Existential closedness of $\overline{\mathbb{Q}}$

as a globally valued field

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Height on $\mathbb{Q}$

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One can define it without referring to a presentation as a quotient of two integers. Note that

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Example

$\mathrm{ht}(\frac{2}{3}) = \max(\mathrm{ord}_2(\frac{2}{3}), 0) \log 2 + \max(\mathrm{ord}_3(\frac{2}{3}), 0) \log 3 + \max(-\log \frac{2}{3}, 0)) = \log 2 + \log \frac{3}{2} = \log 3.$

Height on $\overline{\mathbb{Q}}$

For $q \in \overline{\mathbb{Q}}^{\times}$ one defines the *Weil logarithmic height* by

$$\mathrm{ht}(q) = \frac{1}{[K:\mathbb{Q}]}\Bigg(\sum_{p\text{: prime in } \mathcal{O}_K} \max(\mathrm{ord}_p(q), 0) \log \#\kappa(p) + \sum_{\sigma: K \to \mathbb{C}} \max(-\log|\sigma(q)|, 0)\Bigg)$$

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Here (and in the rest of this presentation) K is any number field with $q \in K$ and $\mathrm{ht}(q)$ does not depend on the choice of such K.
Let Val_K be a set of valuations (both non-Archimedean and Archimedean, i.e., minus logarithms of norms coming from embeddings into $\mathbb{C}$) on K.
Let μ be the discrete measure

$$\mu := \frac{1}{[K:\mathbb{Q}]}\Bigg(\sum_{p \in \mathrm{Spec}(\mathcal{O}_K)} \delta_{\mathrm{ord}_p} \cdot \log \#\kappa(p) + \sum_{\sigma : K \to \mathbb{C}} \delta_{-\log|\sigma(-)|}\Bigg).$$

Height on $\overline{\mathbb{Q}}$ continued

Then for $q \in K$ we can simple write

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$$\mathrm{ht}(x) = \int_{\mathrm{Val}_K} \max_i(v(x_i)) d\mu(v).$$

It does not depend on the choice of coordinates for x because of the following.

Projection formula

$$\int_{\mathrm{Val}_K} v(q) d\mu(v) = 0.$$

Classical applications

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However, in these theorems the degree is bounded/the number field is fixed.

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The structures

Definition [BH22]

A GVF is a field F together with a (class of) measure μ on the space of "valuations" Val_F which satisfies the product formula, i.e.,

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Let t be a $\mathbb{Q}$-tropical polynomial, i.e., a term in the language $+, \min, 0, (q\cdot)_{q\in\mathbb{Q}}$. For example $t(x, y) = max(x, max(x + y, y + 3))$.

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One can write universal axioms on R_t's, so that a field equipped with predicates satisfying these axioms comes from a measure as above.

Examples

For $a \in F^\times$ we define its height (with respect to some GVF structure) $\mathrm{ht}(a) = \int_{\mathrm{Val}_F} \max(v(a), 0) d\mu(v)$. Here are a few GVFs:

- $\overline{\mathbb{Q}}$ with the predicates defined as Weil logarithmic heights. It is denoted $\overline{\mathbb{Q}}[1]$ (if we multiply all predicates by $r > 0$ we get $\overline{\mathbb{Q}}[r]$).

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- If $(K_i)_{i \in I}$ we can take ultraproduct which by definition consists of classes of sequences with bounded height.
- For any field k we can equip $\overline{k(t)}$ with a unique GVF structure where the measure concentrates on valuations trivial on k and $\mathrm{ht}(t) = 1$. In [BH21] it is shown that $\overline{k(t)}$ is existentially closed, i.e., whenever $\overline{k(t)} \subset F$ is a GVF extension, then F embeds into some ultrapower of $\overline{k(t)}$ over $\overline{k(t)}$.

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- If X is a variety over k, a movable curve (or $\dim X - 1$ ample divisors $D_1, \dots$) induces a GVF structure on $k(X)$. Moreover, the space of GVF structures on $k(X)$ is homeomorphic to $\varprojlim N_1^+(X')$ for the system of blowups $X' \to X$.

Arithmetic example

Bilu equidistribution

Let $a_n \in \overline{\mathbb{Q}}^\times$ be a sequence with $\deg(a_n) \to \infty$ and $\mathrm{ht}(a_n) \to 0$. Define measures

$$\mu_n := \frac{1}{\deg(a_n)} \sum_{x \in G \cdot a_n} \delta_x,$$

where $G = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ and δ_x is the Dirac delta at x. Then μ_n weakly converge to the Lebesgue measure on the unit circle in $\mathbb{C}$.

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From the point of view of GVFs this follows from the following fact.

[BH21, Lemma 6.5]

There is a unique GVF structure on $\overline{\mathbb{Q}}(x)$ extending $\overline{\mathbb{Q}}[1]$ with $\mathrm{ht}(x) = 0$.

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More precisely, the measure μ defining restriction of such GVF structure to $\mathbb{Q}(x)$, if restricted to the set of complex places of $\mathbb{Q}(x)$, is the Lebesgue measure on the unit circle in $\mathbb{C}$.

Existential closeness of $\overline{\mathbb{Q}}[1]$

Theorem (Sz.)

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Corollary

Assume that X is an affine variety over $\mathbb{Q}$ and assume that we are given morphisms $f_1, \ldots, f_n : X \to \mathbb{A}^m$. Equip $F = \mathbb{Q}(X) = \mathbb{Q}(\overline{a})$ with a GVF structure and denote $R_{t_i}(f_i(\overline{a})) = r_i$ for some $\mathbb{Q}$-tropical polynomials $t_1, \ldots, t_n$.

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Let $\varepsilon > 0$. Then we can find a (sufficiently generic) $x \in X(\overline{\mathbb{Q}})$ such that for all $i = 1, \ldots, n$ we have

$$|R_{t_i}(f_i(\overline{a}|_x)) - r_i| < \varepsilon.$$

Applications

The existential closedness of $\overline{\mathbb{Q}}$ can be used in the following situations.

- A direct application is an L^1 Fekete-Szegő type result for varieties of arbitrary dimension, i.e., [BH22, Theorem 3.11] for number fields.

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- (in progress) If E is an elliptic curve can one find optimal bounds on $\mathrm{ht}((2P)_x) - 4\,\mathrm{ht}(P_x)$ by finding a GVF measure on the function field $\mathbb{Q}(E)$? More general questions about extremes of heights...

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Definition

An *arithmetic divisor* $\overline{\mathcal{D}} = (\mathcal{D}, g)$ on $\mathcal{X}$ is a divisor $\mathcal{D}$ (linear combination of codimension one subvarieties) on $\mathcal{X}$ together with a Green function $g : (\mathcal{X} \setminus \mathrm{supp}(\mathcal{D}))^{\mathrm{an}} \to \mathbb{R}$.

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A function $g : (\mathcal{X} \setminus \mathrm{supp}(\mathcal{D}))^{\mathrm{an}} \to \mathbb{R}$ is a *Green function* for $\mathcal{D}$, if for any open $\mathcal{U} \subset \mathcal{X}$ on which $\mathcal{D}$ is given by equation $d = 0$ the function $g + \log|d|$ extends to a continuous function on the complex analytification $\mathcal{U}^{\mathrm{an}}$.

Lattice structure

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Let $\overline{\mathcal{D}}, \overline{\mathcal{E}}$ be adelic divisors on $\mathcal{X}$ with $\mathcal{D}, \mathcal{E}$ effective. If $\mathcal{D} \cap \mathcal{E}$ is a (Cartier) divisor, one defines:

$$\overline{\mathcal{D}} \wedge \overline{\mathcal{E}} := (\mathcal{D} \cap \mathcal{E}, \min(g_{\mathcal{D}}, g_{\mathcal{E}})).$$

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If the intersection is not a divisor, one can pass to to a blowup $\mathcal{X}' \to \mathcal{X}$ to ensure that it is the case. The function field $\mathbb{Q}(\mathcal{X}') = \mathbb{Q}(\mathcal{X}) = \mathbb{Q}(X)$ stays the same.

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We can form (modulo technicalities) a real vector space of adelic divisors on blowups of $\mathcal{X}$ denoted by $\mathrm{ADiv}(\mathbb{Q}(\mathcal{X}))$. It has lattice operations $\wedge, \vee$. By definition $\overline{\mathcal{D}} \vee \overline{\mathcal{E}} := -((-\overline{\mathcal{D}}) \wedge (-\overline{\mathcal{E}}))$ and $\mathcal{D}$ is called *effective* ($\overline{\mathcal{D}} \geq 0$), if $\overline{\mathcal{D}} \vee 0 = \overline{\mathcal{D}}$.

Height with respect to an adelic divisor

Definition

If $\overline{\mathcal{D}}$ is an adelic divisor on $\mathcal{X}$ (with $\mathcal{D}$ effective) and $x \in X(\overline{\mathbb{Q}})$ we define the height $h_{\overline{\mathcal{D}}}(x)$ as the number

$$h_{\overline{\mathcal{D}}}(x) := \frac{1}{[\kappa(x):\mathbb{Q}]}\Big(\log \#(\mathcal{O}_{\mathcal{C}}(\mathcal{D})/\mathcal{O}_{\mathcal{C}}) + \sum_{\sigma:\kappa(x)\to\mathbb{C}} g(x^{\sigma})\Big),$$

where $\mathcal{C}$ is the closure of $\{x\}$ in $\mathcal{X}$.

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where $\mathcal{C}$ is the closure of $\{x\}$ in $\mathcal{X}$.

Fact

Let $\overline{\mathcal{D}} = t(\widehat{\mathrm{div}}(\overline{a}))$ for some $\overline{a} \in \mathbb{Q}(\mathcal{X})$ and a $\mathbb{Q}$-tropical polynomial t. Pick $x \in X(\overline{\mathbb{Q}})$ such that $x \notin \mathrm{supp}(\mathcal{D})$. Then in $\overline{\mathbb{Q}}[1]$

$$h_{\overline{\mathcal{D}}}(x) = R_t(\overline{a}|_x).$$

Comparison

Definition

Let $I : \mathrm{ADiv}(\mathbb{Q}(\mathcal{X})) \to \mathbb{R}$ be a linear map over $\mathbb{R}$. It is called a normalised GVF functional if it:

- sends $\widehat{\mathrm{div}}(f)$ to 0 for every $f \in \mathbb{Q}(\mathcal{X})$ (product formula),
- sends effective arithmetic divisors to $\mathbb{R}_{\geq 0}$ (non-negativity of the measure),
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Theorem (Sz.)

There is a bijection between GVF structures on $\mathbb{Q}(\mathcal{X})$ extending $\mathbb{Q}[1]$ and normalised GVF functionals on $\mathrm{ADiv}(\mathbb{Q}(\mathcal{X}))$ given by

$$I(\overline{\mathcal{D}}) = R_t(\overline{a}) \text{ for } \overline{\mathcal{D}} = t(\widehat{\mathrm{div}}(\overline{a})).$$

Arithmetic intersection theory

The height is a part of more general family of intersection theoretic invariants. Namely if $\overline{\mathcal{D}}_0, \ldots, \overline{\mathcal{D}}_k$ are adelic divisors on $\mathcal{X}$ and $\mathcal{Z} \subset \mathcal{X}$ is a $k+1$-dimensional subvariety, then one can define

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The arithmetic degree is multilinear and defined inductively by the following formula.

Arithmetic intersection theory continued

Write $\mathcal{D}_k \cap \mathcal{Z} = \sum_i a_i \mathcal{W}_i$ as a cycle. Assume that the intersection is transversal. Then

Arithmetic intersection theory continued

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Arithmetic intersection theory continued

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&\widehat{\deg}(\overline{\mathcal{D}}_0, \dots, \overline{\mathcal{D}}_k | \mathcal{Z}) \\
&= \sum_i a_i \widehat{\deg}(\overline{\mathcal{D}}_0, \dots, \overline{\mathcal{D}}_{k-1} | \mathcal{W}_i) \\
&+ \int_{\mathcal{Z}(\mathbb{C})} g_{\overline{\mathcal{D}}_k} c_1(\overline{\mathcal{D}}_0) \wedge \cdots \wedge c_1(\overline{\mathcal{D}}_{k-1}).
\end{aligned}$$

If $\dim \mathcal{X} = d + 1$ and $\overline{\mathcal{D}}_0, \dots, \overline{\mathcal{D}}_d$ are arithmetic divisors on $\mathcal{X}$, we write $\overline{\mathcal{D}}_0 \cdot \ldots \cdot \overline{\mathcal{D}}_d$ for the intersection product with respect to $\mathcal{X}$.

Table of Contents

The main result

Let $\mathrm{ADiv}(\mathcal{X})$ be the real vector space of arithmetic divisors on $\mathcal{X}$.The existential closedness of $\overline{\mathbb{Q}}[1]$ translates to the following.

Theorem (Sz.)

Let $\overline{\mathcal{D}}_1, \ldots, \overline{\mathcal{D}}_n$ be arithmetic divisors on $\mathcal{X}$. Assume that $I : \mathrm{ADiv}(\mathbb{Q}(\mathcal{X})) \to \mathbb{R}$ is a normalised GVF functional. Then there is a generic sequence of $\overline{\mathbb{Q}}$-points $x_n \in X$ such that for all $i = 1, \ldots, n$

$$\lim_n h_{\overline{\mathcal{D}}_i}(x_n) = I(\overline{\mathcal{D}}_i).$$

If one of $\overline{\mathcal{D}}_i$ is big, then I can be only defined on the real span of $\overline{\mathcal{D}}_i$'s.

Arithmetic volume

The crucial ingredient of the proof is the *arithmetical volume function*, i.e.,

$$\widehat{\mathrm{vol}}(\overline{\mathcal{D}}) := \limsup_n \frac{\log \# \widehat{H}^0(n\overline{\mathcal{D}})}{n^{d+1}/(d+1)!},$$

where $\widehat{H}^0(n\overline{\mathcal{D}})$ is the set of effective arithmetic divisors rationally equivalent to $n\overline{\mathcal{D}}$ (i.e. their difference is spanned by $\widehat{\mathrm{div}}(f)$'s).

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It measures how big values of $h_{\overline{\mathcal{D}}}$ are expected to be. We call $\overline{\mathcal{D}}$ *big*, if $\widehat{\mathrm{vol}}(\overline{\mathcal{D}}) > 0$.

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If $\overline{\mathcal{D}}$ is big, then $\widehat{\mathrm{vol}}$ has directional derivatives at every direction at $\overline{\mathcal{D}}$ by [Che11]. Also, arithmetic volume is $(d+1)$-homogeneous, i.e., $\widehat{\mathrm{vol}}(n\overline{\mathcal{D}}) = n^{d+1}\widehat{\mathrm{vol}}(\overline{\mathcal{D}})$.

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Moreover, if $\overline{\mathcal{D}}$ is arithmetically ample, then $\widehat{\mathrm{vol}}(\overline{\mathcal{D}}) = \overline{\mathcal{D}}^{d+1}$ (arithmetic intersection product).

Proof sketch

Assume that V is $\mathrm{Span}(\overline{\mathcal{D}}_1, \ldots, \overline{\mathcal{D}}_n)$ divided by rational equivalence and consider $l : V \to \mathbb{R}$. Fix $\varepsilon > 0$. The proof follows the following steps:

- Perturb l by less than ε so that it is strictly positive on all big $\overline{\mathcal{D}} \in V$.

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- Consider the function $\frac{\widehat{\mathrm{vol}}^{1/d+1}}{l}$. It is well defined on $V \setminus \{0\}$ by the first point and it is determined by its values on the unit sphere in V by homogeneity. At the maximum on the sphere, the derivatives of $\widehat{\mathrm{vol}}^{1/d+1}$ and l coincide.

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- We get that up to (multiplicative) constant $D_{\overline{\mathcal{D}}}\widehat{\mathrm{vol}} = l$. But $D_{\overline{\mathcal{D}}}(\overline{\mathcal{D}}^{d+1})(\overline{\mathcal{M}}) = (d+1)\overline{\mathcal{D}}^d \cdot \overline{\mathcal{M}}$. This means that up to a constant l is given by multiplication with $\overline{\mathcal{D}}^d$.

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- Use arithmetic Bertini Theorems [Cha17], [Wil22] to pick d sections of powers of $\overline{\mathcal{D}}$ whose intersection is an irreducible curve in $\mathcal{X}$.
- Let $x \in X(\overline{\mathbb{Q}})$ be the generic point of that curve. By the assumptions of the theorem we can deal with multiplicative constant and x works!

Arithmetic Bertini theorem

The crucial step is passing from the equation $\overline{\mathcal{D}}^d \cdot (-) = I(-)$ to the existence of a point x such that $h_{(-)}(x) \approx I(-)$.

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$$\overline{\mathcal{M}} \cdot n\overline{\mathcal{D}}^d = \widehat{\deg}(\overline{\mathcal{M}} \cdot \overline{\mathcal{D}}^{d-1} | \mathcal{E})$$

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The point it that we can neglect the integral part coming from $g_{\overline{\mathcal{E}}}$.

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(*) Some (multiplicative) constants are skipped in this sketch, for the simpler exposition!

Thank you!

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