Around NIP Noetherian domains

Will Johnson

Fudan University

May 30, 2023

Section 1

Why NIP Noetherian domains?

NIP

A theory T with monster model $\mathbb{M}$ has the *independence property* (IP) if there is an indiscernible sequence $a_1, a_2, \ldots$ and definable set D with

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$$a_i \in D \iff i \equiv 1 \pmod 2.$$

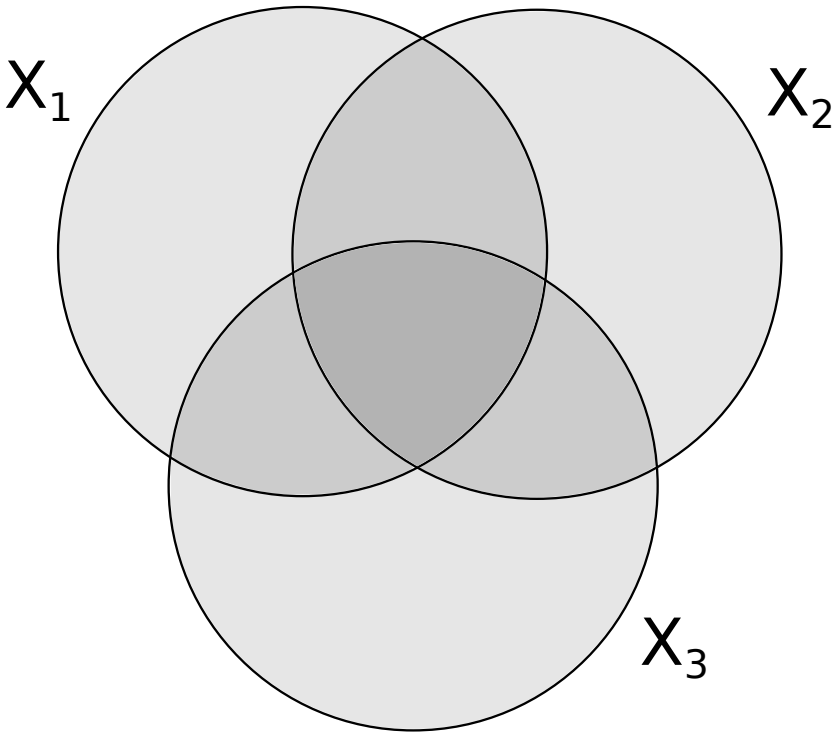
Otherwise, T is *NIP*.

IP and "independence"

T has the IP if and only if there is an infinite family of uniformly definable sets

$$X_1, X_2, X_3, \ldots \subseteq \mathbb{M}$$

which is "independent", in the sense that it freely generates a boolean algebra.



Examples of NIP theories

- Stable theories, such as ACF and SCF
- O-minimal theories, such as RCF
- Many henselian valued fields (Delon, ...)
  - ACVF, RCVF, pCF, $\mathbb{C}((t))$, ...

Dp-rank

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Dp-rank is a reasonable subadditive notion of dimension.

- dp-rk$(X) \leq$ dp-rk(Y) if there is a definable injection $X \to Y$.
- dp-rk$(X \times Y) =$ dp-rk$(X) +$ dp-rk(Y).
- ...

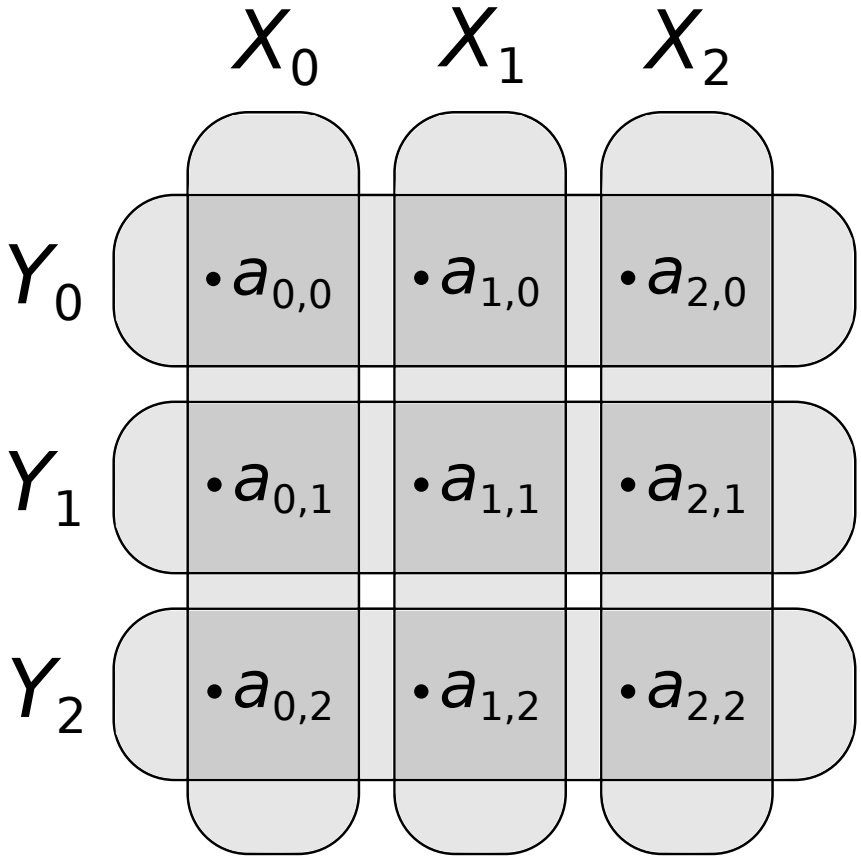
Dp-rank 2

Dp-rank ≥ 2 means there are uniformly definable sets

$$X_0, X_1, X_2, \ldots$$
$$Y_0, Y_1, Y_2, \ldots$$

such that for every $i, j < \omega$, there is an $a_{i,j}$ such that

$$a_{i,j} \in X_k \iff k = i$$
$$a_{i,j} \in Y_k \iff k = j$$



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Definition

T is *dp-finite* if $\text{dp-rk}(\mathbb{M}) \in \{0, 1, 2, 3, \ldots\}$.

A natural question

In this talk, "ring" means "commutative unital ring."

Question

Which rings are NIP?

A natural question

In this talk, "ring" means "commutative unital ring."

Question

Which rings are NIP?

Too hard!

Question

Which Noetherian rings are NIP?

Dp-minimal Noetherian domains

Theorem (J)

The dp-minimal Noetherian domains are the elementary substructures of the following:

- *Dp-minimal fields.*
- $K[[t]]$ *for dp-minimal* K *with* $\mathrm{char}(K) = 0$.
- *Finite-index subrings of* $\mathcal{O}_K$ *for finite extensions* $K/\mathbb{Q}_p$.

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Dp-minimal domains are unclassified, but see (d'Elbée-Halevi).

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- Option 2: assume the conjectures on NIP fields.

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I don't know how to prove this for general NIP domains.

Section 2

Basic facts about NIP Noetherian rings

Prime ideals

Fact (Simon)

If R is an NIP ring, the poset of prime ideals has finite width.

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If R is an NIP Noetherian ring, then...

- *R is semilocal.*
- *R has finitely many prime ideals.*
- *R has Krull dimension ≤ 1.*

Breadth

Definition

A ring R has *breadth* $\mathrm{br}(R) \leq k$ if for any $a_0, \ldots, a_k \in R$, there is i with

$$(a_0, a_1, \ldots, a_k) = (a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_k).$$

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If R is Noetherian, then $\mathrm{br}(R) < \infty \iff$ *(R is semilocal and* $\dim(R) \leq 1$*).*

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If R is an NIP Noetherian ring, then $\mathrm{br}(R) < \infty$*.*

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Lemma

If $\mathrm{br}(R) < \infty$, *then...*

- *Every f.g. ideal is generated by k generators.*
- *Every ideal is externally definable.*

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Externally definable rings are still NIP! (by a theorem of Shelah)

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Theorem (J)

If R is an NIP Noetherian domain and $R \neq \mathrm{Frac}(R)$, then $\mathrm{Frac}(R)$ *has characteristic zero.*

The number of maximal ideals

Conjecture

If $\mathcal{O}_1, \mathcal{O}_2$ are two valuation rings on K and $(K, \mathcal{O}_1, \mathcal{O}_2)$ is NIP, then $\mathcal{O}_1 \subseteq \mathcal{O}_2$ or $\mathcal{O}_2 \subseteq \mathcal{O}_1$.

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3. *The integral closure $\tilde{R}$ is a henselian valuation ring.*

Henselianity again

Theorem (J)

Let R be a NIP Noetherian ring. Suppose $\text{dp-rk}(R) < \aleph_0$ *or the henselianity conjecture holds.*

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Conjecture (Generalized henselianity conjecture)

This holds for any NIP ring R.

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True in these cases:

- Positive characteristic.
- Finite dp-rank.

Section 3

Dp-finite Noetherian domains

Dp-finite fields and valued fields

Theorem

A valued field (K, v) is dp-finite iff the following conditions hold:

1. *v is henselian and defectless.*
2. *The value group vK and residue field Kv are dp-finite.*
3. *If Kv is finite and vK is non-trivial, then $\mathrm{char}(K) = 0$ and the interval $[-v(p), v(p)] \subseteq vK$ is finite.*
4. *If Kv is infinite with characteristic $p > 0$, then the interval $[-v(p), v(p)] \subseteq vK$ is p-divisible.*

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Theorem (J)

A field K is dp-finite iff...

Roughly speaking, the upper theorem generates all dp-finite fields, starting from ACF and RCF.

Dp-finite DVRs

Corollary

A DVR $\mathcal{O}$ is dp-finite iff it is elementarily equivalent to one of the following:

- $K[[t]]$, *where* K *is dp-finite and* $\mathrm{char}(K) = 0$.
- $\mathcal{O}_K$, *where* K *is a finite extension of* $\mathbb{Q}_p$.

A trichotomy

Theorem (J)

Let R be a dp-finite Noetherian domain and $\mathcal{O}$ be its integral closure. Then one of three things happens:

- *R is a field.*
- *R and $\mathcal{O}$ have residue characteristic 0, and $\mathcal{O} \equiv K[[t]]$ for some dp-finite field K of characteristic 0.*
- *R and $\mathcal{O}$ have finite residue fields, and $\mathcal{O} \equiv \mathcal{O}_K$ for some finite extension $K/\mathbb{Q}_p$.*

The R-adic topology

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When R is a valuation ring, this is the valuation topology.

The lucky case

Let R be a dp-finite Noetherian domain with $R \neq \mathrm{Frac}(R)$.

Theorem(?)

If R induces a V-topology on $\mathrm{Frac}(R)$, then one of the following holds, up to elementary equivalence:

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2. *R is "something like"*

$$\mathbb{R} + t\mathbb{R} + t^2\mathbb{R} + t^3\mathbb{C}[[t]].$$

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So $K[[t]] \subseteq R \subseteq L[[t]]$ and $\dim_K L[[t]]/R < \infty$, where K is dp-finite and L/K is a finite extension.

The dp-minimal case

Fact (J)

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If $K = \mathrm{Frac}(R)$*, then* $(R, +, \cdot)$ *and* $(K, R, +, \cdot)$ *have the same dp-rank.*

Corollary

If R *is a dp-minimal domain, then the* R*-adic topology on* $\mathrm{Frac}(R)$ *is a V-topology.*

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By the way...

Fact (d'Elbée-Halevi)

If R *is a dp-minimal integral domain with infinite residue field, then* R *is a valuation ring.*

Section 4

Prospects for dp-rank 2

A mysterious example

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- The R-adic topology on $\mathbb{Q}_p$ is *not* a V-topology.
- R isn't "N-1": its integral closure $\mathbb{Z}_p$ isn't finite over R.

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- ...losing Noetherianity.
- ...(And the proof is terrible!)

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  This has QE, and dp-rank 2.

Some vague conjectures

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There are non-excellent Noetherian domains of dp-rank 2 coming from differential valued fields.

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If R is a Noetherian domain of dp-rank 2 and the R-adic topology isn't a V-topology, then R arises from this construction.

Some evidence

Fact (J)

Any definable field topology on a dp-minimal field is a V-topology.

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Any definable field topology on a dp-minimal field is a V-topology.

Theorem (J)

Let K be a highly saturated field of dp-rank 2, and let τ be a definable field topology on K. Then τ is a V-topology OR *there is a valuation ring $\mathcal{O} \subseteq K$ and a derivation $\partial : K \to K$ such that τ is the R-adic topology, for*

$$R = \{x \in \mathcal{O} : \partial x \in \mathcal{O}\}.$$

References

Will Johnson.
Dp-finite fields IV: the rank 2 picture.
arXiv:2003.09130v1 [math.LO], March 2020.

Will Johnson.
Dp-finite fields VI: the dp-finite Shelah conjecture.
arXiv:2005.13989v1 [math.LO], May 2020.

Will Johnson.
Dp-finite and Noetherian NIP integral domains.
arXiv:2302.03315v1 [math.LO], February 2023.

Questions?