Limit groups, viewed by a logician
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The material presented in these notes, is certainly well-known to all specialists with some logic culture. It presents a model-theoretical proof of why a finitely presented group has finitely many maximal limit quotients.

Let G be a finitely presented group, generated by $\bar{a} = (a_1, \ldots, a_m)$, and with relations $w_1(\bar{a}), \ldots, w_r(\bar{a})$. We let F be the free group on k generators.

I want to show that G has only finitely many maximal limit quotients. The strategy of the proof is simple: first of all, show that limit quotients of G correspond exactly to m-tuples in a (fixed) non-principal ultrapower of F which satisfy certain equations. Then, general logic results give that G only has finitely many limit quotients. If one requires that these limit quotients come from converging sequences of homomorphisms which are distinct modulo composition by elements of $Aut(F)$, one can do so, by imposing that the m-tuples $\bar{b}$ realise a certain type over F, see at the end. I first need the lemma on descending chains of closed sets (well-known, I believe).

Lemma. Consider the free group F on k generators, let $\bar{a}$ be a tuple of elements of F, and $\Sigma(\bar{x}, \bar{a})$ a system of group equations. Then there are formulas $\varphi(\bar{x}, \bar{a}), \varphi_1(\bar{x}, \bar{a}), \ldots, \varphi_s(\bar{x}, \bar{a})$, which are conjuctions of equations in $(\bar{x}, \bar{a})$, and with the equations appearing in φ coming from $\Sigma(\bar{x}, \bar{a})$, such that:

(1) If a tuple $\bar{b}$ of F satisfies $\varphi(\bar{x}, \bar{a})$ then it satisfies all equations of $\Sigma(\bar{x}, \bar{a})$, and it also satisfies some $\varphi_i(\bar{x}, \bar{a})$. If a tuple $\bar{b}$ satisfies some $\varphi_i(\bar{x}, \bar{a})$ then it satisfies $\varphi(\bar{x}, \bar{a})$.

(2) Let $\mathcal{U}$ be a non-principal ultrafilter on $\mathbb{N}$, and $F^* = F^{\mathbb{N}}/\mathcal{U}$. For each i, there is a tuple $\bar{b}$ in F^* such that the equations $w(\bar{x}, \bar{a}) = 1$ satisfied by $\bar{b}$ are exactly those such that $F \models \forall \bar{x}(\varphi_i(\bar{x}, \bar{a}) \to w(\bar{x}, \bar{a}) = 1)$.

Proof. We know that F embeds in $SL_2(\mathbb{Z})$ and therefore in $SL_2(\mathbb{C})$. Hence, F^* embeds in $(SL_2(\mathbb{C}))^{\mathbb{N}}/\mathcal{U} = SL_2(\mathbb{C}^{\mathbb{N}}/\mathcal{U})$. In $SL_2(\mathbb{C}^{\mathbb{N}}/\mathcal{U})^m$, the Zariski closed set defined by the equations of $\Sigma(\bar{x}, \bar{a})$ is in fact defined by a finite number of these equations (because the Zariski topology is Noetherian), and this gives us the formula $\varphi(\bar{x}, \bar{a})$. Consider now the Zariski closure in $SL_2(\mathbb{C}^{\mathbb{N}}/\mathcal{U})^m$ of the set of tuples of F^m which satisfy $\varphi(\bar{x}, \bar{a})$. This set has finitely many irreducible components, say $Z_1, \ldots, Z_s$. For each of these components, we consider the set $\Sigma_i(\bar{x}, \bar{a})$ of group equations satisfied by the elements of Z_i. By Noetherianity, there a finite conjunction of equations $\varphi_i(\bar{x}, \bar{a})$ which implies all equations of $\Sigma_i(\bar{x}, \bar{a})$. The first assertion is therefore clear. Fix i. By irreducibility of Z_i, the union of finitely many Zariski-closed proper subsets of Z_i is a proper subset of Z_i, and therefore its complement in Z_i contains a point of F^m. By ω_1-saturation of F^*, there is some $\bar{b} \in F^*$, $\bar{b}$ a generic of Z_i over $\mathbb{Q}(\bar{a})$. This gives us the second assertion.

Remark. This does not show that every finitely generated subgroup of $F^{\mathbb{N}}/\mathcal{U}$ is finitely presented, only that, modulo the axioms of T (the set of universal sentences true in F), finitely many equations suffice. The theory T will indeed contain many sentences of the form $\forall \bar{x}(w(\bar{x}) = 1 \to w'(\bar{x}) = 1))$, so that one given equation could imply infinitely many.

Let h_n, $n \in \mathbb{N}$, be a convergent sequence of homomorphisms $h_n : G \to F$. The homomorphism $h^* : G \to F^* = F^{\mathbb{N}}/\mathcal{U}$, defined by $h^*(a_j) = (h_n(a_j))_{\mathcal{U}}$ for $j = 1, \ldots, m$, has kernel exactly $\bigcup_{m\in\mathbb{N}}\bigcap_{n\geq m} Ker(h_n)$. Hence the subgroup of F^* generated by $\bar{b}=h^*(\bar{a})$ is isomorphic to the limit group associated to the sequence (h_n).

Conversely, let $\bar{b}$ be an m-tuple of elements of F^* satisfying the equations $w_1(\bar{b}) = \cdots = w_r(\bar{b}) = 1$. If $\bar{b}_i$, $i\in\mathbb{N}$, are m-tuples of F such that $\bar{b}=(\bar{b}_i)_{\mathcal{U}}$, then the set $J=\{i\in\mathbb{N}\mid w_1(\bar{b}_i)=\ldots=w_r(\bar{b}_i)=1\}$ belongs to $\mathcal{U}$. [It is here that we use the finite presentation of G. This can be avoided by going to an uncountable index set I, taking an ultrafilter $\mathcal{F}$ on I which is closed under countable intersection, and then working in $F^I/\mathcal{F}$ instead of $F^{\mathbb{N}}/\mathcal{U}$]. Without loss of generality we may assume that $J=\mathbb{N}$. For each $i\in\mathbb{N}$ we then have a homomorphism $h_i: G\to F$, defined by $h_i(\bar{a})=\bar{b}_i$. Moreover, we have a homomorphism $h^*: G\to F^*$, defined by $h^*(\bar{a})=\bar{b}$.

It remains to extract from the h_i's a converging subsequence. We fix an enumeration $\{w_n(\bar{a})\mid n\in\mathbb{N}\}$ of the elements of G. We will construct by induction on n, an infinite descending chain S_n, $n\in\mathbb{N}$, of subsets of $\mathbb{N}$, having infinite intersection. To ensure it has infinite intersection, we construct at the same time an increasing sequence I_n, $n\in\mathbb{N}$, with each I_n a subset of S_n of size n. We start with $S_{-1}=\mathbb{N}$, $I_{-1}=\emptyset$. Our induction hypothesis is the following:

(a) $S_n\in\mathcal{U}$, $I_n\subseteq S_n$ has size n and contains I_{n-1}

(b) if $h^*(w_n(\bar{a}))=1$, then the set $\{i\in S_n\mid w_n(\bar{b}_i)\neq 1\}$ is finite, and if $h^*(w_n(\bar{a}))\neq 1$, then the set $\{i\in S_n\mid w_n(\bar{b}_i)=1\}$ is finite,

Assume S_n is defined, and consider $w_{n+1}(\bar{a})$. As $S_n\in\mathcal{U}$, S_n is infinite , and we let I_{n+1} be obtained by adjoining an element of $S_n\setminus I_n$ to I_n. Consider the set $U=\{i\in\mathbb{N}\mid w_{n+1}(\bar{b}_i)=1\}$. If $U\in\mathcal{U}$, we let $S_{n+1}=(U\cap S_n)\cup I_{n+1}$. If $U\notin\mathcal{U}$, then $V=(\mathbb{N}\setminus U)\in\mathcal{U}$, and we let $S_{n+1}=(V\cap S_n)\cup I_{n+1}$. By induction hypothesis, $S_n\in\mathcal{U}$, and this implies that $S_{n+1}\in\mathcal{U}$.

We let $S=\bigcap_n S_n$. It then contains $\bigcup_n I_n$, and is therefore infinite. From the construction, it is clear that the sequence of homomorphisms $\{h_i\mid i\in S\}$ is convergent, and that the limit group associated to this sequence is isomorphic to the subgroup of F^* generated by $\bar{b}$.

We apply the lemma (without parameters) to the system of equations $\Sigma(\bar{x}): w_1(\bar{x}) = \cdots = w_r(\bar{x})=1$, and will use the notation of the lemma. We fix m-tuples $\bar{b}_1,\ldots,\bar{b}_s$ of F^*, with $\bar{b}_i$ satisfying the conclusion of (3) for Z_i, and let L_i be the subgroup of F^* generated by $\bar{b}_i$, $f_i: G\to L_i$ the homomorphism defined by $f_i(\bar{a})=\bar{b}_i$.

If $h: G\to L$ is a limit quotient of G, then we may, by the first step, assume that $L\subset F^*$. Then $h(\bar{a})$ will belong to some Z_i, and therefore will satisfy all the group equations satisfied by $\bar{b}_i$. This means that there is $f: L_i\to L$ such that $h=f_i f$. Thus the maximal quotients of G are among the groups $L_1,\ldots,L_s$.

Note that the first step of the proof shows that the finitely generated subgroups of F^* are ω-residually free, and that conversely every finitely generated ω-residually free group embeds in F^*.

If we are only interested in the limit groups originating from convergent sequences where the elements are distinct modulo composition by an inner automorphism of F, the trick is as follows. Let (h_n), $n\in\mathbb{N}$, be such a sequence of homomorphisms, and let $h^*: G\to F^*$ be defined by $h^*(\bar{a})=(h_i(\bar{a}))_{\mathcal{U}}$. Composing each h_i by an inner automorphism of F is equivalent to conjugating the element $h^*(\bar{a})=\bar{b}$ by an element of F^*. Hence we may assume that for each i, the tuple $h_i(\bar{a})$ is closest to 1 among its conjugates (i.e., the max of the values of the distances to 1 of the tuples of $h_i(\bar{a})$ is minimal). Then the same is true in the 2-sorted structure $(F^*, \mathbb{N}^*, d^*)$ (where $\mathbb{N}^* = \mathbb{N}^{\mathbb{N}}/\mathcal{U}$, $d^*((g_i)_\mathcal{U}, (g'_i)_\mathcal{U}) = (d(g_i, g'_i))_\mathcal{U})$. As the sequence h_i is infinite and consists of distinct elements, we have that necessarily $d^*(1, \bar{b})$ is greater than all elements of $\mathbb{N} \subset \mathbb{N}^*$.

Having made these observations, it is immediate that $\bar{b}$ realises the following type $\Gamma(\bar{x})$:

– $\forall y\, d^*(1, y^{-1}\bar{x}y) \geq d^*(1, \bar{x})$.

– $d^*(1, \bar{x}) \geq c$ for all $c \in \mathbb{N}$. Note that each element c of $\mathbb{N}$ is definable (in $\mathbb{N}$ or in $\mathbb{N}^*$), as being the c-th successor of the smallest element of $\mathbb{N}$ (or of $\mathbb{N}^*$).

Conversely, given an m-tuple $\bar{b}$ of F^*, satisfying $\Gamma(\bar{x})$, one then defines as above homomorphisms $h_i : G \to F$. Note that no conjugate of $\bar{b}$ is in the copy of F sitting inside F^*, and that for every $c \in \mathbb{N}$, the set $T_c = \{i \mid d(1, h_i(\bar{a})) \geq c\}$ is in $\mathcal{U}$. Hence, when building the decreasing sequence S_n, one makes sure that the elements of $S_{n+1} \setminus I_n$ are in T_{n+1}. Then, from $\bigcap_n S_n$ one can extract an infinite subset J such that if $i < j$, then $d(1, h_i(\bar{a})) < d(1, h_j(\bar{b}))$. The sequence $(h_i)_{i \in J}$ is our desired convergent sequence.