An Imaginary Ax-Kochen/Ershov Principle: the equicharacteristic zero case

Mariana Vicaría
(joint with: Silvain Rideau-Kikuchi)

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Notation

Let $v : K \to \Gamma \cup \{\infty\}$ be a valuation.

1. $\mathcal{O} = \{x \in K \mid v(x) \geq 0\}$ is the valuation ring,
2. $\mathcal{M} = \{x \in K \mid v(x) > 0\}$ is its maximal ideal.

We refer to $\mathbf{k} = \mathcal{O}/\mathcal{M}$ as the *residue field* and we call Γ the *value group*.

It all started...

Theorem (*Haskell, Hrushovski, Macpherson*)

The theory of ACVF eliminates imaginaries once the geometric sorts are added, this is:

1. *For each natural number n, we add a sort S_n which consists of the codes of the $\mathcal{O}$-sublattices of K^n (i.e. the free $\mathcal{O}$-submodules of K^n on n generators).*
2. *For any $s \in S_n$, we define $red(s) = s/\mathcal{M}s$ (the reduction of s modulo $\mathcal{M}$). And we let $T_n = \bigcup\{red(s) \mid s \in S_n\}$.*

$$\mathrm{Geo} = K \cup \bigcup_{n\in\mathbb{N}} S_n \cup \bigcup_{n\in\mathbb{N}} T_n.$$

Hard worked continued

Down to the geometric sorts:

- p-adics and their ultraproducts (E. Hrushovski, B. Martin and S. Rideau-Kikuchi),
- separably closed valued fields of finite imperfection degree (M. Hils, M. Kamensky, S. Rideau-Kikuchi),
- real closed valued fields (T. Mellor).

An-Imaginary Ax-Kochen/Ershov principle: a very good question

SPINE PHILOSOPHY: AX-KOCHEN/ERSHOV PRINCIPLE

Theorem (Ax-Kochen/Ershov)

Let (K, k, Γ) *and* (K', k', Γ') *be two henselian valued fields of equicharacteristic zero, then* $K \equiv K'$ *if and only if* $k \equiv_{fields} k'$ *and* $\Gamma \equiv_{OAG} \Gamma'$.

Principle

The model theory of a henselian valued field of equicharacteristic zero is controlled by its residue field and its value group.

Conjecture (Hrushovski, 2000)

Is there an Imaginary Ax-Kochen/ Ershov principle for henselian valued field encompassing all the previous results?

Two orthogonal strategies:

Following the Ax-Kochen principle one could try to break the question:

1. Hils,Rideau-Kikuchi: Make the value group very tame (definably complete) and understand the problems that the residue field will bring to the picture.

2. Me: make the residue field as tame as possible (e.g. algebraically closed) and understand the problems that the complexity of the value group will contribute to the problem.

Context: The Hils, Rideau-Kikuchi approach

Context

We have (K, v) a henselian valued field of equicharacterisitic zero. And let's assume that:

1. Γ *is either divisible or a $\mathbb{Z}$-group,*
2. $\mathbf{k}$ *is no longer algebraically closed, but an arbitrary field of characteristic zero.*

Question

Which new sorts would be required to add to eliminate imaginaries that the residue field brings to the picture?

Spoiler! One will need to add the $\mathbf{k}$-linear sorts.

The **k**-linear imaginaries

Context: Let $s \subseteq K^n$ be a lattice of rank n.

1. $V_s = s/\mathcal{M}s$ is a **k**-vector space.
2. So we want to consider the two sorted structure $(\mathbf{k}, s/\mathcal{M}s)$.

The language $\mathcal{L}_{\mathrm{Vect}}$: We have two sorts:

- one for the field **k** equipped with the language of rings $\mathcal{L}_{ring} = \{+, \cdot, 0, 1\}$,
- one for the vector space V with the group structure, i.e. equipped with $\mathcal{L}_G = \{0, +\}$.
- A map $\lambda : \mathbf{k} \times V \to V$ interpreted as scalar multiplication.

The **k**-linear imaginaries

1. We consider the $\mathcal{L}_{\mathrm{Vect}}$-theory of dimension n vector spaces over a field.
2. For each X definable quotient of the vector space sort V in $\mathcal{L}_{\mathrm{Vect}}$. For each $s \subseteq K^n$ a lattice of rank n:

$$X^{(\mathbf{k}, s/\mathcal{M}s)} := \text{ the interpretation of } X \text{ in the structure } (\mathbf{k}, s/\mathcal{M}s).$$

We define $T_{n,X} = \sqcup_{s \in S_n} X^{(\mathbf{k}, s/\mathcal{M}s)}$.

Examples of $T_{n,X}$

Recall: $T_{n,X} = \sqcup_{s \in S_n} X^{(\mathbf{k}, s/\mathcal{M}s)}$ where $X^{(\mathbf{k}, s/\mathcal{M}s)}$ is the interpretation of X in $(\mathbf{k}, s/\mathcal{M}s)$.

1. If $X = V$, then $T_{n,X} = T_n$ the usual T_n-sorts,
2. If X is the *one element quotient of* V then $T_{n,X} \cong S_n$.

Definition (The $\mathbf{k}$ linear imaginaries)

$$\mathbf{k}^{leq} := \sqcup_{n,X} T_{n,X}.$$

A very nice paper!

Theorem (Hils,Rideau-Kikuchi)

Let (K, v) be a henselian valued field of equicharacteristic zero. Assume:

1. *The value group Γ is definably complete (ODAG or a $\mathbb{Z}$-group).*
2. *The induced theory of $\mathbf{k}$ eliminates $\exists^\infty$,*

Then K weakly eliminates imaginaries down to the sorts $Geo \cup k^{leq} \cup \Gamma^{eq}$.

Remark:

1. There is quite a lot of redundancy!
2. this just one piece of what they proved, e.g. extended structure and mixed characteristic case.

Context: The Second approach

Context

We have (K, v) a henselian valued field of equicharacterisitic zero. And we assume that:

1. $\mathbf{k}$ *is algebraically closed.*
2. Γ *is a more general OAG, i.e. no longer divisible or a $\mathbb{Z}$-group.*

Question

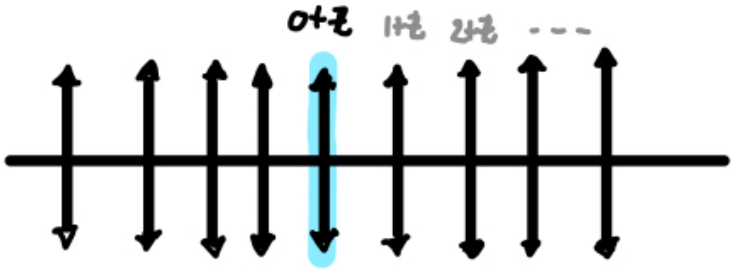
Which new sorts would be required to add if we want to obtain elimination of imaginaries once we take Γ a more general ordered abelian group?

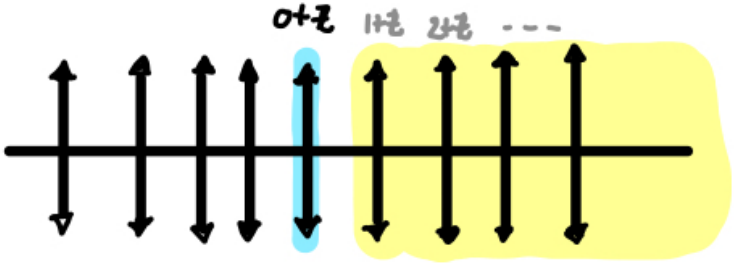
Spoiler: We will need to introduce the *Stabilizer Sorts*.

A little bit of more complexity in OAG

Definably complete: $(\mathbb{Q}, +, <, 0)$ and $(\mathbb{Z}, +, <, 0)$.

Nice property: No definable convex subgroups!





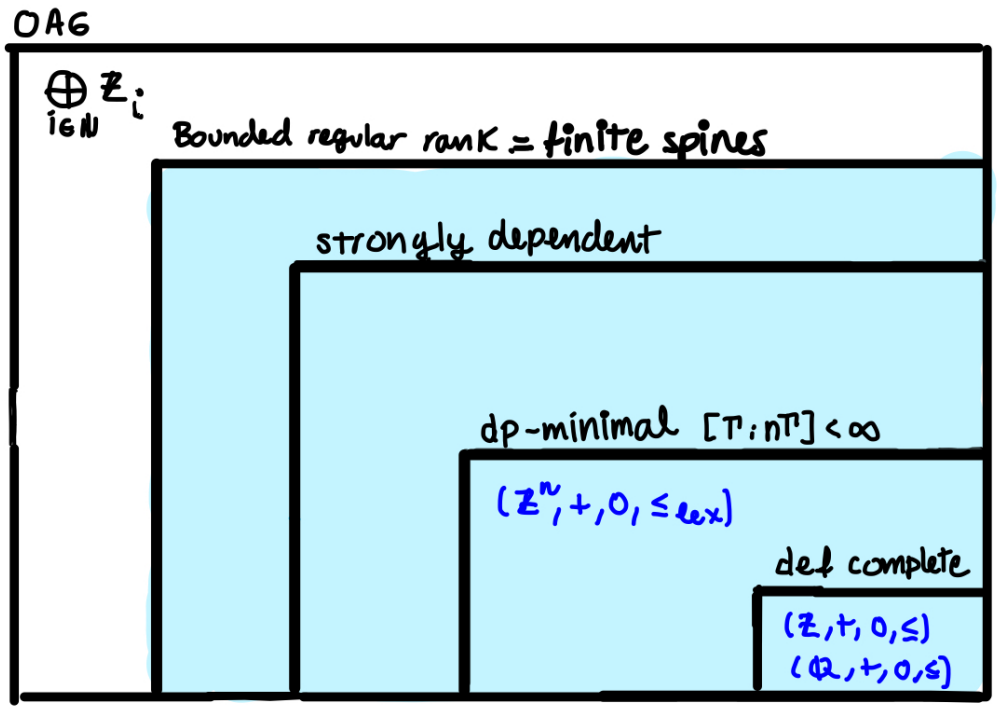
Example: $(\mathbb{Z}^2, <_{lex}, +, 0)$

Picture: More definable end-segments

OAG with bounded regular rank= finite spines

Next step: Having countably many definable convex subgroups, but not uniform definable families of convex subgroups.

Fact:[Farré] Let Γ be an OAG with bounded regular rank, then it has countably many convex definable subgroups.



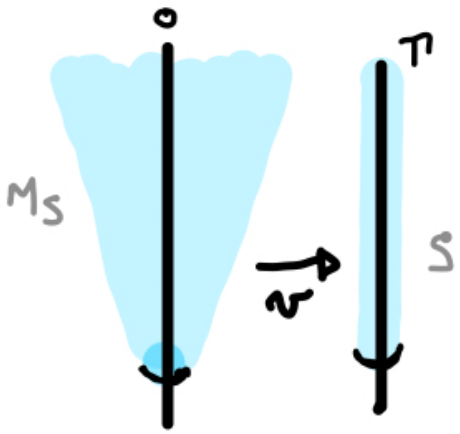
Picture: General map OAG

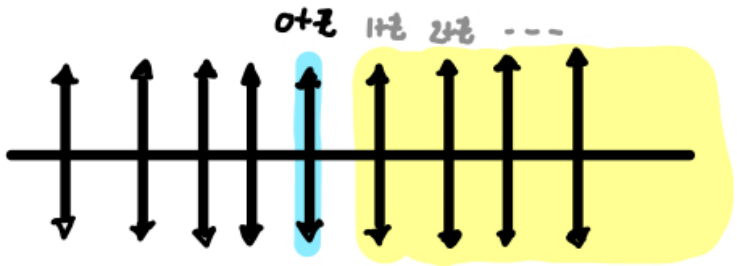
What are the stabilizer sorts?

Key point: They in essence add codes for all the definable $\mathcal{O}$-modules $M \subseteq K^n$.

Question: So which are the new $\mathcal{O}$-submodules that appear when we make the value group with bounded regular rank?

One-dimensional case: There is a one-to-one correspondence between the $\mathcal{O}$-submodules of K and the end-segments of Γ.
Correspondence:





Picture: More definable end-segments

The Stabilizer sorts

The higher dimensional case: If $M \subseteq K^n$ then $M \cong \oplus_{i \leq n} I_i$ where $I_i \subseteq K$ is an $\mathcal{O}$-module.

There is a nicer presentation:

1. Let $B_n(K)$ be the group of upper triangular and invertible $n \times n$ matrices.
2. The *stabilizer of* $C_{(I_1,\ldots,I_n)}$

$$Stab_{(I_1,\ldots,I_n)} = \{M \in B_n(K) \mid MC_{(I_1,\ldots,I_n)} = C_{(I_1,\ldots,I_n)}\}.$$

3. **The stabilizer sorts:**

$$\mathcal{G} = K \cup \{B_n(K)/Stab_{(I_1,\ldots,I_n)} \mid I_i \in \mathcal{I}\}$$

My result!

Theorem (V.)

Let (K, v) be a henselian valued field of equicharacteristic zero, such that:

1. $\mathbf{k}$ *is algebraically closed,*
2. Γ *has bounded regular rank.*

Then K weakly eliminates imaginaries down to $\mathcal{G} \cup \Gamma^{eq}$.

Remark: full elimination of imaginaries when Γ is dp-minimal.

Combining forces

Context

Let (K, v) be a henselian valued field of equicharacteristic zero. Suppose that its value group has bounded regular rank. Let T be is $\mathcal{L}_{div}$-first order theory.

Theorem (Rideau-Kikuchi,V.)

1. *If $[\Gamma : n\Gamma] < \infty$ then T has weak elimination of imaginaries down to $\mathcal{G} \cup \mathbf{k}^{leq} \cup \Gamma^{eq}$.*
2. *If we add an angular component map, then T has weak elimination of imaginaries down to $\mathcal{G} \cup \mathbf{k}^{leq} \cup \Gamma^{eq}$.*

An abstract criterion to prove elimination of imaginaries

Theorem (Hrushovski)

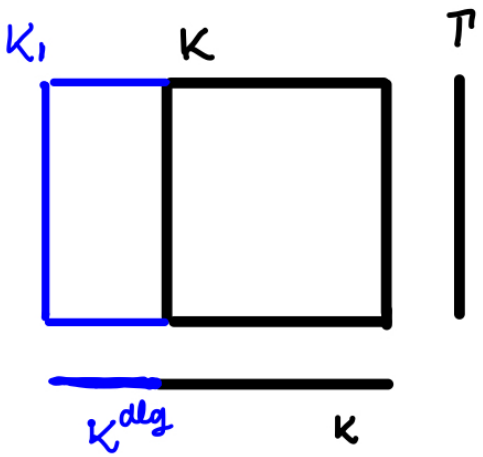
Let T be a first order theory with home sort K (meaning that $\mathfrak{M}^{eq} = dcl^{eq}(K)$). Let $\mathcal{G}$ be some collection of sorts. Suppose that:

- *DENSITY OF DEFINABLE TYPES: For every non-empty definable set $X \subseteq K$ there is an $acl^{eq}(\ulcorner X \urcorner)$-definable type in X,*
- *CODING OF DEFINABLE TYPES: Every definable type in K^n has a code in $\mathcal{G}$ (possibly infinite). This is, if p is any (global) definable type in K^n, then the set $\ulcorner p \urcorner$ of codes of the definitions of p is interdefinable with some (possibly infinite) tuple from $\mathcal{G}$,*

Then T weakly eliminates imaginaries down to $\mathcal{G}$.

The strategy

No longer a chance to have density of definable types, but we can go to the maximal unramifed extension!



1. *Step* 1*: Density of definable types (in a reduct)*
2. *Step* 2*: Invariant extensions*

The strategy

Context

Let K be a henselian valued field of equicharacteristic zero and $K_1 = K^{ur}$ be its maximal unramified extension.

We work in two languages:

1. $\mathcal{L}_1$: For $K_1 = K^{ur}$
   The usual 2-sorted language (K, Γ) with Γ Morleyized.
2. $\mathcal{L}$: For the structure K
   the 3-sorted language (K, RV, Γ) with Γ Morleyized.

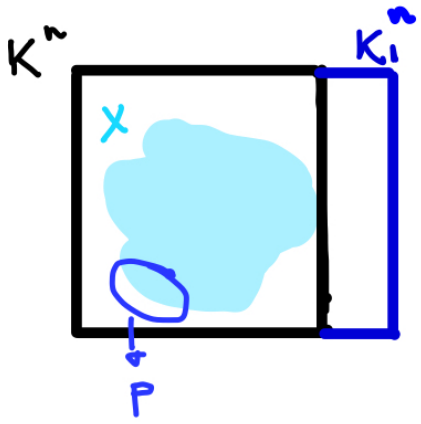
Notation: We denote $\mathrm{tp}_1$and tp, acl_1 and acl, $S_n(K)$ and $S_n^1(K)$, etc... respectively.

Step 1: Density of $\mathcal{L}_1$ definable types

Theorem (Rideau-Kikuchi,V.)

Let (K, v) be a henselian valued field of equicharacteristic zero whose value group has bounded regular rank. Let $A = \mathrm{acl}^{eq}(A) \subseteq K^{eq}$ then for any $\mathcal{L}(A)$-definable subset $X \subseteq K^n$ there is a type $p(x) \in S_n^1(K)$ such that:

1. *$p(x) \cup X$ is consistent.*
2. *It is $\mathcal{L}_1(\mathcal{G}(A) \cup \Gamma^{eq}(A))$-definable.*
   (Its canonical base can be coded in $\mathcal{G} \cup \Gamma^{eq}$)



Sketch of the argument

The one-dimensional case: Let $X \subseteq K$ $\mathcal{L}(A)$-definable.

1. *Step* 1: We first find a generalized ball U that is $\mathcal{L}_1(A)$-definable and such that the generic type $\eta_U(x) \in S^1(K)$ is consistent with X.

2. *Step* 2*:* We complete it to a full definable type in $\mathcal{L}_1(A)$.

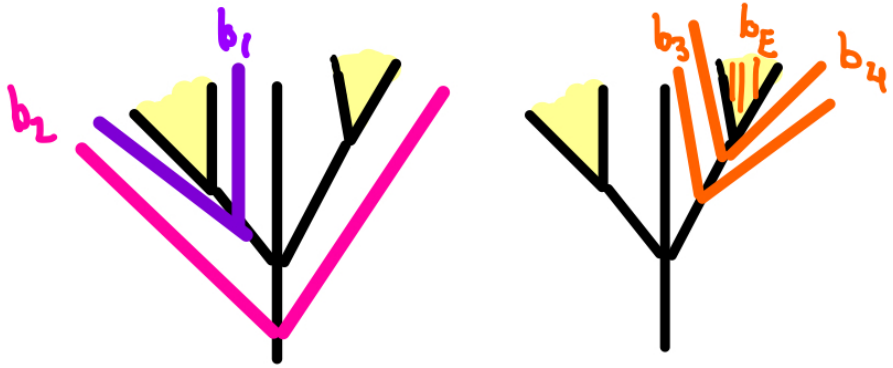
Key point: If $c \models \eta_U(x)$ then for any $a, a' \in U(K)$ one has $v(c-a) = v(c-a') = \gamma$.
Closed: unique extension.
Open: One needs to make a choice for cosets $\Delta + n\Gamma$ in Γ.

Sketch of the argument: how to do step 1?

1. Let $\mathcal{B}$ be the set of closed and open balls. We define the pre-order

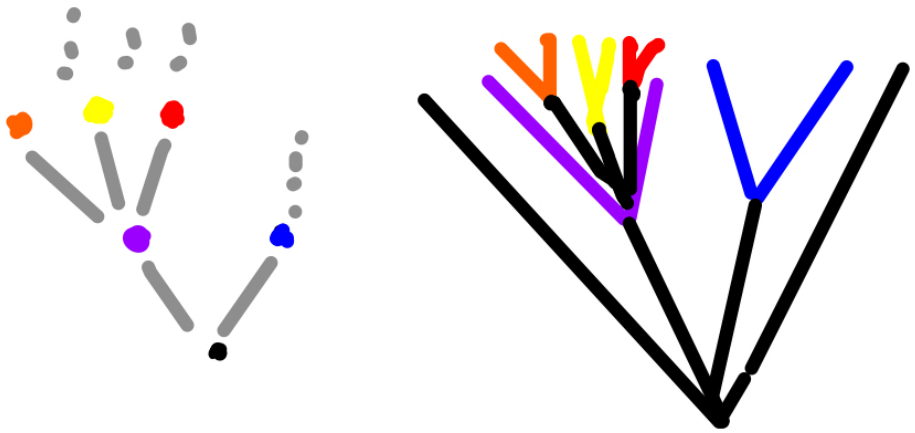
$$b_1 \trianglelefteq b_2 \text{ if and only if } b_1 \cap X \subseteq b_2 \cap X.$$

2. This is a pre-order with associated equivalence relation $\equiv$. The order $\mathcal{T}$ is a tree (remove the class of balls that don't intersect X).
3. For each class E we associate a generalized ball $b_E = \bigcap_{b \in E} b$.



Sketch of the argument: how to do step 1?

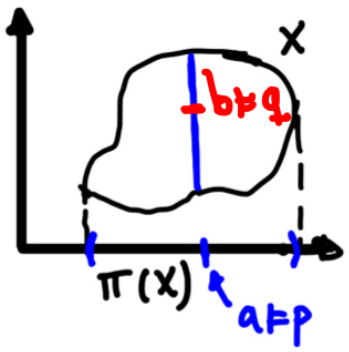
1. For each class E if $\eta_{b_E}(x)$ is not consistent with X, by compactness E has finitely many predecessors for $\trianglelefteq$, each of them in $\mathrm{acl}^{eq}(\ulcorner E \urcorner, A)$.
2. Either the statement holds or the tree has an initial discrete finitely branching tree of $\mathcal{L}_1(A)$-definable classes.



3. By 1-h-minimality. If the second case holds we can find b_E such that $b_E \cap X = b_E$. **Take its generic type!**

Sketch of the argument

1. For the higher dimensional case: we fiber!



Follows an argument on Hrushovski's criterion to show density of definable types in definable sets $X \subseteq K^n$.

Key ingredients to solve the issue: study the potential difference between acl and acl_1.
And solvability induction!

The coding of definable types in $\mathcal{L}_1$

Principle: Coding definable types is the same problem than:

1. Coding subspaces of K^n,
2. Coding valued vector space structures on K-vector spaces,
3. Some additional data in the value group (Γ^{eq} takes care of this)

The coding of definable types in $\mathcal{L}_1$

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2. Coding valued vector space structures on K-vector spaces,
3. Some additional data in the value group (Γ^{eq} takes care of this)

With separated basis: The valued vector structure on K^n is completely determined by the closed balls containing 0.

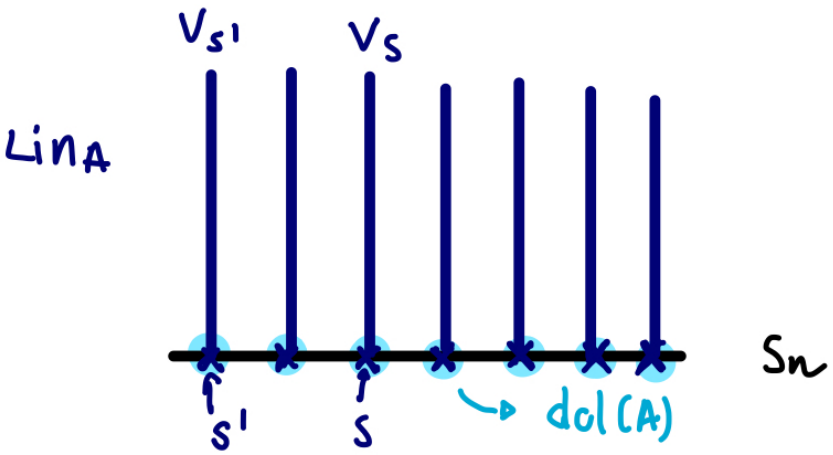
These are $\mathcal{O}$-submodules of K^n!

Step 2: Invariant extensions

Recall that for each rank n lattice $s \subseteq K^n$,
$V_s = s/\mathcal{M}s$ is a $\mathbf{k}$-vector space.

Define:

$$\mathrm{Lin}_A = \bigsqcup_{s \in \mathrm{dcl}_{ACVF}(A)} s/\mathcal{M}s.$$



Step 2: Invariant extensions

How to go from an $\mathcal{L}_1$ definable type to a complete type?

Theorem (Rideau-Kikuchi, V.)

Let $M \prec N \vDash T$ sufficiently saturated and homogeneous. Let $A = \mathrm{acl}^{eq}(A) \subseteq M$.
Let $a \in K(N)$ and assume that $\mathrm{tp}_1(a/M)$ is $\mathrm{Aut}(M/A)$-invariant.

Then $\mathrm{tp}(a/M)$ is $\mathrm{Aut}(M/ARV(M)\mathrm{Lin}_A(M))$-invariant.

Possible complain: $RV \cup \mathrm{Lin}_A$ is big!

Some facts

Even though it is very big! is stably embedded!

Proposition (Hils, Rideau-Kikuchi)

Let M be sufficiently saturated and homogeneous and D be a multi-sorted structure that is stably embedded. Let $e \in M$, then if e is fixed by every $\sigma \in \mathrm{Aut}(M/D(M))$ then $e \in \mathrm{dcl}(D(M))$.

In our context: $D = RV \cup \mathrm{Lin}_A$.

Weakly coding

Theorem (Rideau-Kikuchi, V.)

Let M be a henselian valued field of equicharacteristic zero whose value group has bounded regular rank. Let $e \in M^{eq}$ and $A = \mathrm{acl}^{eq}(e)$. Then:

$$e \in \mathrm{dcl}^{eq}(\mathcal{G}'(A) \cup (\mathrm{RV} \cup \mathrm{Lin}_{\mathcal{G}'(A)})^{eq}(A)).$$

$\mathcal{G}'(A) = \mathcal{G}(A) \cup \Gamma^{eq}(A)$.

Weakly coding

- Let $M \vDash T$ sufficiently saturated and homogeneous, $e \in M^{eq}$ and $A = \mathrm{acl}^{eq}(e)$.
- There is $\mathcal{L}$-definable map g and tuple $b \in K(M)$ such that $g(b) = e$. Let $X = g^{-1}(e)$.
- **Apply step** 1**!**
  We can find $p \in S^1_x(M)$ such that:
  - $p \cup X$ is consistent.
  - p is $\mathcal{L}_1(\mathcal{G}'(A))$-definable.
- Take $a \vDash p \cup X$ then $\mathrm{tp}_1(a/M)$ is $\mathcal{L}_1(\mathcal{G}'(A))$-definable.

Weakly coding

- **Apply step** 2:
  We had found $a \models p \cup X$ then $\mathrm{tp}_1(a/M)$ is $\mathcal{L}_1(\mathcal{G}'(A))$-definable.
  Then **The full type** $\mathrm{tp}(a/M)$- is $\mathrm{Aut}(M/\mathcal{G}'(A)RV(M)\mathrm{Lin}_{\mathcal{G}'(A)}(M))$ invariant!

- Since $e = g(a)$ for any $\sigma \in \mathrm{Aut}(M/\mathcal{G}'(A)RV(M)\mathrm{Lin}_{\mathcal{G}'(A)}(M))$, $\sigma(e) = e$.

- By **Fact** applied to $D = RV \cup \mathrm{Lin}_{\mathcal{G}'(A)}$ (is stably embedded!)

$$e \in \mathrm{dcl}^{eq}(\mathcal{G}'(A)RV(M)\mathrm{Lin}_{\mathcal{G}'(A)}(M))$$

The conclusion

- Since $e \in \mathrm{dcl}^{eq}(\mathcal{G}'(A)RV(M)\mathrm{Lin}_{\mathcal{G}'(A)}(M))$.
- There is h an $\mathcal{L}(\mathcal{G}'(A))$-definable function a tuple $c \in RV(M)^m \times \mathrm{Lin}^k_{\mathcal{G}'(A)}(M)$ such that $h(c) = e$.
- Take $Z = h^{-1}(e)$. This is an $\mathcal{G}'(A)$-definable set and $Z \subseteq RV \cup \mathrm{Lin}_{\mathcal{G}'(A)}$.
- Thus $\ulcorner Z \urcorner \in (RV \cup \mathrm{Lin}_{\mathcal{G}'(A)})^{eq}(A)$.
- Then $e \in \mathrm{dcl}^{eq}(\mathcal{G}'(A)\ulcorner Z \urcorner) \subseteq \mathrm{dcl}^{eq}(\mathcal{G}'(A) \cup (RV \cup \mathrm{Lin}_{\mathcal{G}'(A)})^{eq}(A))$.

Thanks!

Thank you for your attention!