Group Construction in C-minimal Structures

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C-sets can be understood as reducts of ultrametric spaces: if the distance is d, we keep only the information given by the ternary relation

$$C(x, y, z) \iff d(x, y) = d(x, z) \geq d(y, z).$$

So, there is no longer a space of distances, we can only compare distances to a same point.

Definition 1 *A C-minimal structure M is a C-set with additional structure in which every definable subset is a Boolean combination of open or closed balls, more exactly of their generalizations in the framework of C-relations, cones and 0-level sets. Moreover, this must remain true in any structure N elementary equivalent to M.*

Theorem: *Let $\mathcal{M}$ be a C-minimal structure which is dense, definably complete, geometric and non-trivial. Suppose moreover that in the underlying tree of $\mathcal{M}$ there is no definable bijection between a bounded interval and an unbounded one ("NoBij" condition). Then there is an infinite group definable in $\mathcal{M}$.*

Nice family of fonctions

Definition 2 *Let $V \subseteq M$ be a cone and $\mathcal{F} = \{f_u : u \in U\}$ a definable family of definable functions from V to M, indexed by a cone $U \subseteq M$. The family $\mathcal{F}$ is said to be* a nice family of functions *if there is an element $e \in V$ with the following properties:*

1. All the f_u are C-automorphisms of the cone V.

2. For every $u \in U$, we have $f_u(e) = e$.

3. The property $(*)$:

$$(*)\begin{cases} \text{For any fixed } x \in V \setminus \{e\}, \text{ the function} \\ \qquad \varphi_{\mathcal{F},x} : \begin{array}{rcl} U & \longrightarrow & V \\ u & \longmapsto & f_u(x) \end{array} \\ \text{is a } C\text{-isomorphism from } U \text{ onto some subcone of } V. \end{cases}$$

Then one (and only one) of the following situations occurs:

(1) $I(x) \subseteq T(M)$ is a branch with leaf of $T(M)$: thus $x \in M$ (its type over M is realized).

(2) $I(x) \subseteq T(M)$ is a branch of $T(M)$ without maximal element. We say x is *limit* over M .

(3) $I(x) \subseteq T(M)$ has a maximal element $\mu_x \in T(M) \setminus M$, x is called *residual* over M.

(4) $I(x)$ has a maximal element $\mu_x \notin T(M)$, x is called *valuational* over M . Moreover

- if $I(x) \cap T(M) = \emptyset$ we say x is *of type* $-\infty$.
- otherwise take $y \in M$ such that $\mu_x = x \wedge y$. Then μ_x may determine a proper cut on $Br(y) \cap T(M)$ or not, in which case it is either of type α^- or α^+ for some $\alpha \in T(M)$. Note that this case partition does not depend on y.

Proposition 1 *If ψ_g is defined on a neighborhood of e and is definable, the type $\psi_g(e^-)$ is one of the following:*
(1) realized by an element of U
(2) α^- for some $\alpha \in T(U)$
(3) type at the infinity of some cone of U.
If (NoBij) holds only (1) and (2) with α an element of U are possible.

Definition 3 *We say g is* derivable *if ψ_g is defined on a neighborhood of e and $\psi_g(e^-)$ is either realized in U or of type n^- for a leaf n of $T(U)$, in other words if $\psi_g(x)$ has a limit in U when x tends to e. This limit $u \in U$ is called the* derivative *of g (at e) relatively to the family $\mathcal{F}$. The function f_u is called* the tangent to g in $\mathcal{F}$ *and we write $f_u \sim_{\mathcal{F}} g$, or just $f_u \sim g$ if there is no confusion on $\mathcal{F}$.*

Corollary 2 *If (NoBij) holds then, g is derivable iff ψ_g is defined on some neighborhood of e.*

A derivability criterion for composed and inverse functions

Proposition 3 *In Propostion* **??** *the family may be chosen with the following properties:*
- containing the identity,
- non-dilating.

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