathrm{Fal}}(J))$ is a constant depending only on g, $d := [K : \mathbb{Q}]$ and $h_{\mathrm{Fal}}(J)$. The integer $\rho \geq 0$, called the *Mordell–Weil rank over* K, is defined by $\rho = \mathrm{rank}_{\mathbb{Z}} J(K)$. The Mordell–Weil theorem says that $J(K)$ is a finitely generated abelian group, and hence ρ is finite.

On the other hand, it has been expected that $h_{\mathrm{Fal}}(J)$ does not show up in the bound (0.0.1). This question was asked by Mazur shortly after Faltings's proof of the Mordell conjecture; see [Maz00, pp.223] and [Maz86, top of pp.234].

In a joint project with V. Dimitrov and P. Habegger, we answered this question of Mazur affirmatively. The key new ingredient is to establish a *new Gap Principle* which concerns the distribution of rational and more generally algebraic points on curves of genus at least 2.

This memoire sees a summary of the following aspects on distribution of points on algebraic varieties and their interactions.

- Chapter 1: Bound of $\#C(K)$ as conjectured by Mazur (Theorem 1.1.1), and distribution of algebraic points in curves of genus ≥ 2 (Proposition 1.2.1). This concerns a series of work [GH19, DGH19, Gao20a, DGH20].

- Chapter 2: Small points in abelian varieties. The main result is the Geometric Bogomolov Conjecture over characteristic 0 (Theorem 2.2.1). This concerns [GH19, CGHX21].

[1] These were also used in Faltings's first proof of the Mordell conjecture.

- Chapter 3: Special points in mixed Shimura varieties. The main result is towards the André–Oort conjecture for mixed Shimura varieties (Theorem 3.1.2). This concerns [Gao17b, Gao16]. In this chapter, one also sees a finiteness result *à la Bogomolov* (Theorem 3.2.4 from [Gao20b, Thm.1.4]), which is a useful tool to study unlikely intersection problems in mixed Shimura varieties.

- Chapter 4: Interactions. In this chapter, various aspects used to study the problems in the previous chapters are presented. They are (i) (Ax type) functional transcendence results in mixed Shimura varieties in §4.1 ([Gao17b, Gao20b]); (ii) the degeneracy loci of subvarieties of abelian schemes in §4.2 ([Gao20a]); (iii) the Betti map and its generic rank in §4.3 ([Gao20a]); (iv) the height inequality to compare the fiberwise Néron–Tate height with the height on the base for an abelian scheme in §4.4 ([DGH19, DGH20, Gao20a]).

Introduction (version française)

Une question fondamentale en mathématiques est de trouver les solutions des équations. Par exemple soit $f(X,Y)$ un polynôme en deux variables à coefficients dans $\mathbb{Q}$. Il est naturel de penser à chercher les $\mathbb{Q}$-solutions, c'est-à-dire les nombres rationnels x et y tels que $f(x,y)=0$.

Pourtant, il est en général trop difficile de répondre à cette question, comme l'exemple suivant permet de se rendre compte : soit $f(X,Y)=X^n+Y^n-1$ pour un entier $n\geq 3$, alors la question de trouver toutes les $\mathbb{Q}$-solutions de f est précisément le grand théorème de Fermat, qui n'a été démontré qu'en 1995 par Wiles et Taylor–Wiles [Wil95, TW95].

Or, voici une question qui est toujours fondamentale mais plus abordable :

Question. *Est-ce qu'il existe une majoration simple pour les nombres de $\mathbb{Q}$-solutions ? Comment les $\mathbb{Q}$-solutions se répartissent ?*

Cette question étant facilement formulée, y répondre a pris des décennies voire des siècles aux mathématiciens. En langage moderne, les $\mathbb{Q}$-solutions deviennent des *points rationnels* sur les variétés algébriques.

Les premières étapes pour traiter ce problème sont des opérations standards. Tout d'abord l'on prolonge $\mathbb{C}^2$ dans l'espace projectif $\mathbb{P}^2(\mathbb{C})$. Ensuite, en utilisant les coordonnées homogènes $[X_0 : X_1 : X_2]$ de $\mathbb{P}^2(\mathbb{C})$, l'on définit un polynôme $F(X_0,X_1,X_2) = X_0^{\deg f} f(\frac{X_1}{X_0},\frac{X_2}{X_0})$. Maintenant $F(X_0,X_1,X_2)$ définit une courbe dans $\mathbb{P}^2(\mathbb{C})$, qui est l'adhérence de la courbe affine $\{(x,y)\in\mathbb{C}^2 : f(x,y)=0\}$ dans $\mathbb{P}^2(\mathbb{C})$. Nous appelons cette nouvelle courbe une *courbe projective* définie sur $\mathbb{Q}$. Remarquons que seulement un nombre fini de points sont ajoutés. Après quoi, un entier intrinsèque $g\geq 0$ peut être associé à chaque courbe projective, et cet entier s'appelle le *genre*. Par exemple si la courbe n'admet que des nœuds (doubles ordinaires) comme points singuliers, alors $g=\frac{(\deg F-1)(\deg F-2)}{2}-k$ avec k nombre des points singuliers. En général, chaque courbe projective peut être transformée, par une transformation de Cremona, en une courbe projective n'ayant que des nœuds comme points singuliers.

À partir de cet entier intrinsèque, l'on peut déjà voir plusieurs propriétés de $C(\mathbb{Q})$, qui est l'ensemble des $\mathbb{Q}$-points sur la courbe C en question. Lorsque $g=0$, $C(\mathbb{Q})$ est soit vide soit un ensemble infini ; la structure de $C(\mathbb{Q})$ est plutôt simple dans ce cas. Lorsque $g=1$, $C(\mathbb{Q})$ admet une structure de groupes abéliens. La partie de torsion de $C(\mathbb{Q})$ a été étudiée par Mazur [Maz77] (sur un corps de nombres quelconque par Merel [Mer96]), et l'analyse de la partie sans-torsion est fournie par la conjecture de Birch et Swinnerton-Dyer.

Lorsque $g \geq 2$, la première étape pour comprendre $C(\mathbb{Q})$ est la *conjecture de Mordell* (1922) : soit C une courbe projective lisse géométriquement irréductible de genre $g \geq 2$ définie sur un corps de nombres K, alors $C(K)$ est un ensemble fini ; voir [Mor22]. La conjecture de Mordell a été démontrée par Faltings en 1983 [Fal83] en tant que conséquence de ses démonstrations de la conjecture de Tate et de la conjecture de Shafarevich pour les variétés abéliennes.

Après avoir démontré la finitude de $C(K)$, la prochaine étape est de chercher une majoration pour $\#C(K)$. L'observation suivante n'est pas difficile : $\#C(K)$ dépend forcément du genre g et du degré du corps de nombres $[K : \mathbb{Q}]$. Le résultat suivant a été démontré par Caporaso–Harris–Mazur [CHM97] (version améliorée par Pacelli [Pac97]) : *si la conjecture de Lang est vraie*, alors il existe un nombre $B = B(g, [K : \mathbb{Q}]) > 0$ tel que $\#C(K) \leq B$ pour toute C. Ce résultat entraîne deux avis différents : les uns croient par conséquent qu'une majoration par g et $[K : \mathbb{Q}]$ existe effectivement, tandis que les autres pensent que la conjecture de Lang doit être modifiée puisque cette majoration ne peut pas être vraie.

Dans les années 90, Vojta a re-démontré la conjecture de Mordell par une nouvelle méthode. Sa preuve a été simplifiée et généralisée par Faltings [Fal91a], suivant une simplification approfondie par Bombieri [Bom90a]. Dans cette approche, l'on prolonge tout d'abord la courbe C dans sa jacobienne J par l'immersion d'Abel–Jacobi en un point rationnel (s'il en existe un). Un nombre associé à J, appelé la *hauteur de Faltings* et noté $h_{\mathrm{Fal}}(J)$, joue un rôle important. [1] Dans un certain sens, ce nombre mesure la « complexité » des coefficients des équations qui définissent la courbe C. En se reposant sur cette nouvelle approche, Rémond [Rém00] a démontré une majoration explicite

$$\#C(K) \leq c(g, d, h_{\mathrm{Fal}}(J))^{\rho+1}, \tag{0.0.1}$$

où $c(g, d, h_{\mathrm{Fal}}(J))$ est une constante qui ne dépend que de g, $d := [K : \mathbb{Q}]$ et $h_{\mathrm{Fal}}(J)$. Le nombre entier $\rho \geq 0$, appelé le *rang de Mordell–Weil sur* K, est défini comme $\rho = \mathrm{rank}_{\mathbb{Z}} J(K)$. Le théorème de Mordell–Weil assure que $J(K)$ est un groupe abélien finiment engendré, donc ρ est un nombre fini.

D'autre part, l'on a cherché à éliminer $h_{\mathrm{Fal}}(J)$ dans la majoration (0.0.1). Ceci a été explicitement proposé comme une question par Mazur peu après la démonstration de la conjecture de Mordell par Faltings ; voir [Maz00, page 223] et [Maz86, haut de la page 234].

[1] Ces informations ont aussi été utilisées dans la première démonstration de la conjecture de Mordell par Faltings en 1983.

Dans le cadre d'un projet en collaboration avec V. Dimitrov et P. Habegger, nous avons donné une réponse positive à cette question. La clé est d'établir un *nouveau principe d'écart* qui concerne la répartition des points rationnels et plus généralement des points algébriques sur les courbes de genre au moins 2.

Ce mémoire fait une synthèse des aspects suivants concernant les répartitions des points sur les variétés algébriques et leurs interactions.

- Chapitre 1 : la majoration de $\#C(K)$ conjecturée par Mazur (Theorem 1.1.1), et la répartition des points algébriques sur les courbes de genre ≥ 2 (Proposition 1.2.1). Ceci concerne une série de travaux [GH19, DGH19, Gao20a, DGH20].

- Chapitre 2 : les petits points dans les variétés abéliennes. Le résultat principal est la conjecture de Bogomolov géométrique sur caractéristique nulle (Theorem 2.2.1). Ceci concerne [GH19, CGHX21].

- Chapitre 3 : les points spéciaux dans les variétés de Shimura mixtes. Le résultat principal porte sur la conjecture d'André–Oort pour les variétés de Shimura mixtes (Theorem 3.1.2). Ceci concerne [Gao17b, Gao16]. Dans ce chapitre, l'on trouve aussi un résultat de finitude *à la Bogomolov* (Theorem 3.2.4 issu de [Gao20b, Thm.1.4]), qui est un outil pratique pour étudier les problèmes d'intersections atypiques dans les variétés de Shimura mixtes.

- Chapitre 4 : les interactions entre les résultats abordés dans les chapitres précédents. Dans ce chapitre, plusieurs aspects pour étudier les problèmes dans les chapitres précédents sont présentés, y compris (i) des résultats de transcendance sur les corps de fonctions (de type Ax) dans les variétés de Shimura mixtes, dans la section §4.1 ([Gao17b, Gao20b]); (ii) les lieux de dégénérescence des sous-variétés d'un schéma abélien, dans la section §4.2 ([Gao20a]); (iii) l'application de Betti et son rang générique, dans la section §4.3 ([Gao20a]); (iv) l'inégalité de hauteurs qui compare, pour un schéma abélien, la hauteur de Néron–Tate définie fibre par fibre et la hauteur sur la base, dans la §4.4 ([DGH19, DGH20, Gao20a]).

Chapter 1

Rational and Algebraic Points on Curves of genus ≥ 2

1.1 Main result on the bound of the number of rational points

In collaboration with V. Dimitrov and P. Habegger, we proved the following rather uniform bound on the number of rational points on curves of genus at least 2. This answers affirmatively a question of Mazur [Maz00, pp.223].

Theorem 1.1.1 ([DGH20, Thm. 1.1]). *Let $g \geq 2$ and $d \geq 1$ be two integers. There exists a constant $c = c(g,d) > 0$ with the following property. For any geometrically irreducible smooth projective curve C of genus g which is defined over a number field K of degree d, we have*

$$\#C(K) \leq c^{1+\mathrm{rk}_{\mathbb{Z}} J(K)} \tag{1.1.1}$$

where J is the Jacobian of C.

Moreover, $c = c(g,d)$ depends polynomially on d.

Compared to the classical result (0.0.1), the height $h_{\mathrm{Fal}}(J)$ is no longer involved in the bound.

Here are some results towards this theorem priori to our work. Based on the method of Vojta, David–Philippon [DP07] proved the theorem if J is contained in a copy of elliptic curves, Davia–Nakamaye–Philippon [DNP07] proved theorem for some families of curves, and Alpoge [Alp18] proved that the average number of rational points on a curve of genus 2 is bounded. Based on the Chabauty–Coleman approach Stoll [Sto19] showed that $\#C(F)$ is bounded in terms of g and d if rank $J(F) \leq g-3$ and C is hyperelliptic; Katz–Rabinoff–Zureick-Brown [KRZB16] later, under the same rank hypothesis, removed the hyperelliptic hypothesis.

Our method is based on Vojta's proof [Voj91] of the Mordell Conjecture, which was later simplified by Bombieri [Bom90b] and further developed by Faltings [Fal91b]. Let us briefly recall the method. We will fix an immersion $\iota \colon \mathbb{A}_g \to \mathbb{P}^N$ (defined over $\overline{\mathbb{Q}}$) where $\mathbb{A}_g$ is the moduli space of principally polarized abelian varieties. Then the naive height function on $\mathbb{P}^N(\overline{\mathbb{Q}})$ induces a height function $h = h_{\mathbb{A}_g,\iota} \colon \mathbb{A}_g(\overline{\mathbb{Q}}) \to \mathbb{R}$.

It is known, by Faltings, that $h \sim h_{\mathrm{Fal}}$ (quasi-linear), where $h_{\mathrm{Fal}}(A)$ is the Faltings height of the abelian variety A defined over $\overline{\mathbb{Q}}$.

Let $\widehat{h}\colon \mathrm{Jac}(C)(\overline{\mathbb{Q}}) \to \mathbb{R}$ denote the Néron–Tate height attached to the canonical polarization in $\mathrm{Jac}(C)$. Take $P_0 \in C(K)$, and for simplicity we identify C with its image under the Abel–Jacobi embedding $C \to \mathrm{Jac}(C)$ via P_0. We divide $C(K)$ into two parts:

- Small points $\left\{P \in C(K) : \widehat{h}(P) \leq B(C)\right\}$;
- Large points $\left\{P \in C(K) : \widehat{h}(P) > B(C)\right\}$

where $B(C)$ is allowed to depend on a suitable height of C. Roughly speaking, we can take $B(C) = c_0 \max\{1, h(\iota([\mathrm{Jac}(C)]))\}$ for some $c_0 = c_0(g, \iota) > 0$.

Note that the number of small points is immediately finite as small points are lattice points contained in a bounded set in a Euclidean space. The constant c_0 is chosen in a way that accommodates both the *Mumford inequality* and the *Vojta inequality*. Combining these two inequalities yields an upper bound on the number of large points by $c_1(g)^{1+\operatorname{rank} J(K)}$; see Vojta [Voj91, Thm.6.1].

Thus in order to prove Theorem 1.1.1, it suffices to bound the number of small points.

1.2 A new Gap Principle

Our main contribution is the following new *Gap Principle* on the Néron–Tate distance of points in $C(\overline{\mathbb{Q}})$. Roughly speaking, we proved that algebraic points on C are in general far from each other in a quantitative way.

Proposition 1.2.1 ([DGH20, Prop.7.1])**.** *There are positive constants c_1, c_2, c_3, and c_4, depending only on g and ι, such that if $h(\iota([\mathrm{Jac}(C)])) \geq c_1$ then any $P \in C(\overline{\mathbb{Q}})$ satisfies the following alternative.*

- *Either P lies in a subset of $C(\overline{\mathbb{Q}})$ of cardinality at most c_2,*
- *or* $\left\{Q \in C(\overline{\mathbb{Q}}) : \widehat{h}(Q - P) \leq h(\iota([\mathrm{Jac}(C)]))/c_3\right\} < c_4$.

This dichotomy has the following upshot: If $h(\iota([\mathrm{Jac}(C)])) \geq c_1$, then the small points in $C(K)$ either lie in a set of uniformly bounded cardinality c_2, and are contained in at most $(1 + c_0c_3)^{\operatorname{rank} J(K)}$ balls in the Néron–Tate metric, with each ball containing at most c_4 points. In either case this will yield the bound

$\#C(K) \leq c(g)^{1+\operatorname{rank} J(K)}$ *if* $h(\iota([\mathrm{Jac}(C)])) \geq c_1$. Then one can apply Rémond's estimate [DP02, pp.643] to conclude. Alternatively by the Northcott property (and the Torelli theorem), there are only finitely many curves $C_{\overline{\mathbb{Q}}}$ which are defined over a number field K of degree d and has height at most c_1. One can then either apply Silverman's result [Sil93] on the Mordell conjecture for twists of curves to each of these finitely many $C_{\overline{\mathbb{Q}}}$'s.

1.3 Setup for the proof and "the height inequality"

The key to prove Proposition 1.2.1, and hence Theorem 1.1.1, is the following height inequality. Let $\mathbb{M}_g$ be the moduli space of smooth projective curves of genus g, viewed as a subvariety of $\mathbb{A}_g$ via the Torelli map. Up to adding level structures we assume that $\mathbb{M}_g$ and $\mathbb{A}_g$ are both fine moduli spaces, and hence admit the universal curve $\mathfrak{C}_g \to \mathbb{M}_g$ and the universal abelian variety $\pi\colon \mathfrak{A}_g \to \mathbb{A}_g$. We can furthermore assume that there is a symmetric relatively ample line bundle $\mathfrak{L}_g$ on $\mathfrak{A}_g/\mathbb{A}_g$. Let S be an irreducible variety with a (not necessarily dominant) quasi-finite morphism $S \to \mathbb{M}_g$, and define

$$\mathscr{D}_M\colon \mathfrak{C}_S^{[M+1]} = \underbrace{\mathfrak{C}_g \times_{\mathbb{M}_g} \ldots \times_{\mathbb{M}_g} \mathfrak{C}_g}_{(M+1)\text{-copies}} \times_{\mathbb{M}_g} S \longrightarrow \underbrace{\mathfrak{A}_g \times_{\mathbb{A}_g} \ldots \times_{\mathbb{A}_g} \mathfrak{A}_g}_{M\text{-copies}} \times_{\mathbb{A}_g} S$$

to be fiberwise defined by $(P_0, P_1, \ldots, P_M) \to (P_1 - P_0, \ldots, P_M - P_0)$.

Theorem 1.3.1 ([DGH20, Thm.1.6]+ [Gao20a, Thm.1.2']). *Assume $g \geq 2$ and $M \geq 3g-2$. Then there exist constants $c > 0$ and $c' \geq 0$ and a Zariski open dense subset U of $\mathscr{D}_M(\mathfrak{C}_S^{[M+1]})$ with*

$$\widehat{h}_{\mathfrak{L}_g}(x) \geq c h_{\mathbb{A}_g,\iota}(\pi(x)) - c' \quad \textit{for all} \quad x \in U(\overline{\mathbb{Q}}). \tag{1.3.1}$$

Notice that on fibers over $s \in S(\overline{\mathbb{Q}})$ with $ch_{\mathbb{A}_g,\iota}(s) \leq c'$, the height inequality (1.3.1) does not give new information because the left hand side is always non-negative. This leads to the constant $c_1 = c_1(g,\iota) > 0$ in Proposition 1.2.1.

But as soon as $h_{\mathbb{A}_g,\iota}(s) \geq c_1$, (1.3.1) becomes very strong. Each point $x \in U(\overline{\mathbb{Q}})$ equals $(P_1 - P, \ldots, P_M - P)$ for some $s \in S(\overline{\mathbb{Q}})$ and $(P, P_1, \ldots, P_M) \in \mathfrak{C}_s^{M+1}(\overline{\mathbb{Q}})$. The left hand side of (1.3.1) is then $\sum_{i=1}^M \widehat{h}(P_i - P)$. Thus (1.3.1) has the following upshot: Given any $s \in S(\overline{\mathbb{Q}})$ of large height and any $P \in \mathfrak{C}_s(\overline{\mathbb{Q}})$, then for $P_1, \ldots, P_M \in \mathfrak{C}_s(\overline{\mathbb{Q}})$ in general position we cannot have $\widehat{h}(P_i - P) < (c/M)h_{\mathbb{A}_g,\iota}(s)$ for each $i \in \{1, \ldots, M\}$. This ultimately leads to Proposition 1.2.1 up to some technical argument.

The height inequality (1.3.1) is proved by applying [DGH20, Thm.1.6] to [Gao20a, Thm.1.2']. The proof of [Gao20a, Thm.1.2'] uses the Ax–Schanuel theorem for $\mathfrak{A}_g$ [Gao20b, Thm.1.1] and a finiteness result *à la Bogomolov* [Gao20b, Thm.1.4] .

More discussion and a more general form of this height inequality will be given in §4.4. It is expected to have more applications in Diophantine geometry.

1.4 Towards uniform Mordell–Lang

To close this part, I would like to point out that our method can be applied to situations beyond rational points. This is reasonable because the new Gap Principle we proved, Proposition 1.2.1, is not confined to rational points; it applies to algebraic points on curves. We proved the following result towards a more general question posed by Mazur [Maz86, top of pp.234], sometimes known as the *uniform Mordell–Lang* conjecture for curves.

Theorem 1.4.1 ([DGH20, Thm.1.2]). *There exist two constants $\delta = \delta(g,\iota) > 0$ and $c = c(g,\iota) > 0$ with the following property. Let C be a geometrically irreducible smooth projective curve of genus g defined over $\overline{\mathbb{Q}}$, let $P_0 \in C(\overline{\mathbb{Q}})$, and let Γ be a finite rank subgroup of $\mathrm{Jac}(C)(\overline{\mathbb{Q}})$. If C satisfies $h_{\mathbb{A}_{g,\iota}}([\mathrm{Jac}(C)]) > \delta$, then we have*

$$\#(C(\overline{\mathbb{Q}}) - P_0) \cap \Gamma \le c^{1+\mathrm{rank}\,\Gamma}, \tag{1.4.1}$$

where $C - P_0$ is the image of the Abel–Jacobi embedding $C \to \mathrm{Jac}(C)$ via P_0.

Chapter 2

Small points on abelian varieties: the Geometric Bogomolov Conjecture

2.1 Background and setup

In 1998, Ullmo [Ull98] and S. Zhang [Zha98] proved the Bogomolov conjecture over number fields. However its analogue over function fields, which came to be known as the Geometric Bogomolov Conjecture, remained open in full generality both over characteristic 0 and $p > 0$.

In a joint work with S. Cantat, P. Habegger and J. Xie [CGHX21], we proved the Geometric Bogomolov Conjecture over characteristic 0. The statement is as follows.

Let k be an algebraically closed field. For the moment we do not make assumptions on its characteristic. Let B be an irreducible normal projective variety over k of dimension $d \geq 1$. Let $K = k(B)$ be the function field of B. Let A be an abelian variety of dimension g defined over K. Fix a symmetric ample line bundle L on A. Fix an ample line bundle $\mathcal{M}$ on B if $d \geq 2$.

We can define a height function on $\widehat{h}_{L,M} \colon A(\overline{K}) \to \mathbb{R}_{\geq 0}$ as follows. Take $x \in A(\overline{K})$. Let $\rho' \colon B' \to B$ be the normalization of B in $K(x)$. Let $(\mathcal{A}, \mathcal{L})$ be a model of (A, L), *i.e.* $\pi \colon \mathcal{A} \to B$ such that the generic fiber is A and $\mathcal{L}$ an ample line bundle on $\mathcal{A}$ such that the generic fiber is L. It is possible to choose an $\mathcal{A}$ normal. Set $\mathcal{A}' = \mathcal{A} \times_B B'$ and let $\rho \colon \mathcal{A}' \to \mathcal{A}$ be the projection to the first factor. Then set

$$h_{L,M}(x) = \frac{1}{[K(x) : K]} \Big(\Delta_x \cdot c_1(\rho^* \mathcal{L}) \cdot \rho^* \pi^* (c_1(\mathcal{M}))^{d-1} \Big)$$

where Δ_x is the Zariski closure of x in $\mathcal{A}'$. Now $\widehat{h}_{L,M}$ is the normalized height function with respect to $h_{L,M}$ via the Tate-Limit process, *i.e.*

$$\widehat{h}_{L,M}(x) = \lim_{N \to \infty} \frac{h([N]x)}{N^2}.$$

2.2 Statement of the main result

Theorem 2.2.1 ([CGHX21]; [GH19] if $d = 1$)**.** *Let X be an irreducible subvariety of $A_{\overline{K}}$. Assume X is NOT of the form $\mathrm{tr}(Y \otimes_k \overline{K}) + T$, where $(A^{\overline{K}/k}, \mathrm{tr})$ is the*

$\overline{K}/k$-trace of $A_{\overline{K}}$, Y is a subvariety of $A^{\overline{K}/k}$ and T is a torsion coset of $A_{\overline{K}}$. If $\mathrm{char} k = 0$*, then there exists $\varepsilon > 0$ such that*

$$X(\varepsilon; L) = \{x \in X(\overline{K}) : \widehat{h}_{L,M}(x) \leq \varepsilon\}$$

is NOT Zariski dense in X.

Our proof uses heavily the *Betti map* (see §4.3), and the method is completely different from earlier work on the conjecture. Priori to our result, the Geometric Bogomolov Conjecture was proved by Gubler [Gub07a] when A is totally degenerate at some place of K. He has no restriction on the characteristic of k. When X is a curve embedded in its Jacobian A and when $\mathrm{trdeg} K/k = 1$, Yamaki dealt with non-hyperelliptic curves of genus 3 in [Yam02] and with hyperelliptic curves of any genus in [Yam08]. If moreover $\mathrm{char}(k) = 0$, Faber [Fab09] proved the conjecture for X of small genus and Cinkir [Cin11] covered the case of arbitrary genus. Priori to these work, Moriwaki also gave some partial results in [Mor98]. Yamaki [Yam18] reduced the Geometric Bogomolov Conjecture to the case where A has good reduction everywhere and has trivial $\overline{K}/k$-trace. He also proved the cases (co)$\dim X = 1$ [Yam17b] and $\dim(A \otimes_K \overline{K}/(A^{\overline{K}/k} \otimes_k \overline{K})) \leq 5$ [Yam17a]. As in Gubler's setup, Yamaki works in arbitrary characteristic. These results involve techniques ranging from analytic tropical geometry [Gub07b] to Arakelov theory; the latter method overlaps with Ullmo and S. Zhang's original approach for number fields.

Chapter 3

Special points in moduli spaces

3.1 The André–Oort Conjecture

Every connected Shimura variety, being the quotient of a Hermitian symmetric domain by an arithmetic group, can be realized as a moduli space for pure Hodge structures plus tensors. Better than the Hermitian symmetric domains themselves, connected Shimura varieties are algebraic varieties. This was proved by Baily-Borel [BB66]. The prototype for all Shimura varieties is the Siegel moduli space $\mathbb{A}_g$ of principally polarized abelian varieties of dimension g with a level structure. The points in this moduli space corresponding to CM abelian varieties, which are called *special points*, play a particularly important role in the theory of Shimura varieties. A major reason is that the Galois action on special points are fairly completely determined by Shimura-Taniyama theorem [Del71, Thm.4.19] and its generalization by Milne-Shih [DMOS82] to Galois conjugates of CM abelian varieties. The concept of special points and the results concerning the Galois action on them have been generalized to arbitrary Shimura varieties. Every Shimura variety has a Zariski dense subset of special points [Del71, Prop.5.2], and hence the results above have lead to the concept of the *canonical model* of a Shimura variety over a number field: see Deligne [Del79] and Milne [Mil88].

For various reasons, it is natural and necessary to generalize the notion of Shimura varieties to new objects which parametrize *mixed* Hodge structures plus tensors. This will lead to the notion of *connected mixed Shimura varieties.* In order to distinguish, we will use the term "pure Shimura variety" to denote the Shimura varieties mentioned in the previous paragraph.

Here are two reasons for this generalization. First, mixed Shimura varieties exist as natural objects in geometry. Here are several important examples of mixed Shimura varieties which are not pure:

- the universal family of abelian varieties $\mathfrak{A}_g$ of dimension g with a level structure;
- the $\mathbb{G}_m$-torsor over $\mathfrak{A}_g$ which corresponds to the tautological relatively ample line bundle on $\mathfrak{A}_g \to \mathbb{A}_g$;
- the universal Poincaré bi-extension.

Second, mixed Shimura varieties also arise in the natural problem of looking for suitable compactifications of pure Shimura varieties. Indeed, the first compactification is the Baily-Borel (or minimal) compactification [BB66], which is canonical. However this compactification has very bad singularities along the boundary. Next we have the toroidal compactifications [AMRT10], which are no longer canonical but provide smooth compactifications of Shimura varieties. To construct these compactifications one needs to study the boundary of a Shimura variety. As one approaches the boundary of a Hermitian symmetric domain, pure Hodge sturctures degenerate into mixed Hodge structures, and as one approaches the boundary of a Shimura variety, abelian varieties degenerate into 1-motives. This will lead to mixed Shimura varieties.

In Diophantine problems, there are analogous objects for mixed Shimura varieties and for abelian varieties. The following table lists some of them.

abelian varieties	**connected mixed Shimura variety**	**universal abelian variety $\mathfrak{A}_g$**
torsion / small points	special points	torsion points on CM fibers
torsion cosets	mixed Shimura subvarieties	subgroup schemes over pure Shimura subvarieties

The objects in the third row will be called *special subvarieties.*

As we saw above, it is a classical result that each special subvariety contains a Zariski dense subset consisting of special points. The André–Oort conjecture predicts the converse statement, and thus the distribution of special points on mixed Shimura varieties. It is the analogue of the Manin–Mumford conjecture or the Bogomolov conjecture in the Shimura setting.

Conjecture 3.1.1 (André–Oort)**.** *Let X is an irreducible subvariety of a connected mixed Shimura variety S. If X contains a Zariski dense subset of special points, then X is a special subvariety of S.*

I proved the following result on the André–Oort conjecture for mixed Shimura varieties, generalizing known results on this conjecture for pure Shimura varieties.

Theorem 3.1.2 ([Gao17b, Gao16])**.** *The André–Oort conjecture holds true for any mixed Shimura variety which satisfies a lower bound on the Galois orbit on special points. Thus by [Tsi18, Thm.1.2] it holds true for any mixed Shimura variety of abelian type (so in particular the universal abelian variety $\mathfrak{A}_g$).*

For pure Shimura varieties, André [And98] proved the conjecture for $Y(1)^2$. It has since then been open, until Pila's pioneer work [Pil11] proving the conjecture for $Y(1)^N$. Using results on functional transcendence (see comments below Theorem 4.1.2) and the averaged Colmez's conjecture [YZ18, AGHMP18] and Masser–Wüstholz's isogeny theorem [MW93], Tsimerman [Tsi18] proved the conjecture for $\mathbb{A}_g$.

On the other hand, the André–Oort conjecture was proved under the Generalized Riemann Hypothesis by Klingler, Ullmo and Yafaev [UY14a], [KY14]. This generalizes Edixhoven's approach for $Y(1)^2$ [Edi98].

3.2 A finiteness result *à la Bogomolov*

An important step in the Pila–Zannier method to handle special points or more general unlikely intersection problems is a finiteness result *à la Bogomolov.* It dates back to Bogomolov in the 70s.

Theorem 3.2.1 ([Bog81, Thm.1]). *Let A be a complex abelian variety, and let X be an irreducible subvariety. Then there are only finitely many abelian subvarieties B of A with $\dim B > 0$ satisfying the following two properties:*

(i) $a + B \subseteq X$ for some $a \in A(\mathbb{C})$;

(ii) B is maximal for the property described in (i).

The translates $a + B$ in (i) are called *weakly special subvarieties* of A. Notice that the special subvarieties of A are precisely the weakly special subvarieties with a being a torsion point.

The notion of weakly special subvarieties also exists for mixed Shimura varieties. It generalizes the notion for abelian varieties.

Definition 3.2.2 ([Pin05, Defn.4.1]). *Let S be a connected mixed Shimura variety. A subvariety Y of S is called a weakly special subvariety if there exist mixed Shimura subvariety $T \subseteq S$, a quotient Shimura morphism $[\varphi]\colon T \to T'$ and a point $t' \in T'$ such that $Y = [\varphi]^{-1}(t')$.*

This definition is indeed equivalent to [Pin05, Defn.4.1] by [Gao17b, Defn.5.2, Rmk.5.3, Prop.5.7].

Hence one can make the analogous statement of Theorem 3.2.1 for mixed Shimura varieties. This analogue of Bogomolov's result for pure Shimura varieties was proved by Ullmo [Ull14] to study the André–Oort conjecture. I extended Ullmo's result to all mixed Shimura varieties [Gao17b, §12].

On the other hand, *weakly optimal subvarieties*, as a generalization of weakly special subvarieties, were introduced by Habegger–Pila [HP16] to study the Zilber–Pink conjecture. They also proved the corresponding finiteness result for abelian varieties and $Y(1)^N$, before Daw–Ren [DR18, Prop.3.3] proved the statement for all pure Shimura varieties. Such finiteness results are important in the study of unlikely intersection problems: it is often used to proved that certain "unlikely locus", which *à priori* is a complicated union, is Zariski closed.

Definition 3.2.3. *Let S be a connected mixed Shimura variety. Let X be a fixed irreducible subvariety of S.*

(i) For each irreducible subvariety Z of S, define $\delta_{\mathrm{ws}}(Z) = \dim Z^{\mathrm{ws}} - \dim Z$, where Z^{ws} is the smallest weakly special subvariety of S containing Z.

*(ii) A closed irreducible subvariety Y of X is said to be **weakly optimal** if the following condition holds: $Z \subsetneq Z' \subseteq Y \Rightarrow \delta_{\mathrm{ws}}(Z') > \delta_{\mathrm{ws}}(Z)$, where Z' is assumed to be irreducible.*

Extending and simplifying Daw–Ren's proof, I proved the following theorem, which is precisely the desired finiteness result *à la Bogomolov* to study the Zilber–Pink conjecture for $\mathfrak{A}_g$. It is used in the proof of [Gao20a, Thm.1.2'], and hence Theorem 1.1.1.

Theorem 3.2.4 ([Gao20b, Thm.1.4 and Thm.8.2])**.** *Let X be a fixed irreducible subvariety of a connected mixed Shimura variety of Kuga type S.*

There exist finitely many connected mixed Shimura subvarieties $T_1, \ldots, T_m$ of S, and finitely many quotient Shimura morphisms $[\varphi_{i,j}] \colon T_i \to T'_{i,j}$ for each $i \in \{1, \ldots, m\}$ with the following property. If Z is a weakly optimal subvariety of X, then $Z^{\mathrm{ws}} = [\varphi_{i,j}]^{-1}(t')$ for some i, j and $t \in T'_{i,j}$.

Chapter 4

Interactions

4.1 Functional transcendence

Let S be a connected mixed Shimura variety associated with the connected mixed Shimura datum $(P, \mathcal{X})$ and let $u\colon \mathcal{X} \to S = \Gamma\backslash\mathcal{X}$ be the uniformization. An example for S to keep in mind is the universal abelian variety $\mathfrak{A}_g$ (with some level structures). For any $(P, \mathcal{X})$, there exists a complex algebraic variety $\mathcal{X}^\vee$ such that $\mathcal{X}$ can be realized as a semi-algebraic subset of $\mathcal{X}^\vee$ which is open in the usual topology. Thus we can make the following definition: A subset $\widetilde{Y}$ of $\mathcal{X}$ to be *algebraic* if it is a complex analytic irreducible component of the intersection of $\mathcal{X}$ and an algebraic subvariety of $\mathcal{X}^\vee$. This make the system $u\colon \mathcal{X} \to S$ a *bi-algebraic* system.

In [Gao17b, Cor.8.3] (pure case by Ullmo–Yafaev [UY11]), I proved that the bi-algebraic objects are precisely the weakly special subvarieties as defined in Definition 3.2.2. More preciesly given a subvariety $F \subseteq S$, then $\widetilde{F}$ (a complex analytic irreducible component of $u^{-1}(F)$) is algebraic in $\mathcal{X}$ if and only if F is a weakly special subvariety of S.

Using this language of bi-algebraic system, one can state the Ax–Schanuel conjecture.

Conjecture 4.1.1 (Ax–Schanuel)**.** *Let $\Delta \subset \mathcal{X} \times S$ be the graph of u, and let $p_S\colon \mathcal{X} \times S \to S$ be the natural projection to S. Let $\mathscr{Z}$ be a complex analytic irreducible subset of Δ. Then*

$$\dim \mathscr{Z}^{\mathrm{Zar}} - \dim \mathscr{Z} \geq \dim p_S(\mathscr{Z})^{\mathrm{ws}}.$$

Here $p_S(\mathscr{Z})^{\mathrm{ws}}$ means the smallest weakly special subvariety of S which contains $p_S(\mathscr{Z})$.

My results towards this conjecture are:

Theorem 4.1.2. *The Ax–Schanuel conjecture holds true in the following cases:*

(i) [Gao17b, Thm.8.1] If $p_S(\mathscr{Z})$ is algebraic;

(ii) [Gao17b, Thm.1.2] If $p_{\mathcal{X}}(\mathscr{Z})$ is algebraic, where $p_{\mathcal{X}}\colon \mathcal{X} \times S \to \mathcal{X}$ is the natural projection;

(iii) [Gao20b] If $S = \mathfrak{A}_g$ (universal abelian variety) or more general if S is of Kuga type.

Part (ii) is called the *Ax–Lindemann theorem*; it plays a crucial role in the unconditional proof of the André–Oort conjecture. The Ax–Lindemann theorem was proved for $Y(1)^N$ by Pila [Pil11], for projective Shimura varieties by Ullmo–Yafave [UY14b], $\mathbb{A}_g$ by Pila–Tsimerman [PT14a] and pure Shimura varieties by Klingler–Ullmo–Yafaev [KUY16]. The Ax–Schanuel conjecture was proved by Pila–Tsimerman for the j-function [PT14b], by Mok–Pila–Tsimerman for any pure Shimura variety [MPT19]. My results are extensions of these results and are based on them. It is not a simple reduction. Rather, I went into the details of their proofs to adapt each part to variation of mixed (not pure) Hodge structures. Several new ideas were required.

4.2 Unlikely Intersection on abelian schemes: degeneracy loci

Let S be an irreducible variety over $\mathbb{C}$, and let $\pi\colon \mathcal{A} \to S$ be an abelian scheme of relative dimension g, namely a proper smooth group scheme whose fibers are abelian varieties.

Let X be an irreducible subvariety of $\mathcal{A}$.

Definition 4.2.1. *Let $\langle X \rangle$ denote the smallest subvariety of $\mathcal{A}$ which satisfies the following properties:*

(i) $\pi(\langle X \rangle) = \pi(X)$, which we denote by B;

(ii) there exist a finite covering $B' \to B$ such that for the natural projection $\rho\colon \mathcal{A}' := \mathcal{A} \times_B B' \to \mathcal{A}$, we have

$$X = \rho(\mathcal{B} + \sigma + \sigma') \tag{4.2.1}$$

where $\mathcal{B}$ is an abelian subscheme of $\mathcal{A}' \to B'$, ρ is a torsion section of $\mathcal{A}' \to B'$ and ρ' is a constant section of $\mathcal{A}' \to B'$.

Constants sections of $\mathcal{A}' \to B'$ are defined to be, for the largest constant abelian subscheme $C \times B'$ of $\mathcal{A}' \to B'$, sections $\sigma'\colon B' \to \mathcal{A}'$ such that $\sigma'(B') = \{c\} \times B' \subseteq C \times B' \subseteq \mathcal{A}'$.

It should be understand that $\langle X \rangle$ is the "smallest abelian scheme" contained in $\mathcal{A}$ which contains X. In [Gao20a], it is denoted by $\langle X \rangle_{\mathrm{sg}}$.

Having this notion, we can define the *t-th degeneracy locus of X* for each $t \in \mathbb{Z}$. But the most interesting cases are $t \leq 0$ and $t = 1$.

Definition 4.2.2. *The t-th degeneracy locus of X is defined to be the following union*

$$X^{\mathrm{deg}}(t) = \bigcup_{\substack{Y\subseteq X \text{ irreducible},\ \dim Y>0 \\ \dim\langle Y\rangle - \dim\pi(Y) < \dim Y + t}} Y. \tag{4.2.2}$$

We also denote for simplicity by $X^{\mathrm{deg}} := X^{\mathrm{deg}}(0)$.

Notice that when $t \leq 0$, then $\dim Y > 0$ is redundant in the definition.

The t-th degeneracy locus turns out to be very useful to study Diophantine problems related to families of abelian varieties. For example, $X^{\mathrm{deg}}(t)$ is closely related to the generic rank of the Betti rank (see §4.3) when $t \leq 0$, and $X^{\mathrm{deg}}(1)$ is closely related to the relative Manin–Mumford conjecture and eventually the relative Bogomolov conjecture (see [Gao20a, §11]).

The union in (4.2.2) is *à priori* an infinite union, and thus it is *à priori* not clear that $X^{\mathrm{deg}}(t)$ is closed in X even in the usual topology. I proved the following result, which is the first step to understand $X^{\mathrm{deg}}(t)$.

Theorem 4.2.3 ([Gao20a, Thm.1.8])**.** *$X^{\mathrm{deg}}(t)$ is a Zariski closed subset of X for each $t \in \mathbb{Z}$. In particular, X^{deg} is Zariski closed in X.*

Let us briefly explain how Theorem 4.2.3 is proved. First we reduce the theorem to the case where $\mathcal{A} \to S$ is the universal abelian variety $\mathfrak{A}_g \to \mathbb{A}_g$. Next using the geometric description of weakly special subvarieties of $\mathfrak{A}_g$ from [Gao17a], we can replace $\dim\langle Y\rangle - \dim\pi(Y)$ by $\dim Y^{\mathrm{ws}} - \dim\pi(Y)^{\mathrm{ws}}$ in (4.2.2). But then it is not hard to prove that each maximal Y in the union in (4.2.2) is a weakly optimal subvariety of X (Definition 3.2.3), and thus we can invoke the finiteness theorem *à la Bogomolov* in this case, Theorem 3.2.4, to prove that the union in (4.2.2) can be written as a finite union. Therefore $X^{\mathrm{deg}}(t)$ is Zariski closed.

Proposition 4.2.4. *If X and $\pi\colon \mathcal{A} \to S$ are defined over an algebraically closed field K of characteristic 0, then $X^{\mathrm{deg}}(t)$ is defined over K for each $t \in \mathbb{Z}$.*

Proof. This follows from the proof of [Gao20a, Thm.1.8]. Let us explain more details.

[Gao20a, Lem.9.1] reduces this proposition to the case where $\mathcal{A} \to S$ is the universal abelian variety $\mathfrak{A}_g \to \mathbb{A}_g$. Hence we are in the situation of [Gao20a, Thm.7.1].

The key point to prove [Gao20a, Thm.7.1] to apply the finiteness result *à la Bogomolov* (Theorem 3.2.4). More precisely, applying Theorem 3.2.4 to X, we obtain finitely many connected mixed Shimura subvarieties $T_1, \ldots, T_m$ of $\mathfrak{A}_g$, and finitely many quotient Shimura morphisms $[\varphi_{i,j}]\colon T_i \to T'_{i,j}$ for each $i \in \{1, \ldots, m\}$. It is proved in [Gao20a, §7.3] that $X^{\mathrm{deg}}(t)$ is the (finite) union of the following sets:

$$E_{g_{i,j}-t} := \{x \in X(\mathbb{C}) : \dim_x([\varphi_{i,j}]|_{X\cap T_i})^{-1}([\varphi_{i,j}](x)) > g_{i,j} - t\},$$

where $g_{i,j}$ is an integer associated with $[\varphi_{i,j}]\colon T_i \to T'_{i,j}$.

But every $[\varphi_{i,j}]\colon T_i \to T'_{i,j}$ is defined over $\overline{\mathbb{Q}}$ because it is a Shimura morphism between mixed Shimura varieties. As $K \supseteq \overline{\mathbb{Q}}$, each $E_{g_{i,j}-t}$ above is defined over K. So $X^{\mathrm{deg}}(t)$ is defined over K. □

Example 4.2.5. *Let us see the example where* $\dim S = 1$ *and* $t = 0$. *In this case, the condition* $\dim\langle Y\rangle - \dim \pi(Y) < \dim Y$ *is equivalent to* $\dim \pi(Y) = 1$ *and* $\langle Y\rangle = Y$. *So in this case, the argument of [GH19, §3] shows that* X^{deg} *is generically special as defined by [GH19, Defn.1.2] and is precisely* $X \setminus X^*$ *for* X^* *defined below [GH19, Defn.1.2].*

4.3 Betti map and its generic rank

Let S be an irreducible variety over $\mathbb{C}$, and let $\pi\colon \mathcal{A} \to S$ be an abelian scheme of relative dimension g, namely a proper smooth group scheme whose fibers are abelian varieties.

For any $s \in S(\mathbb{C})$, there exists an open neighborhood $\Delta \subseteq S^{\mathrm{an}}$ of s with a real-analytic map, called the *Betti map*,

$$b_\Delta\colon \mathcal{A}_\Delta = \pi^{-1}(\Delta) \to \mathbb{T}^{2g},$$

where $\mathbb{T}^{2g}$ is the real torus of dimension $2g$, defined as follows. Up to shrinking Δ we may assume that it is simply-connected. Then one can define a basis $\omega_1(s), \ldots, \omega_{2g}(s)$ of the period lattice of each fiber $s \in \Delta$ as holomorphic functions of s. Now each fiber $\mathcal{A}_s = \pi^{-1}(s)$ can be identified with the complex torus $\mathbb{C}^g/\mathbb{Z}\omega_1(s) \oplus \cdots \oplus \mathbb{Z}\omega_{2g}(s)$, and each point $x \in \mathcal{A}_s(\mathbb{C})$ can be expressed as the class of $\sum_{i=1}^{2g} b_i(x)\omega_i(s)$ for real numbers $b_1(x), \ldots, b_{2g}(x)$. Then $b_\Delta(x)$ is defined to be the class of the $2g$-tuple $(b_1(x), \ldots, b_{2g}(x)) \in \mathbb{R}^{2g}$ modulo $\mathbb{Z}^{2g}$.

The Betti map turns out to be a powerful tool in Diophantine geometry.

Let X be an irreducible subvariety of $\mathcal{A}$. Studying the following generic rank of $b_\Delta|_{X\cap\mathcal{A}_\Delta}$,

$$\mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_X) := \max_{x \in X^{\mathrm{sm}}(\mathbb{C}) \cap \mathcal{A}_\Delta} (\mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_{X\cap\mathcal{A}_\Delta})_x),$$

has seen many applications.

First of all, $\mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_X)$ has a trivial upper bound by

$$\mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_X) \leq 2\min\{\dim \iota(X), g\}, \tag{4.3.1}$$

where $\iota\colon \mathcal{A} \to \mathfrak{A}_g$ is the modular map. This is because the Betti map factors through the universal abelian variety $\mathfrak{A}_g \to \mathbb{A}_g$ (hence the bound $\leq 2\dim \iota(X)$) and has target $\mathbb{T}^{2g}$ (hence the upper bound $\leq 2g$).

Here are some applications of studying the generic rank of the Betti map, or more precisely to determine whether $\mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_{\Delta}|_X) = 2l$ for some particular l and X. The geometric Bogomolov conjecture over char 0 by Gao–Habegger [GH19] and Cantat–Gao–Habegger–Xie [CGHX21] (Theorem 2.2.1) with $l = \dim X - \dim S$; the denseness of torsion points on sections by André–Corvaja–Zannier [ACZ20] with $l = g$ (and they proved the denseness when $\dim S \geq g$ and $\mathrm{End}(\mathcal{A}/S) = \mathbb{Z}$); application of André–Corvaja–Zannier's result to Lagrangian fibrations to study the Chow ring of hyper-Kähler fourfolds by Voisin [Voi18] with $l = g \geq 4$.

In this memoire, we focus on the case where $\mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_{\Delta}|_X) = 2\dim X$. More general cases are treated in [Gao20a].

It is sometimes convenient to make the following mild assumption:

$$\texttt{(Hyp)} \ : \ \mathbb{Z}X := \textstyle\bigcup_{N\in\mathbb{Z}}\{[N]x : x \in X(\mathbb{C})\} \text{ is Zariski dense in } \mathcal{A}.$$

This assumption is mild because otherwise we may replace $\mathcal{A}$ by $\mathcal{A}\times_S \pi(X)$ (which is again an abelian scheme), and then by a smaller abelian scheme which is an irreducible component of a subgroup scheme of $\mathcal{A} \to S$ (up to taking a finite covering of S).

First of all we make the following observation. If $\mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_{\Delta}|_X) = 2\dim X$ holds true, then by (4.3.1), we must have (i) $\iota|_X$ is generically finite for the modular map $\iota\colon \mathcal{A} \to \mathfrak{A}_g$; (ii) $\dim X \leq g$.[1]

But these conditions are not sufficient. In fact, here is the criterion for $\mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_{\Delta}|_X) = 2\dim X$ proved in [Gao20a]. The proof makes essential use of the mixed Ax–Schanuel theorem for $\mathfrak{A}_g$ (Theorem 4.1.2.(iii)) and a refinement of Theorem 4.2.3.

Theorem 4.3.1. *The following statements are equivalent.*

(i) $\mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_{\Delta}|_X) = 2\dim X$*;*

(ii) $X \neq X^{\deg}$*, with* $X^{\deg}$ *from* (4.2.2) *(with* $t = 0$*) which is Zariski closed by Theorem 4.2.3.*

Moreover if `(Hyp)` *holds, then (i) and (ii) are furthermore equivalent to the following condition. For each abelian subscheme* $\mathcal{B}$ *of* $\mathcal{A} \to S$ *(whose relative dimension is denoted by* $g_{\mathcal{B}}$*), we have* $\dim(\iota_{/\mathcal{B}} \circ p_{\mathcal{B}})(X) \geq \dim X - g_{\mathcal{B}}$*, where* $p_{\mathcal{B}}\colon \mathcal{A} \to \mathcal{A}/\mathcal{B}$ *is*

[1] At this stage, it is good to make the remark that each Y appearing in the union in (4.2.2) has generic Betti rank smaller than $2\dim Y$ by the naive dimension reason (ii). This suggests that $X^{\deg}(0)$ is closely related to this quesiton.

the quotient abelian scheme and $\iota_{/\mathcal{B}}$ is the modular map.[2]

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{p_{\mathcal{B}}} & \mathcal{A}/\mathcal{B} & \xrightarrow{\iota_{/\mathcal{B}}} & \mathfrak{A}_{g-g_{\mathcal{B}}} \\ {\scriptstyle\pi}\downarrow & & \downarrow & \lrcorner & \downarrow \\ S & \xrightarrow{\mathrm{id}_S} & S & \xrightarrow{\iota_{/\mathcal{B},S}} & \mathbb{A}_{g-g_{\mathcal{B}}} \end{array} \tag{4.3.2}$$

Proof. If $\iota|_X$ is generically finite, then (i) and (ii) are equivalent by [Gao20a, Thm.1.7 and Thm.1.8] applied to $l = \dim \iota(X) = \dim X$.

If $\iota|_X$ is not generically finite, then $\mathrm{rank}_{\mathbb{R}}(\mathrm{d}b_\Delta|_X) \leq 2\dim\iota(X) < 2\dim X$, so (i) cannot hold. But in this case, we have $X = X^{\deg}(0)$ by [Gao20a, Lem.9.1] applied to $t = 0$ and $r > 0$. So (ii) cannot hold, either.

Thus (i) and (ii) are equivalent by the previous two paragraphs.

If `(Hyp)` holds, then (i) and the condition above (4.3.2) are equivalent by [Gao20a, Thm.1.1.(i)] applied to $l = \dim X$. □

4.4 The height inequality

We have explained that in the proof of Theorem 1.1.1, a key new ingredient is the height inequality Theorem 1.3.1. In this section, we give this height inequality in its most general form.

Let S be an irreducible variety, and let $\pi\colon \mathcal{A} \to S$ be an abelian scheme of relative dimension g, namely a proper smooth group scheme whose fibers are abelian varieties. Let $\mathcal{L}$ be a relatively ample line bundle on $\mathcal{A}/S$, and let $\mathcal{M}$ be an ample line bundle on some compactification $\overline{S}$ of S. All objects are assumed to be defined over $\overline{\mathbb{Q}}$.

We have then two height functions. The fiberwise defined Néron–Tate height function $\widehat{h}_{\mathcal{A},\mathcal{L}}\colon \mathcal{A}(\overline{\mathbb{Q}}) \to \mathbb{R}_{\geq 0}$, $x \mapsto \widehat{h}_{\mathcal{A}_{\pi(x)},\mathcal{L}_{\pi(x)}}(x)$; and the height function on the base $h_{\overline{S},\mathcal{M}}\colon \overline{S}(\overline{\mathbb{Q}}) \to \mathbb{R}$ given by the Height Machine.

Let X be an irreducible subvariety of $\mathcal{A}$. Recall the degeneracy locus $X^{\deg}$ defined as in (4.2.2) with $t = 0$; it is Zariski closed in X by Theorem 4.2.3.

The following definition of *non-degenerate subvarieties* is equivalent to the one used in [DGH20] ([DGH20, Defn.1.5 and Defn.B.4]) by Theorem 4.3.1.

Definition 4.4.1. *An irreducible subvariety X of $\mathcal{A}$ is said to be non-degenerate if $X \neq X^{\deg}(0)$.*

[2]This geometric criterion, which is checkable by hand, can be simplified in some cases; see [Gao20a, (1.4)].

Theorem 4.4.2. *Let X be an irreducible subvariety of $\mathcal{A}$. Set $X^* := X \setminus X^{\deg}$. By the discussion above, X^* is Zariski open in X and is non-empty if and only if X is non-degenerate.*

There exist constants $c > 0$ and c', depending on X and the data ($\mathcal{A}/S$, $\mathcal{L}$, $\mathcal{M}$), such that

$$\widehat{h}_{\mathcal{A},\mathcal{L}}(x) \geq c h_{\overline{S},\mathcal{M}}(\pi(x)) - c' \quad \text{for all } x \in X^*(\overline{\mathbb{Q}}). \tag{4.4.1}$$

Remark 4.4.3. *When $\dim S = 1$, this theorem is precisely [GH19, Thm.1.4] by Example 4.2.5. For more general S, this is a refinement of [DGH20, Thm.1.6 (and Thm.B.1)] (by plugging in Theorem 4.3.1 and Theorem 4.2.3).*

Proof. We prove the theorem by induction on $\dim X$. If $\dim X = 0$, then result is trivial. So let us assume $\dim X \geq 1$.

If $X = X^{\deg}$, then $X^* = \emptyset$ and there is nothing to prove. Otherwise, X is non-degenerate and hence we can invoke [DGH20, Thm.B.1]. So there exist constants $c > 0$ and $c' \geq 0$ and a Zariski open dense subset U of X defined over $\overline{\mathbb{Q}}$ such that

$$\widehat{h}_{\mathcal{A},\mathcal{L}}(x) \geq c h_{\overline{S},\mathcal{M}}(\pi(x)) - c' \tag{4.4.2}$$

for all $x \in U(\overline{\mathbb{Q}})$. Let $X \setminus U = Z_1 \cup \cdots \cup Z_r$ be the decomposition into irreducible components. Since $\dim Z_i \leq \dim X - 1$, we may do induction on the dimension. By induction hypothesis, the inequality (4.4.2) holds true for all points in $\bigcup_i (Z_i \setminus Z_i^{\deg})(\overline{\mathbb{Q}})$ up to modifying the constants c and c'.

Therefore, in order to prove (4.4.1), it remains to prove

$$X \setminus X^{\deg} \subseteq U \cup \bigcup_i (Z_i \setminus Z_i^{\deg}),$$

or equivalently

$$X^{\deg} \supseteq (X \setminus U) \cap \bigcap_i Z_i^{\deg} = (Z_1 \cup \cdots \cup Z_r) \cap \bigcap_i Z_i^{\deg} = \bigcap_i Z_i^{\deg}.$$

But $Z_i \subseteq X$, and hence $Z_i^{\deg} \subseteq X^{\deg}$ by definition of the degeneracy locus (4.2.2). So we are done. □

In practice, to apply Theorem 4.4.2 one needs to show that $X \neq X^{\deg}$. The criterion is given by the "Moreover" part of Theorem 4.3.1. When $\dim S = 1$ or $\mathcal{A} \to S$ has simple algebraic monodromy group, the criterion can be simplified; see [Gao20a, (1.4)] (also [GH19, Thm.1.4] for $\dim S = 1$).

The following theorem, as an application of Theorem 4.3.1, turns out to be very useful in applications.

As the convention of [Gao20a] (§2 of *loc.cit.*) is somewhat different from standard notation, we state the theorem in terms of the geometric generic fiber.

Theorem 4.4.4 ([Gao20a, Thm.10.1]). *Let X be an irreducible subvariety of $\mathcal{A}$ such that $\pi|_X$ is dominant to S. Assume that $X_{\overline{\eta}}$ (the geometric generic fiber of $X \to S$) is irreducible.*[3]

Assume furthermore

(a) $\dim X > \dim S$.

(b) $X_{\overline{\eta}}$ *is not contained in any proper subgroup of* $\mathcal{A}_{\overline{\eta}}$.

(c) On the geometric generic fiber $\mathcal{A}_{\overline{\eta}}$ *of* $\mathcal{A} \to S$, *the neutral component of* $\mathrm{Stab}_{\mathcal{A}_{\overline{\eta}}}(X_{\overline{\eta}})$ *(the stabilizer of* $X_{\overline{\eta}}$*) is contained in the* $\overline{\mathbb{C}(\eta)}/\mathbb{C}$*-trace of* $\mathcal{A}_{\overline{\eta}}$.

Then as subvarieties of $\mathcal{A}^{[M]}$, *we have that*

(i) $X^{[M]}$ *is non-degenerate if* $M \geq \dim S$ *and* $\iota^{[M]}|_{X^{[M]}}$ *is generically finite.*

(ii) $\mathscr{D}^{\mathcal{A}}_M(X^{[M+1]})$ *is non-degenerate if* $M \geq \dim X$ *and* $\iota^{[M]}|_{\mathscr{D}^{\mathcal{A}}_M(X^{[M+1]})}$ *is generically finite, where*

$$\mathscr{D}_m \colon \underbrace{\mathcal{A} \times_S \mathcal{A} \times_S \cdots \times_S \mathcal{A}}_{(m+1)\text{-copies}} \to \underbrace{\mathcal{A} \times_S \cdots \times_S \mathcal{A}}_{m\text{-copies}}$$

is fiberwise defined by $(P_0, P_1, \ldots, P_m) \mapsto (P_1 - P_0, \ldots, P_m - P_0)$.

Here $X^{[M]} = X \times_S \cdots \times_S X$ *(M-copies) for each integer* $M \geq 1$.

Proof. The hypotheses (a)–(c) above are clearly equivalent to the hypotheses (a)–(c) of [Gao20a, Thm.10.1]. Thus by [Gao20a, Thm.10.1.(i)] applied to $t = 0$, we have

$$\mathrm{rank}_{\mathbb{R}}(\mathrm{d}b^{[m]}_{\Delta}|_{X^{[m]}}) \geq 2\dim \iota^{[m]}(X^{[m]}) \text{ for all } m \geq \dim S.$$

If furthermore $\iota^{[m]}|_{X^{[m]}}$ is generically finite, then the right hand side equals $2\dim X^{[m]}$. Hence part (i) of the current theorem is established.

Similarly, by [Gao20a, Thm.10.1.(ii)] applied to $t = 0$, we have

$$\mathrm{rank}_{\mathbb{R}}(\mathrm{d}b^{[m]}_{\Delta}|_{\mathscr{D}^{\mathcal{A}}_m(X^{[m+1]})}) \geq 2\dim \iota^{[m]}(\mathscr{D}^{\mathcal{A}}_m(X^{[m+1]})) \text{ for all } m \geq \dim X.$$

If furthermore $\iota^{[m]}|_{\mathscr{D}^{\mathcal{A}}_m(X^{[m+1]})}$ is generically finite, then the right hand side equals $2\dim \mathscr{D}^{\mathcal{A}}_m(X^{[m+1]})$. Hence part (ii) of the current theorem is established. □

[3]This assumption is harmless because it can always be achieved in the following way. There exists a quasi-finite étale morphism $S' \to S$ such that some component X' of $X \times_S S'$ satisfies that $X'_{\overline{\eta}}$ is irreducible. But X' dominates X under the natural projection $X \times_S S' \to X$. In applications, we apply this theorem to $X' \subseteq \mathcal{A} \times_S S' \to S'$.

In practice, to verify the extra generic finiteness required in (i) and (ii), one can sometimes use the following observations. For (i), $\iota^{[M]}|_{X^{[M]}}$ is generically finite if $\iota|_X$ is generically finite. For (ii), $\iota^{[M]}|_{\mathscr{D}_M^{\mathcal{A}}(X^{[M+1]})}$ is generically finite if ι (and not $\iota|_X$) is quasi-finite. This leads to [Gao20a, Thm.1.3].

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