Beyond the Fontaine-Wintenberger theorem

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CIRM Luminy '23

Joint work with Franziska Jahnke.

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$\mathbb{Q}_p$ vs $\mathbb{F}_p((t))$

Two methods of comparing them

First method: $p \to \infty$.

Theorem (Ax-Kochen/Ershov '65)

Let ϕ be a sentence in the language of rings. Then there exists $N = N(\phi) \in \mathbb{N}$ such that

$$\phi \text{ holds in } \mathbb{Q}_p \iff \phi \text{ holds in } \mathbb{F}_p((t))$$

for all $p \geq N$.

Application: Artin conjectured that $\mathbb{Q}_p$ is C_2.
This was disproved by Terjanian '66, '80. Nevertheless:

Corollary (Ax-Kochen '65)

Fix $d \in \mathbb{N}$. Then $\mathbb{Q}_p$ is $C_2(d)$ for $p >> 0$.

Remark:
Later, the Ax-Kochen/Ershov transfer principle between $\mathbb{Q}_p$ and $\mathbb{F}_p((t))$ was vastly generalized in a motivic framework.
(Denef-Loeser '01, Cluckers-Loeser '08, ...)

$\mathbb{Q}_p$ vs $\mathbb{F}_p((t))$

Two methods of comparing them: K vs $K^\flat$

Second method: $e \to \infty$.

"Approximation des corps valués complets de caractéristique p par ceux de caractéristique 0." (Krasner '56)

This works especially well by considering higher and higher *wildly ramified* extensions and passing to the limit.

For instance: Take $\mathbb{Q}_p(p^{1/p^\infty})$ and $\mathbb{F}_p((t))(t^{1/p^\infty})$.

Theorem (Fontaine-Wintenberger '79)

$G_{\mathbb{Q}_p(p^{1/p^\infty})} \cong G_{\mathbb{F}_p((t))(t^{1/p^\infty})}$.

This was generalized within the framework of *perfectoid fields.*

Given a perfectoid field K one defines its *tilt* $K^\flat$.

Slogan: K and $K^\flat$ are very similar.

Theorem (Scholze '12 and Kedlaya-Liu '15)

$G_K \cong G_{K^\flat}$

Fact: Scholze also obtains an appropriate geometric generalization for perfectoid spaces.

Motivating question (K vs $K^\flat$)

Fix a perfectoid field K with tilt $K^\flat$.
Question: (M. Morrow, Luminy '18)
How are K and $K^\flat$ related model-theoretically? More precisely:
(i) To what extent does $Th(K^\flat)$ determine $Th(K)$?
(ii) How close are K and $K^\flat$ to being elementary equivalent?
Answers:
(1) K and $K^\flat$ are bi-interpretable in continuous logic.
(Rideau-Scanlon-Simon)
(!) Their interpretation of K in $K^\flat$ requires *parameters* for the Witt vector coordinates of $\xi_K \in W(\mathcal{O}_{K^\flat})$.
(2) K is "in practice" decidable relative to $K^\flat$. (K. '21)
(Need that $\xi_K \in W(\mathcal{O}_{K^\flat})$ is a *computable* Witt vector)
(3) By passing to an elementary extension of K, we find a well-behaved valuation on it, whose residue field is an elementary extension of $K^\flat$. (joint with F. Jahnke)

Goal of this talk: Explain (3).

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Main theorem

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Summary: By passing to an elementary extension of K, we find a well-behaved valuation on it, whose residue field is an elementary extension of $K^\flat$.

Theorem (Jahnke, K. '23)

- *(K, v) perfectoid field and $\varpi \in \mathfrak{m} \backslash \{0\}$.*
- *U a non-principal ultrafilter on $\mathbb{N}$.*
- *(K_U, v_U) the corresponding ultrapower.*

Let w be the coarsest coarsening of v_U such that $w\varpi > 0$. Then:

(I) *Every finite extension of (K_U, w) is unramified.*

(II) *There is an elementary embedding from $(K^\flat, v^\flat)$ to $(k_w, \overline{v})$.*

Moreover, the isomorphism $G_{K_U} \cong G_{k_w}$ restricts to $G_K \cong G_{K^\flat}$.

Moral:

One eliminates all difficulties related to the defect at the expense of making the residue field more complicated.

The main theorem facilitates the transfer of first-order information between K and $K^\flat$. For instance:

Example

1. Using (I) and (II), we get that

$$G_K \equiv G_{K_U} \cong G_{k_w} \equiv G_{K^\flat}$$

2. Regarding the property of being defectless: (Draw picture.)

$$(K, v) \text{ is defectless } \iff (K_U, v_U) \text{ is defectless } \overset{!}{\iff}$$

$$(k_w, \overline{v}) \text{ is defectless } \iff (K^\flat, v^\flat) \text{ is defectless}$$

3. Regarding the property of being C_i:

$$K \text{ is } C_i \iff K_U \text{ is } C_i \overset{!}{\Longrightarrow} k_w \text{ is } C_i \iff K^\flat \text{ is } C_i$$

   Problem: Suppose $K^\flat$ is C_i. Is K also C_i? ($i = 1$?)

4. Existence of rational points in rationally connected varieties. (In progress)

Goal for the rest of the talk:
Explain (I) and (II) of the main theorem.

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Part I of the main theorem

Recall the statement:

Theorem (Part I of the main theorem)

- (K, v) *perfectoid field and* $\varpi \in \mathfrak{m} \backslash \{0\}$.
- (K_U, v_U) *a non-principal ultrapower.*
- w*="the coarsest coarsening of* v_U *s.t.* $w\varpi > 0$*".*

Then every finite extension of K_U *is unramified with respect to* w*.*

This is a non-standard version of the *almost purity theorem*:

Theorem (Tate '67/Gabber-Ramero '03)

Let K *be a perfectoid field. Then every finite extension of* K *is almost unramified. More precisely, if* K'/K *is a finite extension then* $\mathcal{O}_{K'}/\mathcal{O}_K$ *is almost finite étale.*

More general versions of almost purity exist.
(Faltings '02, Scholze '12)

Part I of the main theorem

A model-theoretic glimpse into almost mathematics

Goals: (for the next 15 minutes)
(1) Explain the key ideas in almost mathematics.
(2) Explain why Part I is a non-standard version of almost purity.
(3) Make precise the connection between our model-theoretic framework and almost mathematics.

Disclaimer:
This will only be a small glimpse into a highly technical subject.

I will try to motivate the key ideas without getting bogged down with too many technical details.

Central notion: Almost étale extension. (Faltings '02)
Let us explain this with an example:

Example

- Let K be the p-adic completion of $\mathbb{Q}_p(p^{1/p^\infty})$. $(p \neq 2)$
- $K' = K(p^{1/2})$. (totally ramified quadratic extension)

The extension $\mathcal{O}_{K'}/\mathcal{O}_K$ is close to being étale. To see this, let us have a look at what happens at a finite level:

1. Consider the local fields $K_n = \mathbb{Q}_p(p^{1/p^n})$ and $K'_n = K_n(p^{1/2})$.
2. One can show that $\mathcal{O}_{K'_n} = \mathcal{O}_{K_n}[p^{1/2p^n}]$.
3. Computing the module of Kähler differentials, we see that

$$p^{1/p^n} \cdot \Omega_{\mathcal{O}_{K'_n}/\mathcal{O}_{K_n}} = 0$$

4. Passing to the limit, we get $\mathfrak{m} \cdot \Omega_{\mathcal{O}_{K'}/\mathcal{O}_K} = 0$.
5. In almost mathematics, such a module is treated as zero.

Almost mathematics (d'après Faltings, Gabber-Ramero)

An introduction

Main idea:

Build a version of ring theory where $\mathfrak{m}$-torsion is ignored.

Basic setup:

- K a perfectoid field, $\mathcal{O}_K$ its valuation ring, $\mathfrak{m}$ the maximal ideal.
- $\mathcal{O}_K$-Mod="the category of $\mathcal{O}_K$-modules"

The arena of almost mathematics is the Serre quotient

$$\mathcal{O}_K^a\text{-Mod} = \mathcal{O}_K\text{-Mod}/(\mathfrak{m} - \text{torsion modules})$$

called the category of *almost modules*.

Consider $(-)^a : \mathcal{O}_K\text{-Mod} \to \mathcal{O}_K^a\text{-Mod} : M \mapsto M^a$.

It is useful to think of M^a as a *slightly generic fiber* of M.

(But beware that this is not literally true.)

The category $\mathcal{O}_K^a$-Mod is a tensor abelian category.

One can start defining almost analogues of notions in ring theory (e.g., almost algebras, flatness, projectivity(!), **étaleness**)

Almost mathematics (a paradigm shift)

What does this have to do with model theory?

Example (revisited)

- Let K be the p-adic completion of $\mathbb{Q}_p(p^{1/p^\infty})$. $(p \neq 2)$
- $K' = K(p^{1/2})$.
- Consider the non-principal ultrapower K_U and the coarsest coarsenings w of v_U with $wp > 0$ (similarly for K').

1. We can also write $K' = K(p^{1/2p^n})$.
2. Thus, $K'_U = K_U(\pi^{1/2})$ where $\pi = \text{ulim } p^{1/p^n}$. (Łoś)
3. $\mathcal{O}_{w'} = \mathcal{O}_w[\pi^{1/2}]$.
4. $\mathcal{O}_{w'}/\mathcal{O}_w$ is an (honest!) étale extension.

Moral:
The almost étale extension $\mathcal{O}_{K'}/\mathcal{O}_K$ transformed into an honest étale extension $\mathcal{O}_{w'}/\mathcal{O}_w$.
This hints at a dictionary between the two frameworks.

Almost mathematics

A dictionary between the two frameworks

Write $S \subseteq \mathcal{O}_{v_U}$ for the set of elements of infinitesimal valuation.

Theorem (Jahnke, K. '23)

We have a faithful exact monoidal functor

$$S^{-1}(-)_U : \mathcal{O}_K^a\text{-Mod} \to \mathcal{O}_w\text{-Mod} : M^a \to S^{-1}M_U$$

Suppose M^a is uniformly almost finitely generated over $\mathcal{O}_K^a$. Then: M^a is flat (resp. almost projective, etc.) over $\mathcal{O}_K^a$ if and only if $S^{-1}M_U$ is flat (resp. projective, etc.) over $\mathcal{O}_w$.

cf: (Gabber's construction)
Given an $\mathcal{O}_K$-module M, let $M^\diamond = M^{\mathbb{N}}/M^{(\mathbb{N})}$ and $S \subseteq \mathcal{O}_K^{\mathbb{N}}$ be the set of sequences $(s_n)_{n\in\omega}$ with $vs_n \to 0$. Then

$$S^{-1}(-)^\diamond : \mathcal{O}_K^a\text{-Mod} \to \mathcal{O}_K^\diamond\text{-Mod} : M \mapsto S^{-1}M^\diamond$$

is a faithful exact functor which preserves and reflects flatness.

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Part II of the main theorem

$K^\flat$ as a residue field of K in disguise

Theorem (Part II of the main theorem)

Let (K, v) be a perfectoid field and (K_U, v_U) be a non-principal ultrapower. Let w be the coarsest coarsening of v_U such that $w\varpi > 0$. Then $(K^\flat, v^\flat)$ embeds elementarily in $(k_w, \overline{v})$.

1. Constructing $\iota : (K^\flat, v^\flat) \to (k_w, \overline{v})$ requires a bit of algebraic yoga but it is not difficult.
2. Once ι has been constructed, it is also not hard to check that ι induces an *elementary embedding* of rings

$$\overline{\iota} : \mathcal{O}_{K^\flat}/(t) \to \mathcal{O}_{\overline{v}}/(\iota(t))$$

   (It is essentially a diagonal embedding into an ultrapower.)
3. But why is $\iota : (K^\flat, v^\flat) \to (k_w, \overline{v})$ also elementary?
4. One needs an Ax-Kochen/Ershov principle down to $\mathcal{O}/\varpi$. But for what kind of valued fields?

Since k_w is not "perfectoid", we need to work with a larger class $\mathcal{C}$ of valued fields. It is desirable that $\mathcal{C}$ be an elementary class.

An elementary class of "almost tame" fields

Definition

Fix a prime p. Let $\mathcal{C}$ be the class of valued fields (K, v) together with a distinguished $\varpi \in \mathfrak{m}_v \backslash \{0\}$, such that:

1. (K, v) is a henselian valued field of residue characteristic p.
2. The ring $\mathcal{O}_v/(p)$ is semi-perfect.
3. The valuation ring $\mathcal{O}_v[\varpi^{-1}]$ is algebraically maximal.

Structures in $\mathcal{C}$ will naturally be viewed as $L_{\mathrm{val}}(\varpi)$-structures.

Facts:

1. Perfectoid fields are in $\mathcal{C}$. (Note that $\mathcal{O}_K[\varpi^{-1}] = K$.)
2. $\mathcal{C}$ is an elementary class in $L_{\mathrm{val}}(\varpi)$. (tricky!)
   (**Warning:** Axiom 3 is *not elementary by itself.*)
3. There is an Ax-Kochen/Ershov principle for valued fields in $\mathcal{C}$ down to $\mathcal{O}/\varpi$.

Part II of the main theorem

$K^\flat$ as a residue field of K in disguise

Theorem (Jahnke, K. '23)

Let $(K, v) \subseteq (K', v')$ be two henselian valued fields of residue characteristic $p > 0$ such that:

- *$\mathcal{O}_v/(p)$ and $\mathcal{O}_{v'}/(p)$ is semi-perfect.*
- *There is $\varpi \in \mathfrak{m}_v$ such that $\mathcal{O}_v[\varpi^{-1}]$ (resp. $\mathcal{O}_{v'}[\varpi^{-1}]$) is algebraically maximal.*

Then the following are equivalent:

1. *$(K, v) \preceq (K', v')$ in L_{val}.*
2. *$\mathcal{O}_v/(\varpi) \preceq \mathcal{O}_{v'}/(\varpi)$ in L_{rings} and $\Gamma_v \preceq \Gamma_{v'}$ in L_{oag}.*

If Γ_v and $\Gamma_{v'}$ are regularly dense, then the value group condition in (2) can be omitted.

Key ingredient: The model theory of tame fields. (Kuhlmann '16)

This applies to show that $\iota : (K^\flat, v^\flat) \to (k_w, \overline{v})$ is elementary.

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Other applications of the AKE principle

This Ax-Kochen/Ershov principle has several other applications.

For instance, recall the following open problem:

Problem

Is $\mathbb{F}_p(t)^h$ an elementary substructure of $\mathbb{F}_p((t))$?

We prove its "perfected" variant:

Corollary (Jahnke, K. '23)

The perfect hull of $\mathbb{F}_p(t)^h$ is an elementary substructure of the perfect hull of $\mathbb{F}_p((t))$.

Remark:

Unfortunately, this does not seem to shed any light on the theory of the perfect hull of $\mathbb{F}_p((t))$.

A perfectoid AKE principle

We show that tilting respects elementary equivalence *over a perfectoid base:*

Corollary (Jahnke, K. '23)

Let K_1, K_2 be two perfectoid fields extending a perfectoid field K. Then:

$$K_1 \equiv_K K_2 \text{ if and only if } K_1^\flat \equiv_{K^\flat} K_2^\flat$$

In particular, we have that $K_1 \preceq K_2$ if and only if $K_1^\flat \preceq K_2^\flat$

Remark:
Without fixing a base, the reverse direction fails in general.
(e.g., $K_1 = \widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ and $K_2 = \widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$.)

Thank you for your attention!