An Imaginary Ax-Kochen/Ershov Principle: the equicharacteristic zero case

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Let $\nu : K \to \Gamma \cup \{\infty\}$ be a valuation.

1. $\mathcal{O} = \{ x \in K \mid \nu(x) \geq 0 \}$ is the valuation ring,
2. $\mathcal{M} = \{ x \in K \mid \nu(x) > 0 \}$ is its maximal ideal.

We refer to $k = \mathcal{O}/\mathcal{M}$ as the residue field and we call $\Gamma$ the value group.
It all started...

Theorem (Haskell, Hrushovski, Macpherson)

The theory of ACVF eliminates imaginaries once the geometric sorts are added, this is:

1. For each natural number \( n \), we add a sort \( S_n \) which consists of the codes of the \( \mathcal{O} \)-sublattices of \( K^n \) (i.e. the free \( \mathcal{O} \)-submodules of \( K^n \) on \( n \) generators).

2. For any \( s \in S_n \), we define \( \text{red}(s) = s/\mathcal{M}s \) (the reduction of \( s \) modulo \( \mathcal{M} \)). And we let \( T_n = \bigcup \{ \text{red}(s) \mid s \in S_n \} \).

\[ \text{Geo} = K \cup \bigcup_{n \in \mathbb{N}} S_n \cup \bigcup_{n \in \mathbb{N}} T_n. \]
Hard worked continued

Down to the geometric sorts:

- $p$-adics and their ultraproducts (E. Hrushovski, B. Martin and S. Rideau-Kikuchi),

- separably closed valued fields of finite imperfection degree (M. Hils, M. Kamensky, S. Rideau-Kikuchi),

- real closed valued fields (T. Mellor).
An-Imaginary Ax-Kochen/Ershov principle: a very good question

**SPINE PHILOSOPHY: AX-KOCHEN/ERSHOV PRINCIPLE**

**Theorem (Ax-Kochen/Ershov)**

Let \((K, k, \Gamma)\) and \((K', k', \Gamma')\) be two henselian valued fields of equicharacteristic zero, then \(K \equiv K'\) if and only if \(k \equiv_{\text{fields}} k'\) and \(\Gamma \equiv_{OAG} \Gamma'\).

**Principle**

The model theory of a henselian valued field of equicharacteristic zero is controlled by its residue field and its value group.

**Conjecture (Hrushovski, 2000)**

Is there an Imaginary Ax-Kochen/Ershov principle for henselian valued field encompassing all the previous results?
Two orthogonal strategies:

Following the Ax-Kochen principle one could try to break the question:

1. Hils, Rideau-Kikuchi: Make the value group very tame (definably complete) and understand the problems that the residue field will bring to the picture.

2. Me: make the residue field as tame as possible (e.g. algebraically closed) and understand the problems that the complexity of the value group will contribute to the problem.
Context: The Hils, Rideau-Kikuchi approach

Context

We have \((K, v)\) a henselian valued field of equicharacteristic zero. And let’s assume that:

1. \(\Gamma\) is either divisible or a \(\mathbb{Z}\)-group,
2. \(k\) is no longer algebraically closed, but an arbitrary field of characteristic zero.

Question

Which new sorts would be required to add to eliminate imaginaries that the residue field brings to the picture?

Spoiler! One will need to add the \(k\)-linear sorts.
The $k$-linear imaginaries

**Context:** Let $s \subseteq K^n$ be a lattice of rank $n$.

1. $V_s = s/Ms$ is a $k$-vector space.
2. So we want to consider the two sorted structure $(k, s/Ms)$.

**The language $\mathcal{L}_{\text{Vect}}$:** We have two sorts:

- one for the field $k$ equipped with the language of rings $\mathcal{L}_{\text{ring}} = \{+, \cdot, 0, 1\}$,
- one for the vector space $V$ with the group structure, i.e. equipped with $\mathcal{L}_G = \{0, +\}$.
- A map $\lambda : k \times V \to V$ interpreted as scalar multiplication.
The $k$-linear imaginaries

1. We consider the $L_{\text{Vect}}$-theory of dimension $n$ vector spaces over a field.

2. For each $X$ definable quotient of the vector space sort $V$ in $L_{\text{Vect}}$. For each $s \subseteq K^n$ a lattice of rank $n$:

   $$X^{(k,s/Ms)} := \text{the interpretation of } X$$
   in the structure $(k, s/Ms)$.

   We define $T_{n,X} = \bigcup_{s \in S_n} X^{(k,s/Ms)}$. 
Examples of $T_{n,\mathcal{X}}$

Recall: $T_{n,\mathcal{X}} = \bigsqcup_{s \in S_n} \mathcal{X}^{(k,s/\mathcal{M}s)}$ where $\mathcal{X}^{(k,s/\mathcal{M}s)}$ is the interpretation of $\mathcal{X}$ in $(k, s/\mathcal{M}s)$.

1. If $\mathcal{X} = V$, then $T_{n,\mathcal{X}} = T_n$ the usual $T_n$-sorts,
2. If $\mathcal{X}$ is the one element quotient of $V$ then $T_{n,\mathcal{X}} \cong S_n$.

Definition (The $k$ linear imaginaries)

$$k^{\text{leq}} := \bigsqcup_{n,\mathcal{X}} T_{n,\mathcal{X}}.$$
Theorem (Hils, Rideau-Kikuchi)

Let \((K, \nu)\) be a henselian valued field of equicharacteristic zero. Assume:

1. The value group \(\Gamma\) is definably complete (ODAG or a \(\mathbb{Z}\)-group).
2. The induced theory of \(k\) eliminates \(\exists^\infty\).

Then \(K\) weakly eliminates imaginaries down to the sorts \(\text{Geo} \cup k^{\text{leq}} \cup \Gamma^{\text{eq}}\).

Remark:

1. There is quite a lot of redundancy!
2. This just one piece of what they proved, e.g. extended structure and mixed characteristic case.
Context: The Second approach

Context
We have $(K, \nu)$ a henselian valued field of equicharacteristic zero. And we assume that:

1. $k$ is algebraically closed.
2. $\Gamma$ is a more general $OAG$, i.e. no longer divisible or a $\mathbb{Z}$-group.

Question
Which new sorts would be required to add if we want to obtain elimination of imaginaries once we take $\Gamma$ a more general ordered abelian group?

Spoiler: We will need to introduce the Stabilizer Sorts.
A little bit of more complexity in OAG

*Definably complete:* $(\mathbb{Q}, +, <, 0)$ and $(\mathbb{Z}, +, <, 0)$.

*Nice property:* No definable convex subgroups!

**Example:** $(\mathbb{Z}^2, <_{\text{lex}}, +, 0)$

**Picture:** More definable end-segments
OAG with bounded regular rank $= \text{finite spines}$

Next step: Having countably many definable convex subgroups, but not uniform definable families of convex subgroups.

**Fact:** [Farré] Let $\Gamma$ be an OAG with bounded regular rank, then it has countably many convex definable subgroups.

**Picture:** General map OAG
What are the stabilizer sorts?

**Key point:** They in essence add codes for all the definable \( \mathcal{O} \)-modules \( M \subseteq K^n \).

**Question:** So which are the new \( \mathcal{O} \)-submodules that appear when we make the value group with bounded regular rank?

**One-dimensional case:** There is a one-to-one correspondence between the \( \mathcal{O} \)-submodules of \( K \) and the end-segments of \( \Gamma \).

**Correspondence:**

**Picture:** More definable end-segments
The Stabilizer sorts

The higher dimensional case: If $M \subseteq K^n$ then $M \cong \oplus_{i \leq n} l_i$ where $l_i \subseteq K$ is an $O$-module.

There is a nicer presentation:

1. Let $B_n(K)$ be the group of upper triangular and invertible $n \times n$ matrices.
2. The stabilizer of $C_{(l_1,\ldots,l_n)}$

$$Stab_{(l_1,\ldots,l_n)} = \{ M \in B_n(K) \mid MC_{(l_1,\ldots,l_n)} = C_{(l_1,\ldots,l_n)} \}.$$  
3. The stabilizer sorts:

$$G = K \cup \{ B_n(K)/Stab_{(l_1,\ldots,l_n)} \mid l_i \in I \}$$
My result!

**Theorem (V.)**

Let $(K, v)$ be a henselian valued field of equicharacteristic zero, such that:

1. $k$ is algebraically closed,
2. $\Gamma$ has bounded regular rank.

Then $K$ weakly eliminates imaginaries down to $\mathcal{G} \cup \Gamma^{eq}$.

**Remark:** full elimination of imaginaries when $\Gamma$ is $dp$-minimal.
Combining forces

Context
Let \((K, \nu)\) be a henselian valued field of equicharacteristic zero. Suppose that its value group has bounded regular rank. Let \(T\) be is \(\mathcal{L}_{\text{div}}\)-first order theory.

Theorem (Rideau-Kikuchi,V.)

1. If \([\Gamma : n\Gamma] < \infty\) then \(T\) has weak elimination of imaginaries down to \(\mathcal{G} \cup k^{\leq} \cup \Gamma^{eq}\).

2. If we add an angular component map, then \(T\) has weak elimination of imaginaries down to \(\mathcal{G} \cup k^{\leq} \cup \Gamma^{eq}\).
An abstract criterion to prove elimination of imaginaries

Theorem (Hrushovski)

Let $T$ be a first order theory with home sort $K$ (meaning that $\mathcal{M}^\text{eq} = \text{dcl}^\text{eq}(K)$). Let $\mathcal{G}$ be some collection of sorts. Suppose that:

- **DENSITY OF DEFINABLE TYPES**: For every non-empty definable set $X \subseteq K$ there is an $\text{acl}^\text{eq}(\Gamma X \neg)$-definable type in $X$,

- **CODING OF DEFINABLE TYPES**: Every definable type in $K^n$ has a code in $\mathcal{G}$ (possibly infinite). This is, if $p$ is any (global) definable type in $K^n$, then the set $\Gamma p \neg$ of codes of the definitions of $p$ is interdefinable with some (possibly infinite) tuple from $\mathcal{G}$,

Then $T$ weakly eliminates imaginaries down to $\mathcal{G}$. 
The strategy

No longer a chance to have density of definable types, but we can go to the maximal unramified extension!

1. Step 1: Density of definable types (in a reduct)
2. Step 2: Invariant extensions
The strategy

Context

Let $K$ be a henselian valued field of equicharacteristic zero and $K_1 = K^{ur}$ be its maximal unramified extension.

We work in two languages:

1. $L_1$: For $K_1 = K^{ur}$
   The usual 2-sorted language $(K, \Gamma)$ with $\Gamma$ Morleyized.

2. $L$: For the structure $K$
   the 3-sorted language $(K, RV, \Gamma)$ with $\Gamma$ Morleyized.

Notation: We denote $\text{tp}_1$ and $\text{tp}$, $\text{acl}_1$ and $\text{acl}$, $S_n(K)$ and $S_n^1(K)$, etc... respectively.
Step 1: Density of $\mathcal{L}_1$ definable types

Theorem (Rideau-Kikuchi,V.)

Let $(K, \nu)$ be a henselian valued field of equicharacteristic zero whose value group has bounded regular rank. Let $A = \text{acl}^{eq}(A) \subseteq K^{eq}$ then for any $\mathcal{L}(A)$-definable subset $X \subseteq K^n$ there is a type $p(x) \in S^1_n(K)$ such that:

1. $p(x) \cup X$ is consistent.
2. It is $\mathcal{L}_1(\mathcal{G}(A) \cup \Gamma^{eq}(A))$-definable.

(It's canonical base can be coded in $\mathcal{G} \cup \Gamma^{eq}$)
The one-dimensional case: Let $X \subseteq K \mathcal{L}(A)$-definable.

1. **Step 1:** We first find a generalized ball $U$ that is $\mathcal{L}_1(A)$-definable and such that the generic type $\eta_U(x) \in S^1(K)$ is consistent with $X$.

2. **Step 2:** We complete it to a full definable type in $\mathcal{L}_1(A)$.

**Key point:** If $c \models \eta_U(x)$ then for any $a, a' \in U(K)$ one has $v(c - a) = v(c - a') = \gamma$.

Closed: unique extension.

Open: One needs to make a choice for cosets $\Delta + n\Gamma$ in $\Gamma$. 
Sketch of the argument: how to do step 1?

1. Let $B$ be the set of closed and open balls. We define the pre-order

   $b_1 \preceq b_2$ if and only if $b_1 \cap X \subseteq b_2 \cap X$.

2. This is a pre-order with associated equivalence relation $\equiv$. The order $\mathcal{T}$ is a tree (remove the class of balls that don’t intersect $X$).

3. For each class $E$ we associate a generalized ball $b_E = \bigcap_{b \in E} b$. 
Sketch of the argument: how to do step 1?

1. For each class $E$ if $\eta_{b_E}(x)$ is not consistent with $X$, by compactness $E$ has finitely many predecessors for $\subseteq$, each of them in $\text{acl}^{eq}(\Gamma E, A)$.

2. Either the statement holds or the tree has an initial discrete finitely branching tree of $L_1(A)$-definable classes.

3. By 1-h-minimality. If the second case holds we can find $b_E$ such that $b_E \cap X = b_E$. Take its generic type!
Sketch of the argument

1. For the higher dimensional case: we fiber!

\[ X \subseteq K^n. \]

Follows an argument on Hrushovski’s criterion to show density of definable types in definable sets \( X \subseteq K^n. \)

Key ingredients to solve the issue: study the potential difference between \( \text{acl} \) and \( \text{acl}_1. \)

And solvability induction!
The coding of definable types in $\mathcal{L}_1$

**Principle**: Coding definable types is the same problem than:

1. Coding subspaces of $K^n$,
2. Coding valued vector space structures on $K$-vector spaces,
3. Some additional data in the value group ($\Gamma^{eq}$ takes care of this)
The coding of definable types in $\mathcal{L}_1$

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**With separated basis:** The valued vector structure on $K^n$ is completely determined by the closed balls containing 0.

These are $\mathcal{O}$-submodules of $K^n$!
Step 2: Invariant extensions

Recall that for each rank $n$ lattice $s \subseteq K^n$, $V_s = s/\mathcal{M}s$ is a $k$-vector space.

Define:

$$\text{Lin}_A = \bigsqcup_{s \in \text{dcl}_{ACVF}(A)} s/\mathcal{M}s.$$
Step 2: Invariant extensions

How to go from an \( L_1 \) definable type to a complete type?

**Theorem (Rideau-Kikuchi, V.)**

Let \( M \prec N \models T \) sufficiently saturated and homogeneous. Let 
\( A = \acl^{eq}(A) \subseteq M \).

Let \( a \in K(N) \) and assume that \( \text{tp}_1(a/M) \) is \( \text{Aut}(M/A) \)-invariant.

Then \( \text{tp}(a/M) \) is \( \text{Aut}(M/ARV(M)\Lin_A(M)) \)-invariant.

Possible complain: \( RV \cup \Lin_A \) is big!
Some facts

Even though it is very big! is stably embedded!

Proposition (Hils, Rideau-Kikuchi)

Let $M$ be sufficiently saturated and homogeneous and $D$ be a multi-sorted structure that is stably embedded. Let $e \in M$, then if $e$ is fixed by every $\sigma \in \text{Aut}(M/D(M))$ then $e \in dcl(D(M))$.

In our context: $D = RV \cup \text{Lin}_A$. 
Weakly coding

Theorem (Rideau-Kikuchi, V.)

Let $M$ be a henselian valued field of equicharacteristic zero whose value group has bounded regular rank. Let $e \in M^{eq}$ and $A = acl^{eq}(e)$. Then:

$$e \in dcl^{eq}(G'(A) \cup (RV \cup \text{Lin}_{G'}(A))^{eq}(A)).$$

$G'(A) = G(A) \cup \Gamma^{eq}(A)$. 
Weakly coding

- Let $M \models T$ sufficiently saturated and homogeneous, $e \in M^{eq}$ and $A = acl^{eq}(e)$.

- There is $\mathcal{L}$-definable map $g$ and tuple $b \in K(M)$ such that $g(b) = e$. Let $X = g^{-1}(e)$.

- **Apply step 1!**
  We can find $p \in S^1_x(M)$ such that:
  - $p \cup X$ is consistent.
  - $p$ is $\mathcal{L}_1(\mathcal{G}'(A))$-definable.

- Take $a \models p \cup X$ then $tp_1(a/M)$ is $\mathcal{L}_1(\mathcal{G}'(A))$-definable.
Weakly coding

- **Apply step 2:**
  We had found $a \models p \cup X$ then $\text{tp}_1(a/M)$ is $\mathcal{L}_1(\mathcal{G}'(A))$-definable.
  Then **The full type** $\text{tp}(a/M)$- is $\text{Aut}(M/\mathcal{G}'(A)\text{RV}(M)\text{Lin}_{\mathcal{G}'(A)}(M))$ invariant!

- Since $e = g(a)$ for any $\sigma \in \text{Aut}(M/\mathcal{G}'(A)\text{RV}(M)\text{Lin}_{\mathcal{G}'(A)}(M))$, $\sigma(e) = e$.

- By **Fact** applied to $D = \text{RV} \cup \text{Lin}_{\mathcal{G}'(A)}$ (is stably embedded!)
  
  $$e \in \text{dcl}^{eq}(\mathcal{G}'(A)\text{RV}(M)\text{Lin}_{\mathcal{G}'(A)}(M))$$
The conclusion

- Since $e \in \text{dcl}^{eq}(G'(A)RV(M)\text{Lin}_{G'(A)}(M))$.

- There is $h$ an $\mathcal{L}(G'(A))$-definable function a tuple $c \in RV(M)^m \times \text{Lin}_{G'(A)}^k(M)$ such that $h(c) = e$.

- Take $Z = h^{-1}(e)$. This is an $G'(A)$-definable set and $Z \subseteq RV \cup \text{Lin}_{G'(A)}$.

- Thus $\models Z \models (RV \cup \text{Lin}_{G'(A)})^{eq}(A)$.

- Then
  $$e \in \text{dcl}^{eq}(G'(A)\models Z) \subseteq \text{dcl}^{eq}(G'(A) \cup (RV \cup \text{Lin}_{G'(A)})^{eq}(A)).$$
Thanks!

Thank you for your attention!