Definable convex and henselian valuations on ordered fields

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Joint with:

- S. Kuhlmann and G. Lehéricy:
  'Ordered fields dense in their real closure and definable convex valuations', *Forum Math.* 33 (2021) 953–972.
- S. Kuhlmann and G. Lehéricy:
  'Strongly NIP almost real closed fields', *MLQ Math. Log. Q.* 67 (2021) 321–328.
- S. Kuhlmann and M. Link:
  'Definability of henselian valuations by conditions on the value group', to appear in *J. Symb. Log.*, 19 pp.
- P. Dittmann, F. Jahnke and S. Kuhlmann:
  'Definable valuations on ordered fields', to appear in *Model Theory*, 17 pp.

1. Motivation
2. Henselian valuations
3. Convex valuations
4. Almost real closed fields

Model theoretic setting

- first-order language
  of rings: $\mathcal{L}_{\mathrm{r}} = \{+, -, \cdot, 0, 1\}$
  of ordered rings: $\mathcal{L}_{\mathrm{or}} = \{+, -, \cdot, 0, 1, <\}$
- throughout
  $K = (K, <) = (K, +, -, \cdot, 0, 1, <)$ denotes a (linearly) ordered field
  $F = (F, +, -, \cdot, 0, 1)$ denotes a field
  $G = (G, +, -, 0, <)$ denotes an ordered abelian group
- $\mathcal{L}$-definability: *with* parameters
  $\emptyset$-$\mathcal{L}$-definability: *without* parameters

Valuations theoretic setting

- valuation $v\colon F \to vF \cup \{\infty\}$, where the value group vF is an *additive* ordered abelian group
- valuation ring $\mathcal{O}_v = \{x \in F \mid v(x) \geq 0\}$
  valuation ideal $\mathcal{M}_v = \{x \in F \mid v(x) > 0\}$
- residue field $Fv = \mathcal{O}_v/\mathcal{M}_v$ with elements $\overline{a} = a + \mathcal{M}_v$ for $a \in \mathcal{O}_v$

Expressive power

Is the language $\mathcal{L}_{\mathrm{or}}$ more expressive than $\mathcal{L}_{\mathrm{r}}$ in the context of ordered fields?

More specifically, are there subsets of K that are definable in the language $\mathcal{L}_{\mathrm{or}}$ but not in the language $\mathcal{L}_{\mathrm{r}}$?

Expressive power: positive cones

Is the language $\mathcal{L}_{\mathrm{or}}$ more expressive than $\mathcal{L}_{\mathrm{r}}$ in the context of ordered fields?

If $<$ is already $\mathcal{L}_{\mathrm{r}}$-definable, then the answer is no.

- O-minimal setting: if K is real closed, then the $\mathcal{L}_{\mathrm{r}}$-formula $\exists y\,(y \neq 0 \wedge y \cdot y = x)$ defines $P_< = \{x \in K \mid 0 < x\}$.
- More generally: if there is some $n \in \mathbb{N}$ such that every non-negative element of K is a sum of n squares, then $P_<$ is $\mathcal{L}_{\mathrm{r}}$-definable (e.g. euclidean fields).

Expressive power: positive cones

Is the language $\mathcal{L}_{\mathrm{or}}$ more expressive than $\mathcal{L}_{\mathrm{r}}$ in the context of ordered fields?

Generally yes, because sometimes $<$ is not $\mathcal{L}_{\mathrm{r}}$-definable:

Consider $K = \mathbb{Q}(t_1, t_2, \ldots)$ ordered by $\mathbb{Q} \ll t_1 \ll t_2 \ll \ldots$. Then for any $\mathcal{L}_{\mathrm{r}}$-formula $\varphi(x_1, x_2, y_1, \ldots, y_m)$ we have

$$K \models \varphi(t_{m+1}, t_{m+2}, t_1, \ldots, t_m) \text{ if and only if } K \models \varphi(t_{m+2}, t_{m+1}, t_1, \ldots, t_m).$$

Convex valuations

Definition

A valuation v on K is **convex** if its valuation ring $\mathcal{O}_v$ is a convex subset of K, i.e. for any $a, b \in \mathcal{O}_v$ and any $c \in K$ with $a < c < b$ also $c \in \mathcal{O}_v$.

- In the study of ordered valued fields, one is mostly interested in convex valuations.
- The set of convex valuation rings on K is linearly ordered by $\subseteq$.
- If v is a convex valuation on K, then Kv is linearly ordered by

$$\overline{a} < \overline{b} :\Leftrightarrow (a < b \wedge \overline{a} \neq \overline{b})$$

for any $a, b \in \mathcal{O}_v$.

Lemma (Knebusch, Wright, 1976)

Any henselian valuation on a field F is convex with respect to any linear ordering on F.

Henselian valuations

We denote by $F((G))$ the **field of generalised power series** with coefficients in F and exponents in G. We denote by $v_{\min}$ the henselian valuation

$$F((G))^{\times} \to G, \quad \sum_{g \geq g_0} s_g t^g \mapsto g_0$$

(for $s_{g_0} \neq 0$).
The field $K((G))$ can be naturally ordered by $\sum_{g \geq g_0} s_g t^g > 0$ if and only if $s_{g_0} > 0$.

Theorem [Ax–Kochen–Ershov Principle] (Farré, 1993)

Let (K, v) be an ordered henselian valued field. Then

$$(K, +, -, \cdot, 0, 1, <, v) \equiv (Kv((vK)), +, -, \cdot, 0, 1, <, v_{\min}).$$

Expressive power: convex valuations

Is the language $\mathcal{L}_{\mathrm{or}}$ more expressive than $\mathcal{L}_{\mathrm{r}}$ in the context of ordered fields *and convex valuations*?

Definition

A valuation v on K is called $\mathcal{L}$-**definable** (for $\mathcal{L} \in \{\mathcal{L}_{\mathrm{r}}, \mathcal{L}_{\mathrm{or}}\}$) if its valuation ring $\mathcal{O}_v$ is an $\mathcal{L}$-**definable** subset of K.

Main questions:

1. Is there an ordered field K and a *henselian* valuation v on K such that v is $\mathcal{L}_{\mathrm{or}}$- but not $\mathcal{L}_{\mathrm{r}}$-definable?
2. Is there an ordered field K and a *convex* valuation v on K such that v is $\mathcal{L}_{\mathrm{or}}$- but not $\mathcal{L}_{\mathrm{r}}$-definable?

Expressive power: convex valuations

Is the language $\mathcal{L}_{\mathrm{or}}$ more expressive than $\mathcal{L}_{\mathrm{r}}$ in the context of ordered fields *and convex valuations*?

Definition

A valuation v on K is called $\mathcal{L}$-**definable** (for $\mathcal{L} \in \{\mathcal{L}_{\mathrm{r}}, \mathcal{L}_{\mathrm{or}}\}$) if its valuation ring $\mathcal{O}_v$ is an $\mathcal{L}$-**definable** subset of K.

Main questions:

1. Is there an ordered field K and a *henselian* valuation v on K such that v is $\mathcal{L}_{\mathrm{or}}$-but not $\mathcal{L}_{\mathrm{r}}$-definable? **No.**
2. Is there an ordered field K and a *convex* valuation v on K such that v is $\mathcal{L}_{\mathrm{or}}$- but not $\mathcal{L}_{\mathrm{r}}$-definable? **Yes.**

Previously known results on definable henselian valuations

Theorem (Ax, 1965)

The henselian valuation $v_{\min}$ is $\emptyset$-$\mathcal{L}_{\mathrm{r}}$-definable in $F((t)) = F((\mathbb{Z}))$. Hence, if F is undecidable, then so is $F((t))$.

Theorem (Hong, 2013/2014)

Let (F, v) be a henselian valued field. Suppose that one of the following holds:

1. *vF is discretely ordered, i.e. has a least positive element.*
2. *vF is densely ordered and contains a convex subgroup that is p-regular but not p-divisible for some prime p.*

Then v is $\mathcal{L}_{\mathrm{r}}$-definable with one parameter from F.

Further results can be found in [Fehm, Jahnke, 2017].

Previously known result on definable convex valuations

Proposition (Jahnke, Simon, Walsberg, 2017)

If K is not dense in its real closure (with respect to the order-topology), then K admits a non-trivial $\mathcal{L}_{\mathrm{or}}$-definable convex valuation.

Answering our first main question

Theorem (Dittmann, Jahnke, K., Kuhlmann, 2022)

Let v be a henselian valuation on K. If v is $\mathcal{L}_{\mathrm{or}}$-definable, then it is already $\mathcal{L}_{\mathrm{r}}$-definable.

A main component of this proof is to show that the value group vK and the *ordered* residue field Kv are stably embedded in $(K, +, -, \cdot, 0, 1, <, v)$.

We can thus restrict ourselves to $\mathcal{L}_{\mathrm{r}}$-definability in the context of henselian valuations!

Open problem:

Does the above also hold for parameter-free definability, i.e. is any $\emptyset$-$\mathcal{L}_{\mathrm{or}}$-definable henselian valuation on an ordered field already $\emptyset$-$\mathcal{L}_{\mathrm{r}}$-definable?

Conditions on the value group

Divisible hull of G: $\{\frac{g}{n} \mid g \in G, n \in \mathbb{N} \setminus \{0\}\}$.

Theorem (K., Kuhlmann, Link, 2022)

*Let (F, v) be a henselian valued field such that vF is **not closed in its divisible hull**. Then v is $\mathcal{L}_{\mathrm{r}}$-definable with one parameter from F.*

This is a strict generalisation of Hong's result:

Theorem (Hong, 2013/2014)

*Let (F, v) be a henselian valued field. Suppose that vF is densely ordered and **contains a convex subgroup that is p-regular but not p-divisible for some prime p**. Then v is $\mathcal{L}_{\mathrm{r}}$-definable with one parameter from F.*

Examples

Closed in divisible hull:

- Every discretely ordered abelian group is closed in its divisible hull.
- If G is non-divisible and H is divisible, then $G \oplus H$ is densely ordered and closed in its divisible hull. (E.g. $\mathbb{Z} \oplus \mathbb{Q}$.)
- A Hahn sum $\bigoplus_{i\in\mathbb{N}} G_i = G_0 \oplus G_1 \oplus \ldots$ is either divisible (i.e. each component G_i is divisible) or closed in its divisible hull.

Examples

Let $A = \{\frac{a}{2^n} \mid a \in \mathbb{Z}, n \in \mathbb{N}\}$ (the 2-divisible hull of $\mathbb{Z}$). Consider the following non-divisible densely ordered abelian groups with divisible hull $\mathbb{Q} \oplus \mathbb{Q}$:

- $\mathbb{Q} \oplus A$ is dense in its divisible hull.
  E.g. $(0, \frac{1}{3}) < (0, \frac{1}{2}) < (0, \frac{2}{3})$.
- $A \oplus \mathbb{Q}$ is closed in its divisible hull.
  E.g. $A \oplus \mathbb{Q}$ contains no element strictly between $(\frac{1}{3}, -1)$ and $(\frac{1}{3}, 1)$, so $(\frac{1}{3}, 0)$ is no limit point of $A \oplus \mathbb{Q}$.
- $A \oplus A$ is neither dense nor closed in its divisible hull.
  E.g. again $(\frac{1}{3}, 0) \in \mathbb{Q} \oplus \mathbb{Q}$ is no limit point of $A \oplus A$ but $(0, \frac{1}{3}) \notin A \oplus A$ is a limit point of $A \oplus A$.

Parameter-free definability

Theorem (Ax / Koenigsmann / Hong)

Let (F, v) be a henselian valued field. Suppose that one of the following holds:

1. *vF is discretely ordered and elementarily equivalent to $\mathbb{Z}$.*
2. *vF is densely ordered, non-divisible and dense in its divisible hull.*

Then v is $\emptyset$-$\mathcal{L}_{\mathrm{r}}$-definable.

Theorem (Hong / K., Kuhlmann, Link)

Let (F, v) be a henselian valued field. Suppose that one of the following holds:

1. *vF is discretely ordered.*
2. *vF is densely ordered and not closed in its divisible hull.*

Then v is $\mathcal{L}_{\mathrm{r}}$-definable with one parameter from F.

Parameter-free definability

Theorem (Hong / K., Kuhlmann, Link)

Let (F, v) be a henselian valued field. Suppose that one of the following holds:

1. *vF is discretely ordered.*
2. *vF is densely ordered and not closed in its divisible hull.*

Then v is $\mathcal{L}_{\mathrm{r}}$-definable with one parameter from F.

In general, we cannot reduce to parameter-free definability!

Theorem (K., Kuhlmann, Link, 2022)

Suppose that $G \neq \{0\}$. Then there exists an ordered henselian valued field (K, v) such that $vK = \ldots \oplus G \oplus G$ and $(K, <)$ admits no non-trivial $\emptyset$-$\mathcal{L}_{\mathrm{or}}$-definable henselian valuation. In particular, v is not $\emptyset$-$\mathcal{L}_{\mathrm{or}}$-definable.

This theorem relies on a construction method of t-**henselian non-henselian** ordered fields with prescribed value groups.

Parameter-free definability

Theorem (Hong / K., Kuhlmann, Link)

Let (F, v) be a henselian valued field. Suppose that one of the following holds:

1. *vF is discretely ordered.*
2. *vF is densely ordered and not closed in its divisible hull.*

Then v is $\mathcal{L}_{\mathrm{r}}$-definable with one parameter from F.

Theorem (K., Kuhlmann, Link, 2022)

Suppose that $G \neq \{0\}$. Then there exists an ordered henselian valued field (K, v) such that $vK = \ldots \oplus G \oplus G$ and K admits no non-trivial $\emptyset$-$\mathcal{L}_{\mathrm{or}}$-definable henselian valuation. In particular, v is not $\emptyset$-$\mathcal{L}_{\mathrm{or}}$-definable.

- Setting $G = \mathbb{Z}$, we obtain that vK is discretely ordered.
- Setting $G = \{\frac{a}{2^n} \mid a \in \mathbb{Z}, n \in \mathbb{N}\}$, we obtain that vK is densely ordered and not closed in its divisible hull.

Density in real closure

Proposition (Jahnke, Simon, Walsberg, 2017)

If K is not dense in its real closure, then it admits a non-trivial $\mathcal{L}_{\mathrm{or}}$-definable convex valuation.

When is K not dense in its real closure? If K is dense in its real closure, then Kv is real closed and vK is divisible.

Theorem (Dittmann, Jahnke, K., Kuhlmann, 2022)

Let v be a convex valuation on K. Suppose that at least one of the following holds:

1. *vK is discretely ordered.*
2. *vK is not closed in its divisible hull.*
3. *Kv is not closed in its real closure.*

Then v is $\mathcal{L}_{\mathrm{or}}$-definable (in the first two cases with one parameter from K).

Generalising previous results

Theorem (Dittmann, Jahnke, K., Kuhlmann, 2022)

Let v be a convex valuation on K. Suppose that at least one of the following holds:

1. *vK is discretely ordered.*
2. *vK is not closed in its divisible hull.*
3. *Kv is not closed in its real closure.*

Then v is $\mathcal{L}_{\mathrm{or}}$-definable (in the first two cases with one parameter from K).

Corollary

Let v be a convex valuation on K. Suppose that at least one of the following holds:

1. *vK is discretely ordered and elementarily equivalent to $\mathbb{Z}$.*
2. *vK is densely ordered, non-divisible and dense in its divisible hull.*
3. *Kv is not real closed but dense in its real closure.*

Then v is $\emptyset$-$\mathcal{L}_{\mathrm{or}}$-definable.

Examples

- Every archimedean field is dense in its real closure. Moreover, density in the real closure transfers via $\mathcal{L}_{\mathrm{or}}$-elementary equivalence.
- The ordered field of generealised power series $K((G))$ is either real closed (if and only if K is real closed and G is divisible) or closed in its real closure.
- $\mathbb{Q}(t)$ with the ordering $0 < t < \mathbb{Q}^{>0}$ is neither dense nor closed in its real closure.

Answering our second main question

Lemma (Dittmann, Jahnke, K., Kuhlmann, 2022)

Let $F = \mathbb{Q}(s_i \mid i \in \mathbb{N})$, where the s_i are algebraically independent over $\mathbb{Q}$. Suppose that v is a valuation on F with $v(s_0) < 0$ and $v(s_i) \geq 0$ for $i > 0$. Then v is not $\mathcal{L}_{\mathrm{r}}$-definable.

Example of a $\emptyset$-$\mathcal{L}_{\mathrm{or}}$-definable but not $\mathcal{L}_{\mathrm{r}}$-definable convex valuation:

- Let $k = \mathbb{Q}(s_1, s_2, \ldots) \subseteq \mathbb{R}$, where the s_i are algebraically independent over $\mathbb{Q}$.
- Set $K = k(t^{-1}) = k(t) \subseteq k(\!(t)\!)$.
- Then $v_{\min} K = \mathbb{Z}$, so $v_{\min}$ **is $\emptyset$-$\mathcal{L}_{\mathrm{or}}$-definable**.
- However, $v_{\min}(t^{-1}) = -1 < 0$ and $v_{\min}(s_i) = 0$ for $i > 0$, so $v_{\min}$ **is not $\mathcal{L}_{\mathrm{r}}$-definable**.

Definable valuations

Definition

Let v be a valuation on F. Then F is called **almost real closed** (with respect to v) if v is henselian and Fv is real closed.

- Every almost real closed field is $\mathcal{L}_{\mathrm{r}}$-elementarily equivalent to $\mathbb{R}((G))$ for some G.
- Every almost real closed field can be ordered, but the ordering is not necessarily unique.
- In almost real closed fields, all convex valuations are already henselian.
- Definable henselian valuations in almost real closed fields were well-studied by Delon–Farré (1996).

Theorem (Dittmann, Jahnke, K., Kuhlmann, 2022)

*Let F be an almost real closed field and let $<$ be any ordering on F. Then **any** $\mathcal{L}_{\mathrm{or}}$-definable valuation on $(F,<)$ is henselian (and thus already $\mathcal{L}_{\mathrm{r}}$-definable).*

Classification of strongly dependent ordered fields

What are necessary and sufficient conditions on an ordered field to be strongly dependent?

o-minimal $\rightarrow$ **weakly o-minimal** $\rightarrow$
dp-minimal $\rightarrow$ **dp-finite** $\rightarrow$ **strongly dependent** $\rightarrow$ **NIP**

Classification of strongly dependent ordered fields

Conjecture (Shelah et al.)

Any infinite strongly dependent field F is either algebraically closed, real closed or admits a non-trivial $\mathcal{L}_{\mathrm{r}}$-definable henselian valuation.

Specialised to ordered fields:

Conjecture

Any strongly dependent ordered field K is either real closed or admits a non-trivial $\mathcal{L}_{\mathrm{r}}$-definable henselian valuation.

Strongly dependent almost real closed fields

Theorem (K., Kuhlmann, Lehéricy, 2021)

The following are equivalent:

1. *Any strongly dependent ordered field is either real closed or admits a non-trivial $\mathcal{L}_{\mathrm{r}}$-definable henselian valuation.*
2. *Any strongly dependent ordered field is almost real closed.*

Theorem (K., Kuhlmann, Lehéricy, 2021)

Let $(K, <)$ be almost real closed field with respect to v. Then $(K, <)$ is strongly dependent if and only if vK is strongly dependent (as an ordered abelian group).

Not every strongly dependent ordered field is almost real closed with respect to some $\mathcal{L}_{\mathrm{r}}$-definable henselian valuation! (K., Kuhlmann, Lehéricy, 2021)

Further definability questions in the study of (N)IP ordered fields

The smallest ordered field: since $\mathbb{Z}$ is $\mathcal{L}_{\mathrm{r}}$-definable in $\mathbb{Q}$, this ordered field has the independence property.

1. In what ordered fields K is $\mathbb{Z}$ definable (in the language $\mathcal{L}_{\mathrm{r}}$ or $\mathcal{L}_{\mathrm{or}}$)?
2. In what ordered fields K is a subring with infinitely many non-associated primes definable (in the language $\mathcal{L}_{\mathrm{r}}$ or $\mathcal{L}_{\mathrm{or}}$)?
3. Is $\mathbb{Z}$ or any other ring as in (2) definable in the euclidean closure of $\mathbb{Q}$?
4. Given a $\mathbb{Q}$-algebraically independent set $B \subseteq \mathbb{R}$, is $\mathbb{Z}$ or any other ring as in (2) definable in $\mathbb{Q}(B)$?

$$\vdots$$

See posters of Laura Wirth and Lasse Vogel.

Primary References

- P. Dittmann, F. Jahnke, L. S. Krapp and S. Kuhlmann, 'Definable valuations on ordered fields', to appear in *Model Theory*, 2022, 17 pp., arXiv:2206.15301.
- L. S. Krapp, S. Kuhlmann and G. Lehéricy, 'Ordered fields dense in their real closure and definable convex valuations', *Forum Math.* 33 (2021) 953–972.
- L. S. Krapp, S. Kuhlmann and G. Lehéricy, 'Strongly NIP almost real closed fields', *MLQ Math. Log. Q.* 67 (2021) 321–328.
- L. S. Krapp, S. Kuhlmann and M. Link, 'Definability of henselian valuations by conditions on the value group', to appear in *J. Symb. Log.*, 2022, 19 pp., arXiv:2105.09234.

Further References

- J. Ax, 'On the undecidability of power series fields', *Proc. Amer. Math. Soc.* 16 (1965) 846.
- F. Delon and R. Farré, 'Some model theory for almost real closed fields', *J. Symb. Log.* 61 (1996) 1121–1152, doi:10.2307/2275808.
- R. Farré, 'A transfer theorem for Henselian valued and ordered fields', *J. Symb. Log.* 58 (1993) 915–930, doi:10.2307/2275104.
- A. Fehm and F. Jahnke, 'Recent progress on definability of Henselian valuations', *Ordered Algebraic Structures and Related Topics*, Contemp. Math. 697 (eds F. Broglia, F. Delon, M. Dickmann, D. Gondard-Cozette and V. A. Powers; Amer. Math. Soc., Providence, RI, 2017), 135–143.
- J. Hong, *Immediate expansions by valuations of fields*, doctoral thesis, McMaster University, 2013.

Further References

- J. Hong, 'Definable non-divisible Henselian valuations', *Bull. Lond. Math. Soc.* 46 (2014) 14–18.
- F. Jahnke, P. Simon and E. Walsberg, 'Dp-minimal valued fields', *J. Symb. Log.* 82 (2017) 151–165, doi:10.1017/jsl.2016.15.
- M. Knebusch and M. J. Wright, 'Bewertungen mit reeller Henselisierung', *J. Reine Angew. Math.* 286/287 (1976) 314–321, doi:10.1515/crll.1976.286-287.314.
- J. Koenigsmann, 'Elementary characterization of fields by their absolute Galois group', *Sib. Adv. Math.* 14 (2004) 16–42.
- S. Shelah, 'Strongly dependent theories', *Israel J. Math.* 204 (2014) 1–83, doi:10.1007/s11856-014-1111-2.